Cyclic Douglas–Rachford Iterations

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A Common Problem

The feasibility problem asks

$$\text{Find } x \in C = \bigcap_{i=1}^{N} C_i,$$

where C and the C_i's are subsets of a Hilbert space, $\mathcal{H}$. Examples are:

- Linear systems of equations; i.e. affine C_i's.
- Matrix completion problems;[1] e.g. PSD matrices, protein structure.
- 3-SAT, TetraVex, Sudoku, nonograms;[2] (NP-complete, combin.)
- Various inverse problems; e.g. phase retrieval.

Projection algorithms are frequently used to solve such problems. At each step, these methods utilise the nearest point projections onto the C_i's (rather than directly onto C).

[1] **Douglas–Rachford feasibility methods for matrix completion problems** with F.J. Aragón Artacho and J.M. Borwein. Submitted Aug. 2013. arXiv:1308.4243

[2] **Recent results on Douglas–Rachford methods for combinatorial optimization** with F.J. Aragón Artacho and J.M. Borwein. Submitted 2013 arXiv:1305.2657

A Variational Toolkit

Let $S \subseteq \mathcal{H}$. The (nearest point) projection onto S is the (set-valued) mapping,

$$P_S x := \operatorname*{argmin}_{s \in S} \|s - x\|.$$

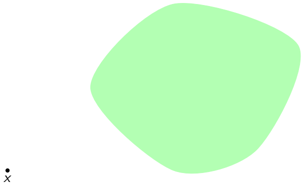
Variational characterisation of convex projections

Let $C \subseteq \mathcal{H}$ be closed and convex. Then $P_C x$ exist uniquely, $\forall x \in \mathcal{H}$, and

$$p = P_C x \iff p \in C \text{ and } \langle x - p, c - p \rangle \leq 0, \ \forall c \in C.$$

The reflection w.r.t. S is the (set-valued) mapping,

$$R_S := 2P_S - I.$$



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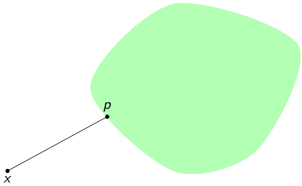

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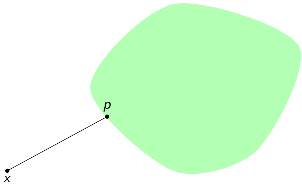

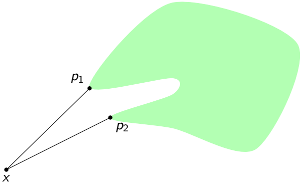

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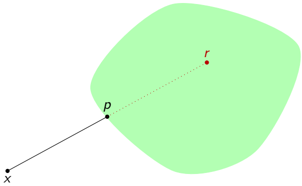

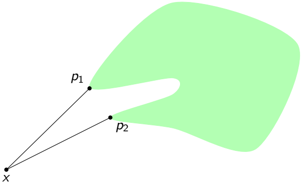

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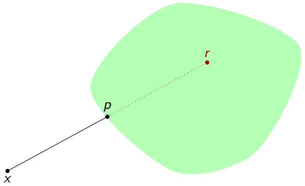

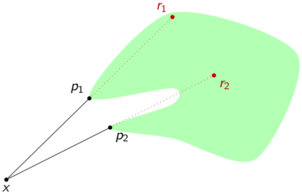

Some Common Projection Methods

A significant portion of the literature focuses on results like the following:

Theorems

Let $x_0 \in \mathcal{H}$. If $C_1, \ldots, C_N$ have certain properties then (x_n) converges in some sense to a point x having some properties.

Scheme	Iteration
Cyclic Projections	$x_{n+1} := \prod_{i=1}^{N} P_{C_i} x_n$
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- In the convex setting, projection methods are *fairly* well understood.
- In the non-convex setting, there are some useful beginnings.
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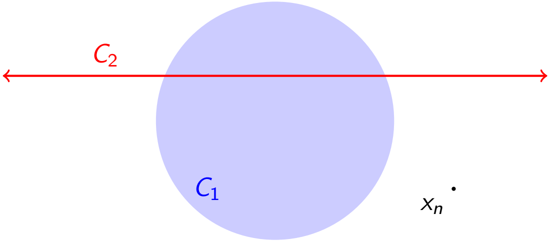
The Classical Douglas–Rachford Scheme

Theorem (Douglas–Rachford, Lions–Mercier)

Suppose $C_1, C_2 \subseteq \mathcal{H}$ are closed and convex with nonempty intersection. For any $x_0 \in \mathcal{H}$ define

$$x_{n+1} := T_{C_1,C_2} x_n \text{ where } T_{C_1,C_2} := \frac{I + R_{C_2} R_{C_1}}{2}.$$

Then $x_n \overset{w}{\rightharpoonup} x$ such that $P_{C_1} x \in C_1 \cap C_2$.



$$C_1 = \{x \in \mathcal{H} : \|x\| \leq 1\}, \quad C_2 = \{x \in \mathcal{H} : \langle a, x \rangle = b\}.$$

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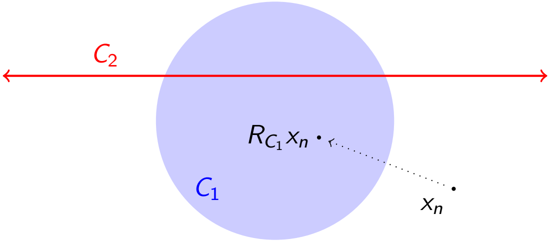

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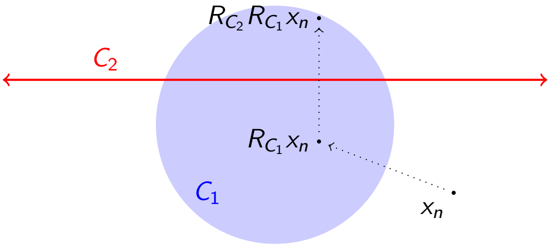

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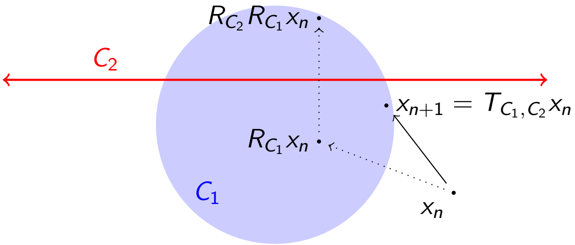

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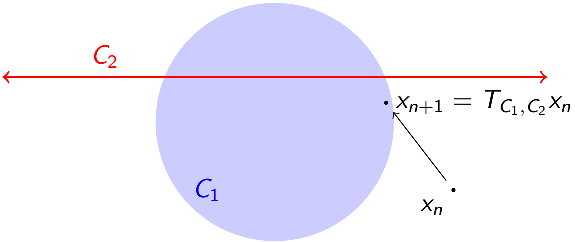

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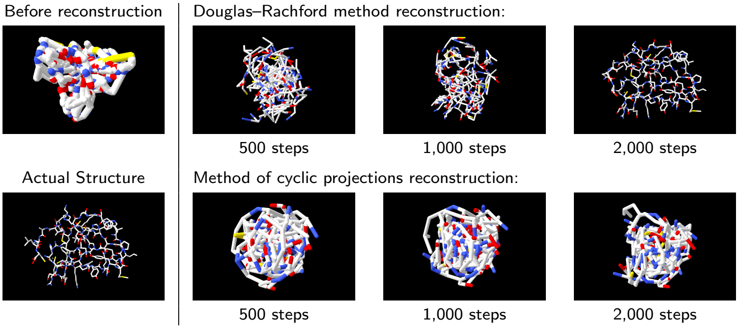

Table: Reconstructions of the protein 1PTQ (404 atoms) from “NMR” data.

- The method of cyclic projections works well in optical aberration correction (Hubble) (a non-convex feasibility problem) why not here?

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Failure of three set Douglas–Rachford (Example due to Brailey Sims)

$$T_{A,B,C} := \frac{1}{2}(I + R_C R_B R_A).$$

Let $x_0 = (-\sqrt{3}, -1), \alpha \in [2, \infty]$.

$$A := \{\lambda(0,1) : |\lambda| \leq \alpha\},$$
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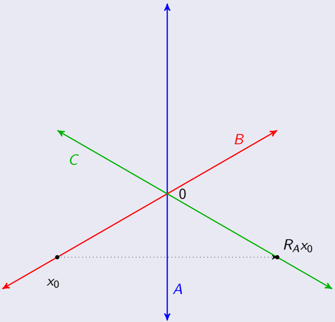

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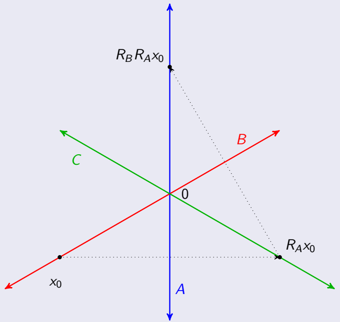

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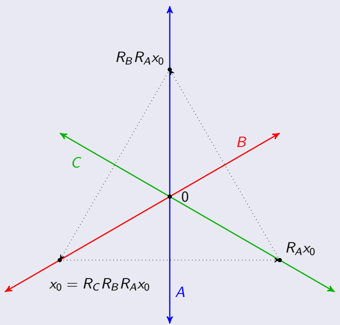

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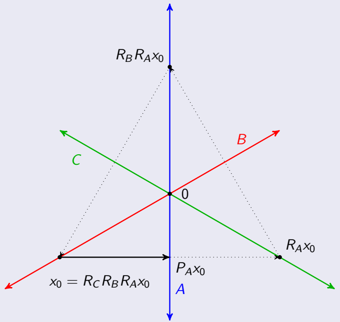

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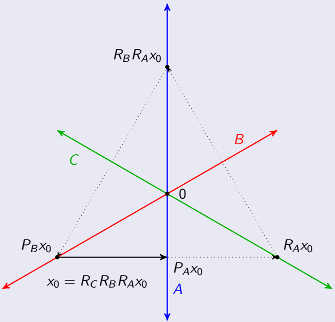

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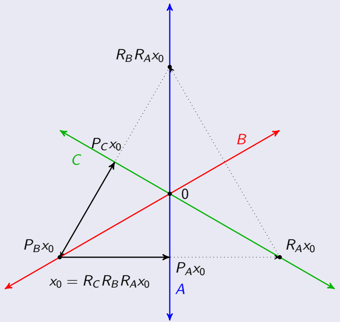

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Our investigation was motivated by the classical Douglas–Rachford scheme's good behaviour on various non-convex problems, and the absence of an obvious extension to feasibility problems with more than two sets. In the remainder of this talk I will discuss our findings. In particular, I will discuss the content of our recent paper:

A Cyclic Douglas–Rachford Iteration Scheme with J.M. Borwein.
Published online in *J. Optim. Theory. Appl.*, August 2013.
DOI: 10.1007/s10957-013-0381-x

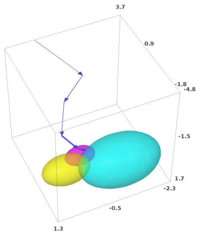

Fig. A cyclic Douglas–Rachford iteration for three balls constraints drawn in *Sage*.

Tools from Nonexpansive Mapping Theory

Let $T : \mathcal{H} \to \mathcal{H}$. Then T is:

- nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

- firmly nonexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

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Proposition (Nonexpansive properties)

The following are equivalent.

- T is firmly nonexpansive.
- $I - T$ is firmly nonexpansive.
- $2T - I$ is nonexpansive.
- $T = \alpha I + (1 - \alpha)R$, for $\alpha \in (0, 1/2]$ and some nonexpansive R.
- Many other characterisations.

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Nonexpansive properties of projections

Let $A, B \subseteq \mathcal{H}$ be closed and convex. Then

- $P_A := \mathrm{argmin}_{s \in S} \|\cdot - s\|$ is firmly nonexpansive.
- $R_A := 2P_A - I$ is nonexpansive.
- $T_{A,B} := \frac{1}{2}(I + R_B R_A)$ is firmly nonexpansive.

Nonexpansive maps are closed under composition, convex combinations, etc. Firmly nonexpansive maps need not be. E.g., Composition of two projections onto subspace in $\mathbb{R}^2$ (Bauschke–Borwein–Lewis, 1997).

Tools from Nonexpansive Mapping Theory (cont.)

- asymptotically regular if, for all $x \in \mathcal{H}$,

$$\|T^{n+1}x - T^n x\| \to 0.$$

Any firmly nonexpansive mapping with at least one fixed point is asymptotically regular.

Tools from Nonexpansive Mapping Theory (cont.)

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A useful Theorem for building iterative schemes:

Theorem (Opial, 1967)

Let $T : \mathcal{H} \to \mathcal{H}$ be nonexpansive and asymptotically regular. Set $x_{n+1} = Tx_n$. Then $x_n \overset{w}{\rightharpoonup} x$ such that $x \in \operatorname{Fix} T$.

Cyclic Douglas–Rachford Scheme

- In some sense, the classical Douglas–Rachford scheme is "unfair".
  - Reflection is always performed first with respect to the same set.
- A "fair" scheme might change the reflection order at each step.
- For two sets,

$$x_{n+1} := T_{C_2,C_1} T_{C_1,C_2} x_n = \left(\frac{I + R_{C_1} R_{C_2}}{2}\right) \left(\frac{I + R_{C_2} R_{C_1}}{2}\right) x_n.$$

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- For three sets,

$$\begin{aligned} x_{n+1} &:= T_{C_3,C_1} T_{C_2,C_3} T_{C_1,C_2} x_n \\ &= \left(\frac{I + R_{C_1} R_{C_3}}{2}\right)\left(\frac{I + R_{C_3} R_{C_2}}{2}\right)\left(\frac{I + R_{C_2} R_{C_1}}{2}\right) x_n. \end{aligned}$$

- And so on . . .

Cyclic Douglas–Rachford Scheme (cont.)

Theorem (Borwein–T 2013)

Let $C_1, \dots, C_N \subseteq \mathcal{H}$ be closed and convex with nonempty intersection. For any $x_0 \in \mathcal{H}$, define[3]

$$x_{n+1} = T_{[C_1\ C_2\ \dots\ C_N]} x_n \text{ where } T_{[C_1\ C_2\ \dots\ C_N]} := \prod_{i=1}^{N} T_{C_i, C_{i+1}}.$$

Then $x_n \xrightarrow{w} x$ such that $P_{C_i} x = P_{C_j} x$, for all indices i, j. In particular,

$$P_{C_j} x \in \bigcap_{i=1}^{N} C_i, \text{ for each index } j.$$

[3] Here and elsewhere, indices are understood modulo N.

Cyclic Douglas–Rachford Scheme (cont.)

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Proof.

First show $\mathrm{Fix}\, T_{[C_1 \ldots, C_N]} = \cap_{i=1}^{N} \mathrm{Fix}\, T_{C_i, C_{i+1}} \neq \emptyset$. Establish weak convergence to a fixed point, and use the variational characterisation of convex projections. □

[3] Here and elsewhere, indices are understood modulo N.

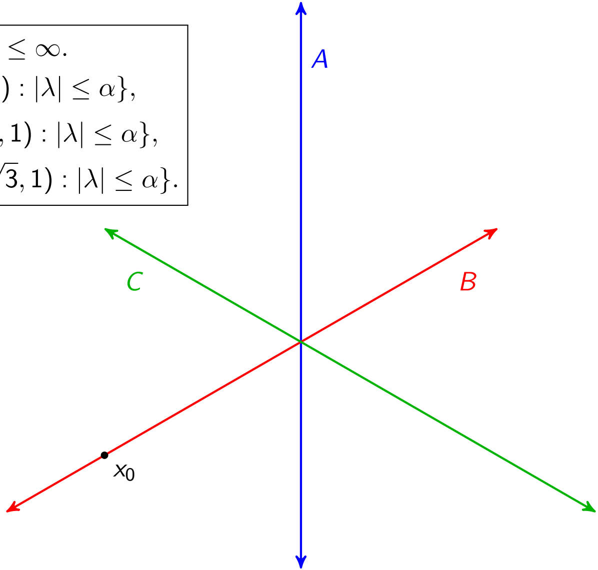
Example Revisited

$$2 \le \alpha \le \infty.$$

$$A := \{\lambda(0,1) : |\lambda| \le \alpha\},$$

$$B := \{\lambda(\sqrt{3},1) : |\lambda| \le \alpha\},$$

$$C := \{\lambda(-\sqrt{3},1) : |\lambda| \le \alpha\}.$$



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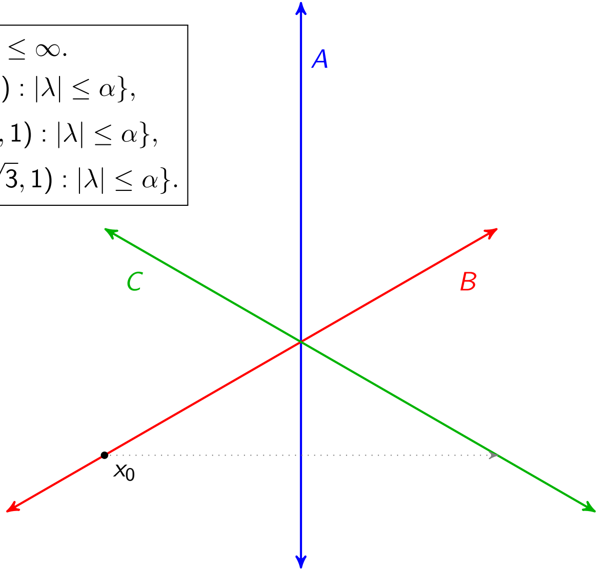

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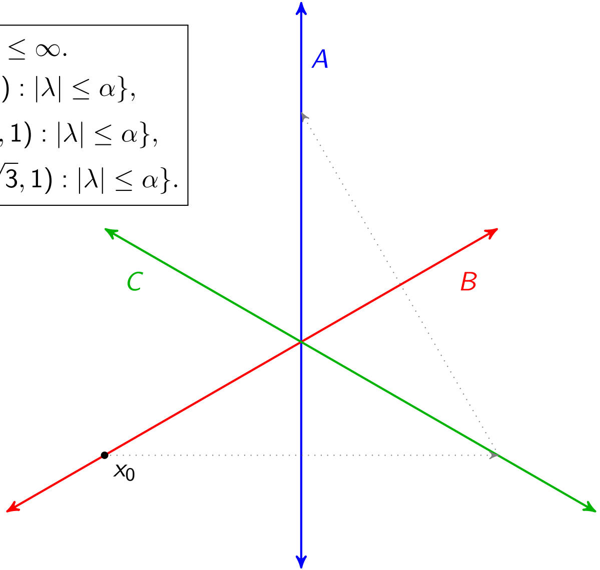

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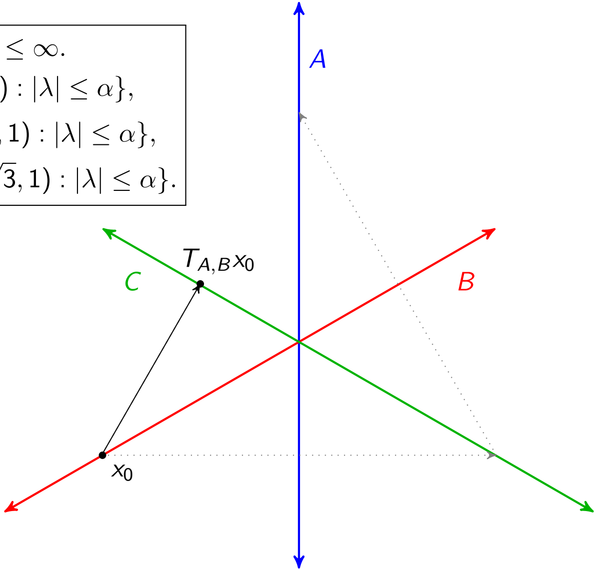

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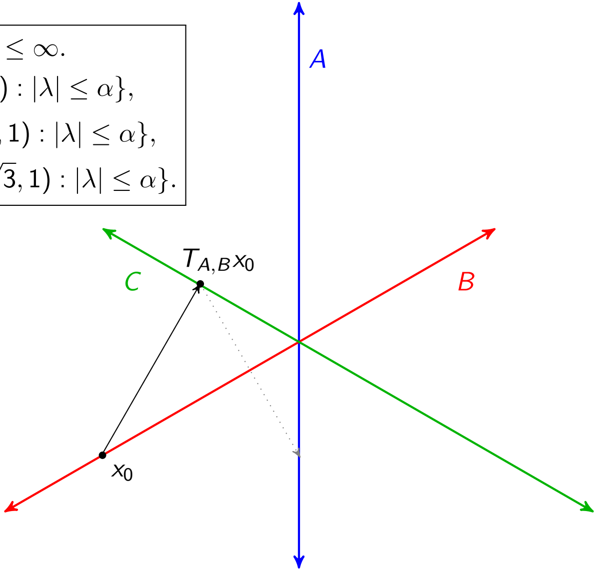

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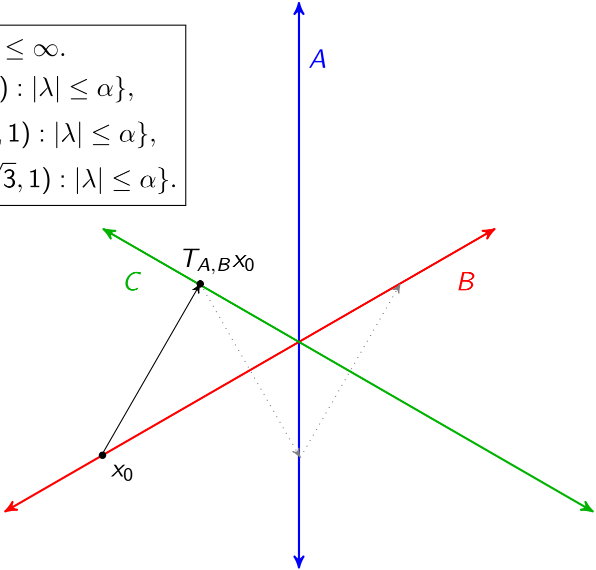

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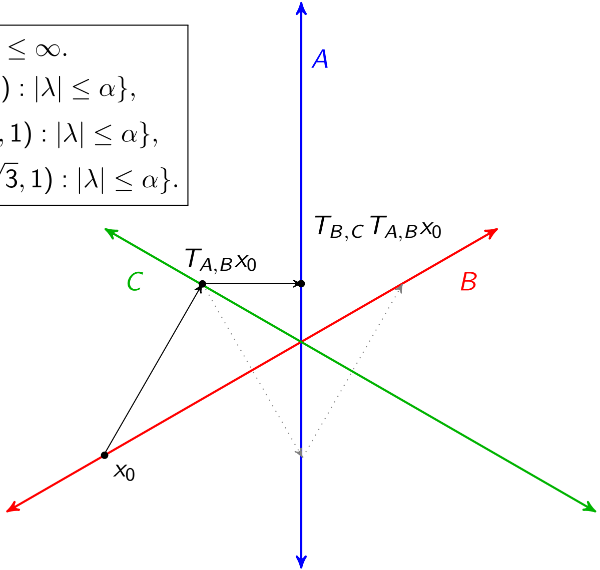

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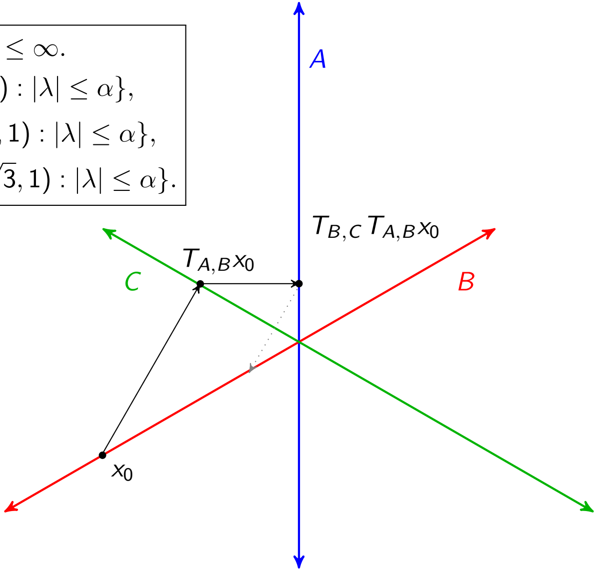

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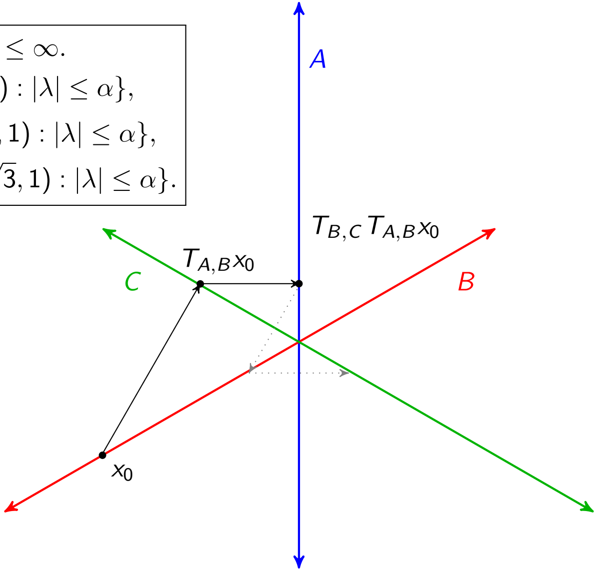

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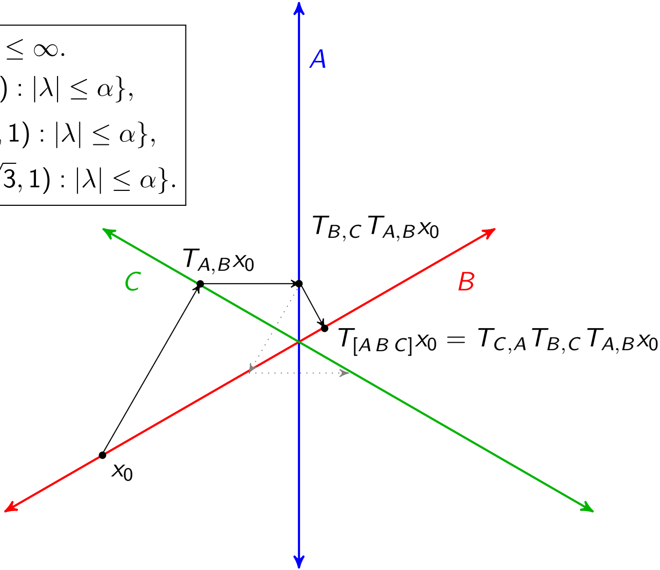

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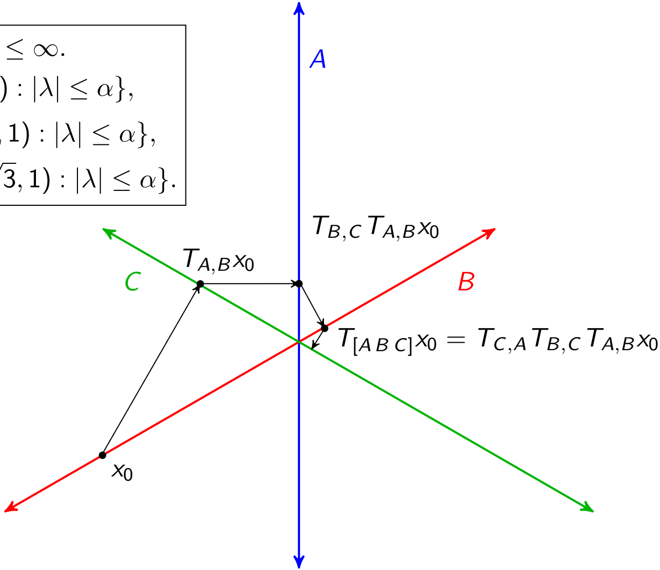

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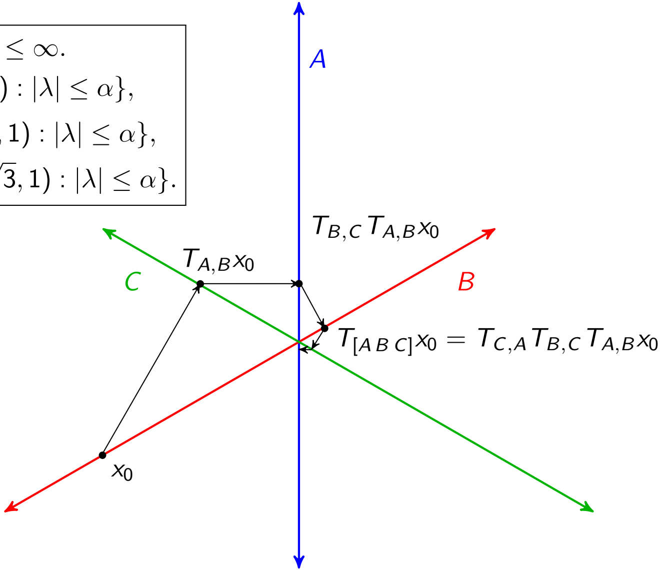

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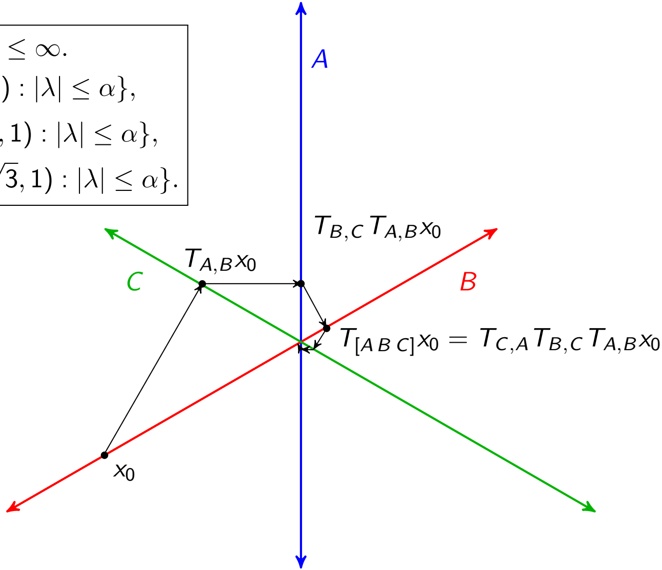

Cyclic Douglas–Rachford (cont.)

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Failure of Norm Convergence (Hundal, Matoušková–Reich, Kopecká)

Let $\mathcal{H} = \ell_2$ and $\{e_i\}$ denote the standard basis. Define

$$C_1 = \{x \in \mathcal{H} : \langle e_1, x \rangle = x_1 = 0\}, \quad C_2 = \text{an "unnatural" cone}.$$

Then $C_1 \cap C_2 = \{0\}$. There exists $x_0 \in \mathcal{H}$ such that $T^n_{[C_1\ C_2]} x_0$ does not converge in norm

- C_1 is a closed subspace, C_2 a closed convex cone.
- For appropriate initial points, the cyclic Douglas–Rachford iterations and the alternating projections iterations coincide.
- Both converge weakly to 0, the unique point in the intersection.
- (Bauschke–Borwein 1993) Conjecture norm convergence if C_1 is affine, finite codimension, and $C_2 = L_2^+(\Omega, \mu)$. True for codim. 1.

General Framework

The cyclic Douglas–Rachford method framework applies more generally.

Theorem (Borwein–T 2013)

Let $C_1, \dots, C_N \subseteq \mathcal{H}$ be closed and convex with nonempty intersection. For any $x_0 \in \mathcal{H}$, define

$$x_{n+1} := Tx_n \text{ where } T := \prod_{i=1}^{M} T_i.$$

Further, suppose

1. T is nonexpansive and asymptotically regular,
2. $\operatorname{Fix} T = \bigcap_{i=1}^{M} \operatorname{Fix} T_i \neq \emptyset$,
3. $P_{C_i} \operatorname{Fix} T_i \subseteq C_{i+1}$, for each index i.

Then $x_n \overset{w}{\rightharpoonup} x$ such that $P_{C_i}x = P_{C_j}x$, for all indices i, j. In particular,

$$P_{C_j}x \in \bigcap_{i=1}^{N} C_i, \text{ for each index } j.$$

Averaged Douglas–Rachford Scheme

Theorem (Borwein–T 2013)

Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex with nonempty intersection. For any $x_0 \in \mathcal{H}$, define

$$x_{n+1} := \frac{1}{N}\left(\sum_{i=1}^{N} T_{C_i,C_{i+1}}\right) x_n.$$

Then $x_n \overset{w}{\rightharpoonup} x$ such that $P_{C_i}x = P_{C_j}x$, for all indexes i,j. In particular,

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Proof. (Performed in $\mathcal{H}^N$).

Apply the previous Theorem to the sequence defined by

$$\mathbf{x}_{n+1} := P_D(T_1, T_2, \ldots, T_N)\mathbf{x}_n,$$

where $D = \{(x, x, \ldots, x) \in \mathcal{H}^N : x \in \mathcal{H}\}$. □

- Other applicable variants! e.g. cyclic project-project-average.

Infeasible Iterations (Alternating Projections)

Consider $A, B \subseteq \mathcal{H}$ with possibly empty intersection. For convenience, we introduce the sequences (a_n) and (b_n) where

$$x_0 \overset{P_A}{\mapsto} a_1 \overset{P_B}{\mapsto} b_1 \overset{P_A}{\mapsto} a_2 \overset{P_B}{\mapsto} b_2 \overset{P_A}{\mapsto} a_3 \overset{P_B}{\mapsto} \dots$$

Further define

$$E := \{x \in A : d(x, B) = d(A, B)\}, \; F := \{x \in B : d(x, A) = d(A, B)\}.$$

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Theorem (Bauschke–Borwein 1994)

Let $A, B \subseteq \mathcal{H}$ be closed and convex. Exactly one of the following alternatives hold.

(a) $E, F = \emptyset$, $\|a_n\|, \|b_n\| \to \infty$.

(b) $E, F \neq \emptyset$, $a_n \overset{w}{\rightharpoonup} a \in E$, $b_n \overset{w}{\rightharpoonup} b \in F$ where $b = P_B a$ and $a = P_A b$. Furthermore, $\|a - b\| = d(A, B)$ and $b_n - a_n, \ b_n - a_{n+1} \to b - a$.

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- Does not generalise to more than two sets: "*There is no variational characterization of the cycles in the method of periodic projections*", Baillion–Combettes–Cominetti (2012).

Infeasible Iterations (cont.)

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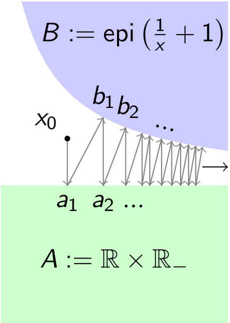
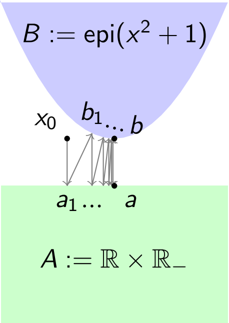


Example of (a).



Example of (b).

Infeasible Iterations (Alternating Douglas-Rachford)

Similarly we introduce the sequences (α_n) and (β_n) where

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The difficulty is $\text{Fix } T_{A,B} \neq \emptyset \iff A \cap B \neq \emptyset$. In the empty case,

$$\text{Fix } T_{[C_1 \dots, C_N]} \supseteq \bigcap_{i=1}^{N} \text{Fix } T_{C_i, C_{i+1}} = \emptyset.$$

Theorem (Borwein–T 201?)

Let $A, B \subseteq \mathcal{H}$ be closed and convex. Exactly one of the following alternatives hold.

(a) $E, F, \text{Fix } T_{[A\,B]}, \text{Fix } T_{[B\,A]} = \emptyset$, and $\|\alpha_n\|, \|\beta_n\| \to \infty$.

(b) $E, F, \text{Fix } T_{[A\,B]}, \text{Fix } T_{[B\,A]} \neq \emptyset$, and

$$\alpha_n \overset{w.}{\rightharpoonup} \alpha \in \text{Fix } T_{[A\,B]}, \quad \beta_n \overset{w.}{\rightharpoonup} \beta \in \text{Fix } T_{[B\,A]},$$

where $\beta = T_{A,B}\alpha$ and $\alpha = T_{B,A}\beta$. Furthermore,

$$\beta - \alpha = P_B\beta - P_A\alpha, \quad \|P_B\beta - P_A\alpha\| = d(A, B),$$

and $\beta_n - \alpha_n, \beta_{n+1} - \alpha \to \beta - \alpha$.

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and $\beta_n - \alpha_n, \beta_{n+1} - \alpha \to \beta - \alpha$.

- cf. Classical Douglas–Rachford: If $A \cap B = \emptyset$ then $\|x_n\| \to \infty$.

Closing Remarks and Future Work

Some avenues for future investigation include:

1. Better understand asymptotics in the (two set) infeasible case.
   - Is there a variational characterisation for more than two sets?
2. Norm convergence assuming regularity *a lá* Bauschke–Borwein.
3. Non-convex settings:
   - Euclidean sphere and affine subspace: Aragón–Borwein–Sims.
   - Local relaxations of firm nonexpansivity: Hesse–Luke.
4. Applications & computational studies: Initial results are promising!
   - 200 ball constraints in $\mathbb{R}^{2000}$, implemented in *Python*:
     - Classical Douglas–Rachford: $\sim$ 30s for a solution with error $\sim 10^{-4}$.
     - Cyclic Douglas–Rachford: $\sim$ 0.5s for a solution with error $\sim 10^{-25}$.

A Cyclic Douglas–Rachford Iteration Scheme with J.M. Borwein. Published online in *J. Optim. Theory. Appl.*, August 2013. DOI: 10.1007/s10957-013-0381-x

Many resources can be found at:

http://carma.newcastle.edu.au/DRmethods