Cyclic Douglas–Rachford Iterations

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Talk Outline

Feasibility Problems

The N-set convex feasibility problem asks:

$$\text{Find } x \in \bigcap_{i=1}^{N} C_i \subseteq \mathcal{H}, \qquad \text{(CFP)}$$

where C_i are closed and convex, $\mathcal{H}$ a Hilbert space.

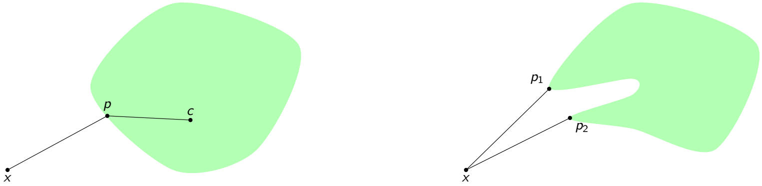
A common approach is the use of projection algorithms.

- von Neumann's alternating projection method (cyclic projections).
- Dysktra's method.
- Douglas–Rachford method.
- Many variants exist!

Projections, Reflections

Let $S \subseteq \mathcal{H}$. The (nearest point) projection of x onto S is the (set-valued) mapping defined by

$$P_S(x) := \arg\min_{s \in S} \|x - s\|.$$



Variational Characterisation of Projections

Let $C \subseteq \mathcal{H}$ be closed and convex. Then $P_C(x)$ exists uniquely $\forall x \in \mathcal{H}$, and

$$P_C(x) = p \iff \langle x - p, c - p \rangle \le 0, \quad \forall c \in C.$$

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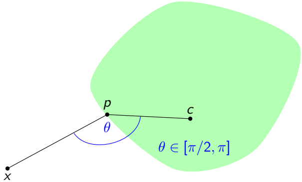

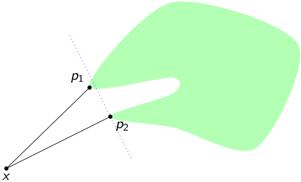

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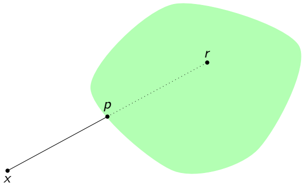
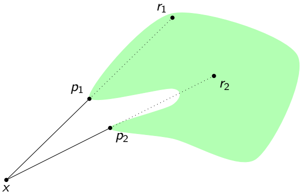
Projections, Reflections

Let $S \subseteq \mathcal{H}$. The reflection of x onto S is the (set-valued) mapping defined by

$$R_S(x) := 2P_S(x) - x.$$





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Classical Douglas–Rachford

Douglas–Rachford Scheme (Douglas–Rachford, 1956 & Lions–Mercier, 1979)

Let $A, B \subseteq \mathcal{H}$ be closed and convex with $A \cap B \neq \emptyset$. For any $x_0 \in \mathcal{H}$, set $x_{n+1} := T_{A,B} x_n$ where

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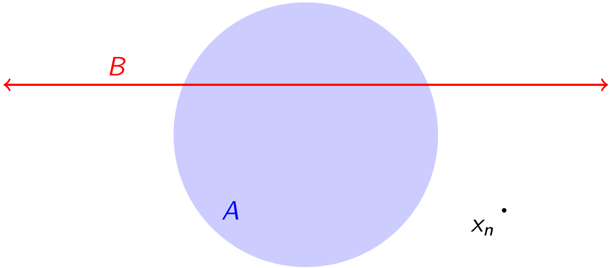
Then (x_n) converges weakly to x such that $P_A x \in A \cap B$.



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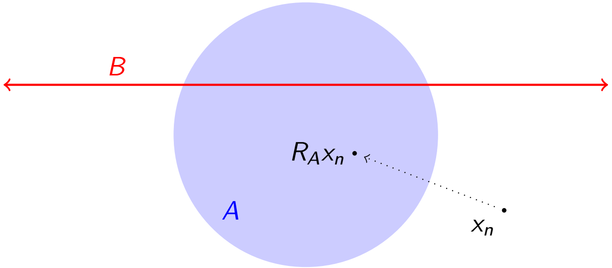

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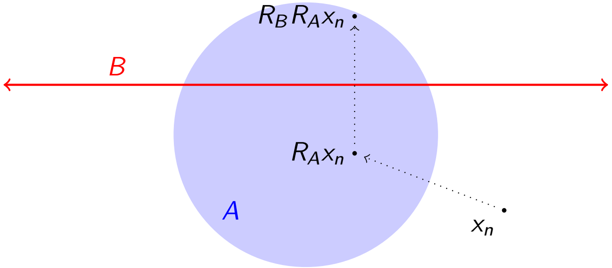

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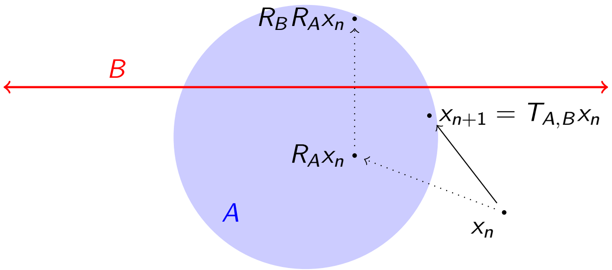

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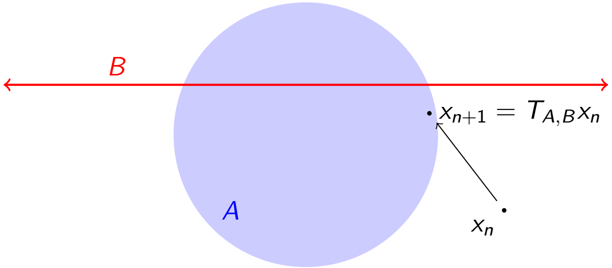

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Some Tools

Let $T : \mathcal{H} \to \mathcal{H}$. Then T is:

- nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

- firmly nonexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

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Proposition (Nonexpansive properties)

The following are equivalent.

- T is firmly nonexpansive.
- $I - T$ is firmly nonexpansive.
- $2T - I$ is nonexpansive.
- $T = \alpha I + (1 - \alpha)R$, for $\alpha \in (0, 1/2]$ and some nonexpansive R.
- Many other characterisations.

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Nonexpansive properties of projections

Let $A, B \subseteq \mathcal{H}$ be closed and convex. Then

- $P_A := \arg\min_{s \in S} \| \cdot - s\|$ is firmly nonexpansive.
- $R_A := 2P_A - I$ is nonexpansive.
- $T_{A,B} := (I + R_B R_A)/2$ is firmly nonexpansive.

Nonexpansive maps are closed under composition, convex combinations, etc. Firmly nonexpansive maps need not be. E.g. Composition of two projections onto subspace in $\mathbb{R}^2$ (Bauschke–Borwein–Lewis, 1997).

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- asymptotically regular if, for all $x \in \mathcal{H}$,

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Theorem (Opial, 1967)

Let $T : \mathcal{H} \to \mathcal{H}$ be nonexpansive and asymptotically regular. Set $x_{n+1} = T^n x_n$. Then (x_n) converges weakly to a point in $\operatorname{Fix} T$.

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Corollary

The Douglas–Rachford scheme converges weakly to a point $x \in \text{Fix}\, T_{A,B}$.

Some Tools

What do fixed points of $T_{A,B}$ look like?

$$
\begin{aligned}
x \in \operatorname{Fix} T_{A,B} &\iff x = \frac{x + R_B R_A x}{2} \\
&\iff x = 2P_B R_A x - R_A x \\
&\iff x = 2P_B R_A x - 2P_A x + x \\
&\iff P_A x = P_B R_A x
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What if the feasibility problem has more than 2 sets? Can we generalise?

Product Reformulation

$$\text{Find } x \in \bigcap_{i=1}^{N} C_i \subseteq \mathcal{H}. \tag{CFP}$$

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$$C := \prod_{i=1}^{N} C_i, \quad D := \{(x, x, \ldots, x) \in \mathcal{H}^N : x \in \mathcal{H}\}.$$

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Moreover, the projections can be computed. If $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_n) \in \mathcal{H}^N$,

$$P_C \mathbf{z} = \prod_{i=1}^{N} P_{C_i} \mathbf{z}_i, \quad P_D \mathbf{z} = \left(\frac{1}{N} \sum_{i=1}^{N} \mathbf{z}_i \right)^N.$$

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- Possible that $P_A x, P_B x, P_C x \notin A \cap B \cap C$.

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Then $A \cap B \cap C = \{0\}$.

We have $x_0 \in \operatorname{Fix} T_{A,B,C}$. However,

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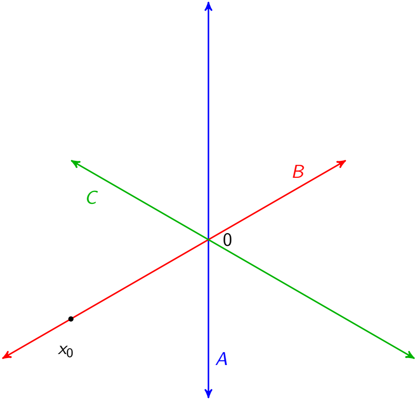

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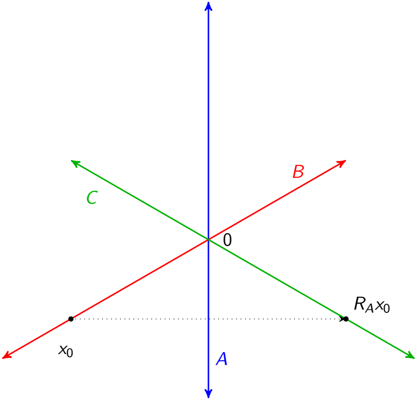

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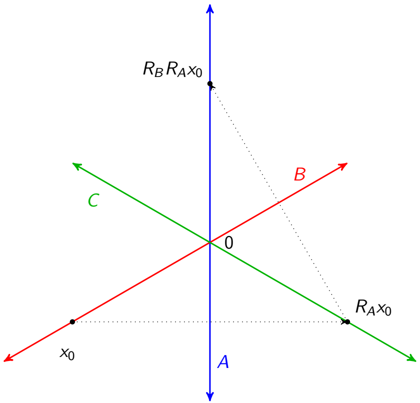

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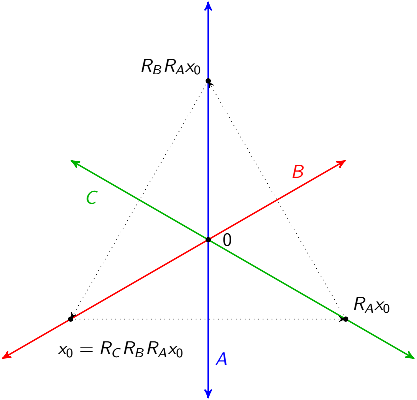

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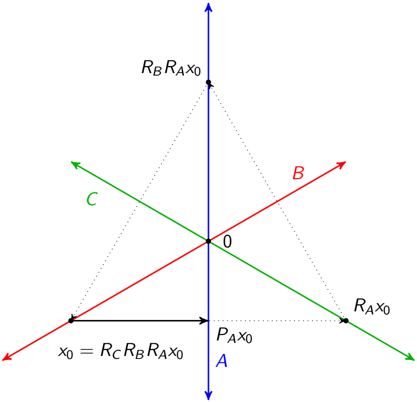

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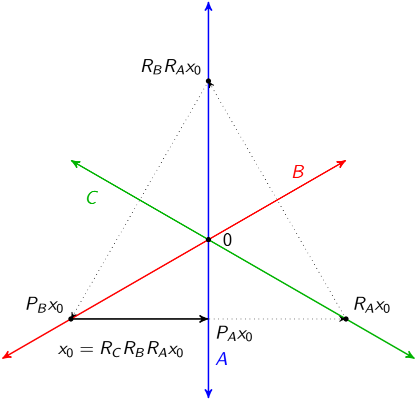

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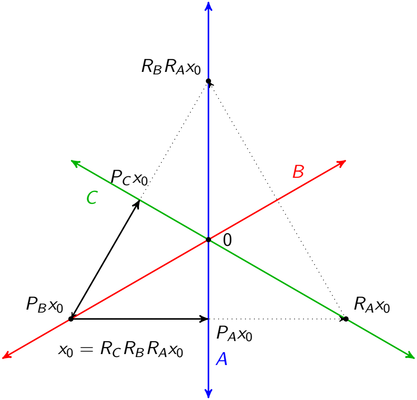

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A cyclic Douglas–Rachford iteration

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$$T_{[C_1\, C_2\, \dots\, C_N]} := T_{C_1,C_N}T_{C_N,C_{N-1}}\dots T_{C_2,C_3}T_{C_1,C_2}.$$

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In the $N = 2$ case, the mapping is:

$$T_{[A\, B]} = T_{B,A} T_{A,B} = \left(\frac{I + R_A R_B}{2} \right) \left(\frac{I + R_B R_A}{2} \right).$$

Cyclic Douglas–Rachford

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Note: The iteration can be applied with or without the product space formulation.

Proof Sketch

$$P_S(x) := \arg\min_{s \in S} \|x - s\|, \quad R_S = 2P_S - I, \quad T_{A,B} = \frac{I + R_B R_A}{2}.$$

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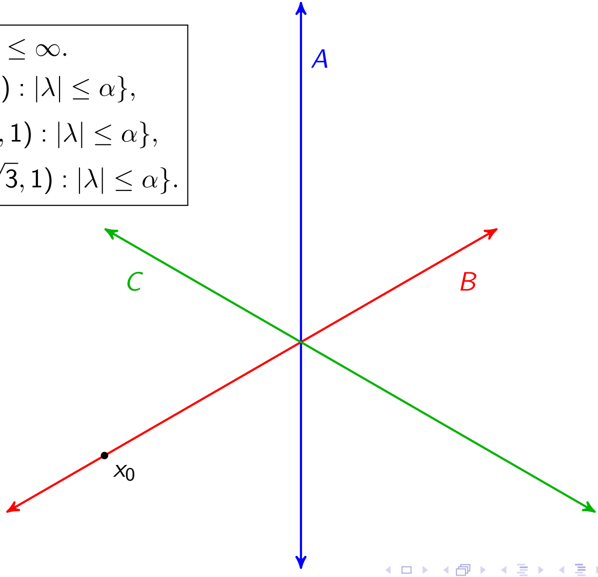

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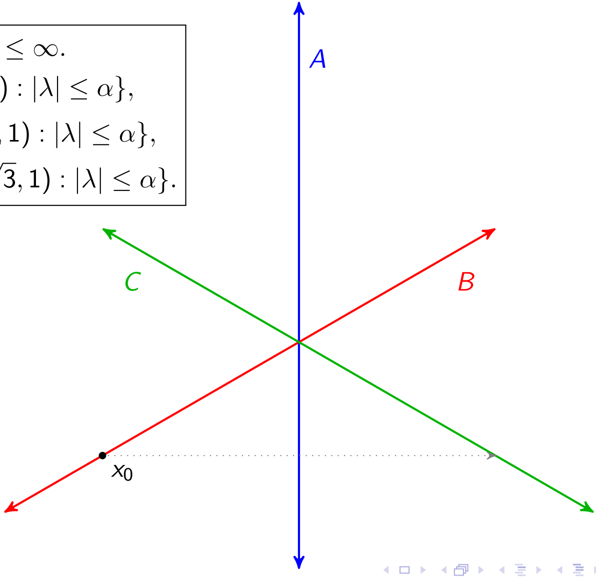

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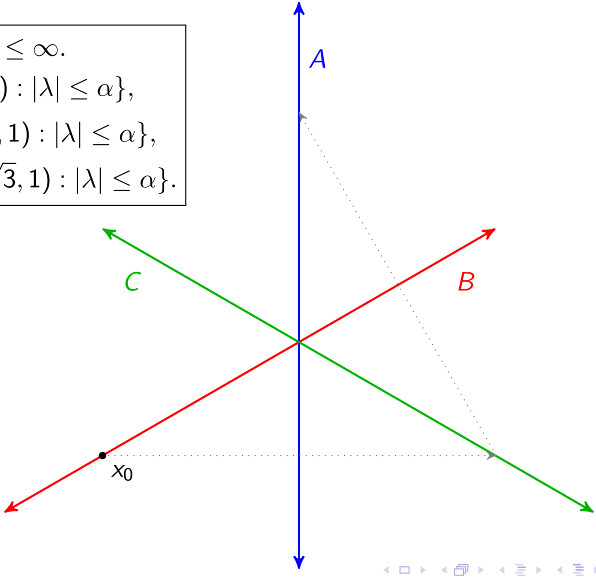

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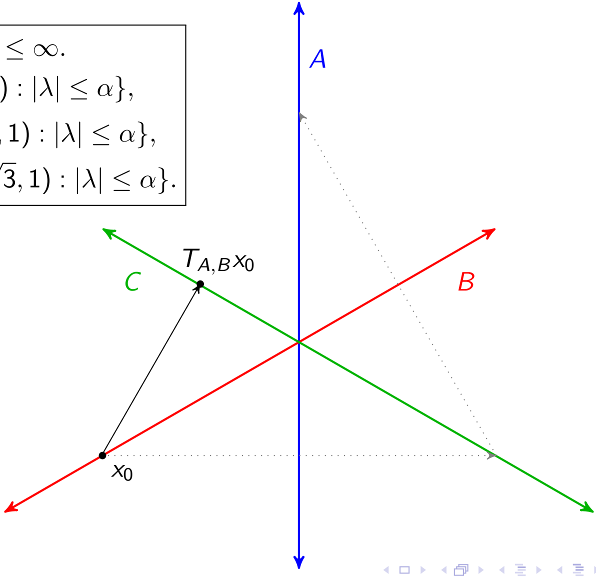

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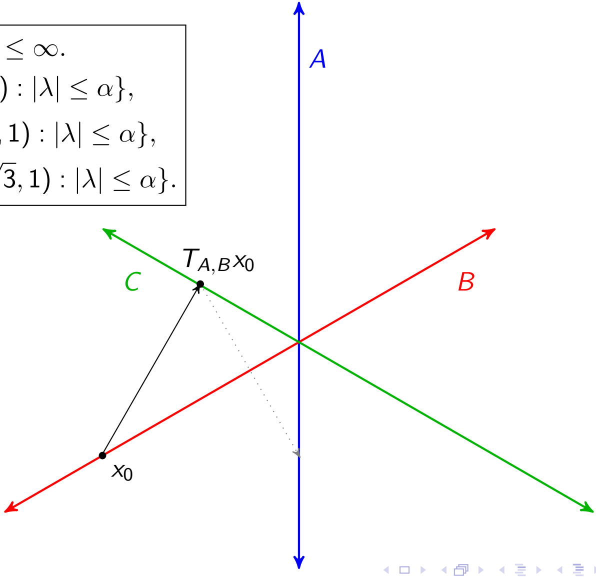

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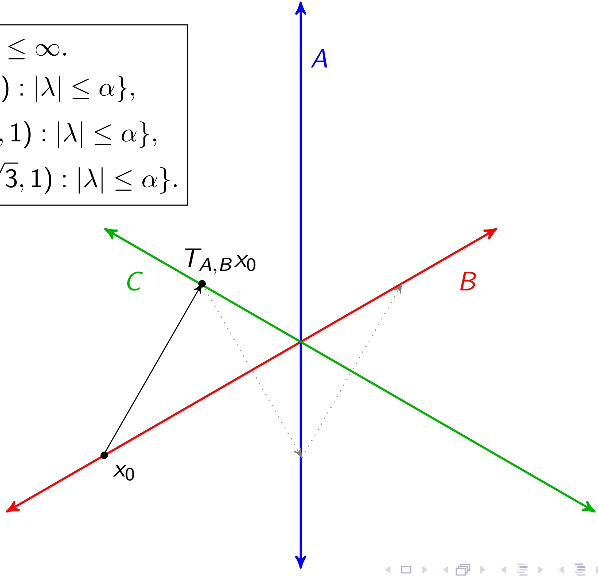

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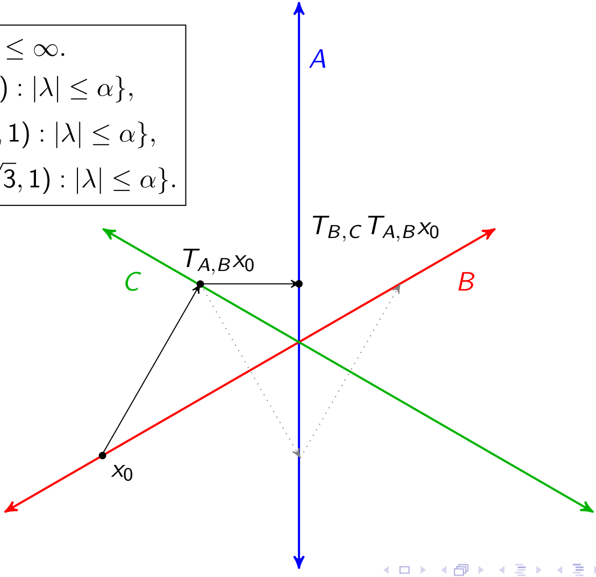

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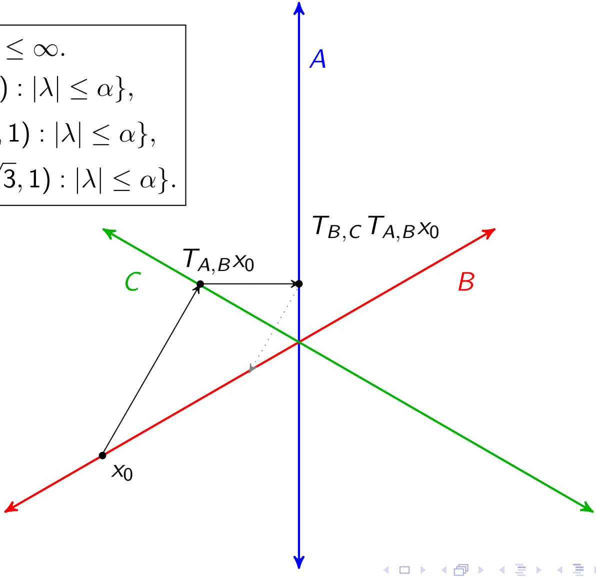

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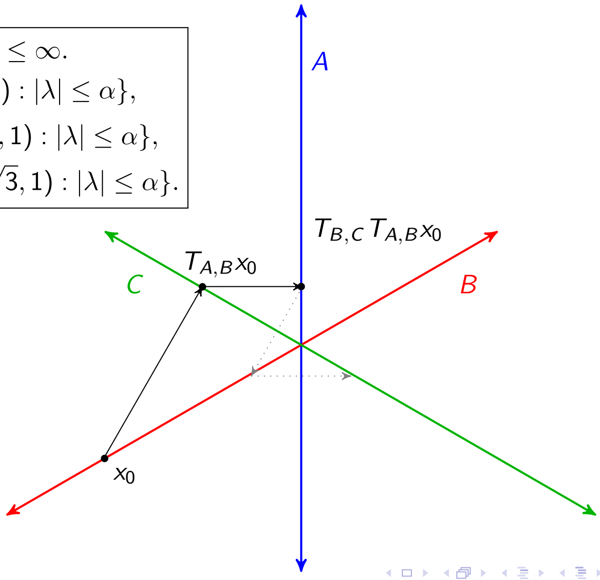

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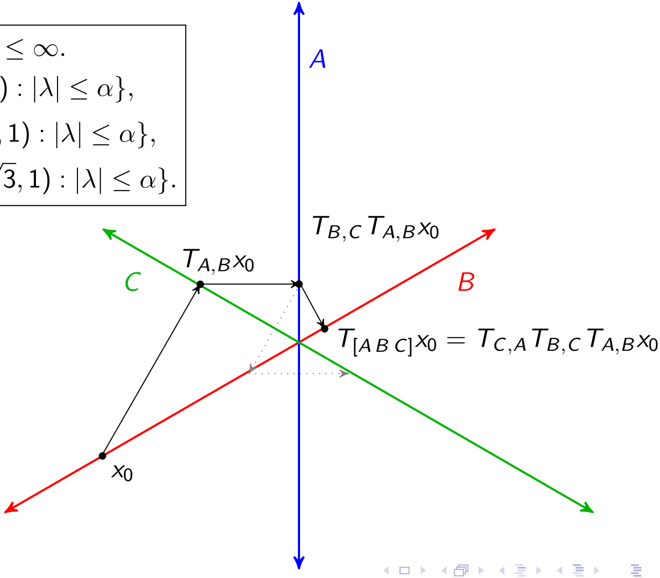

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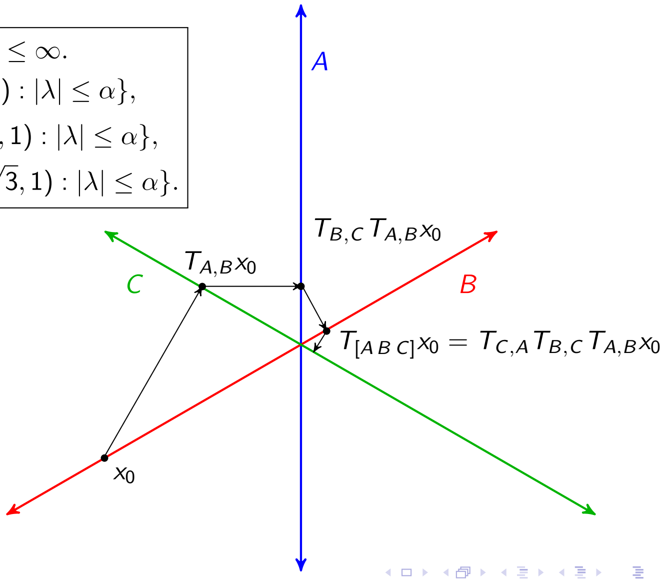

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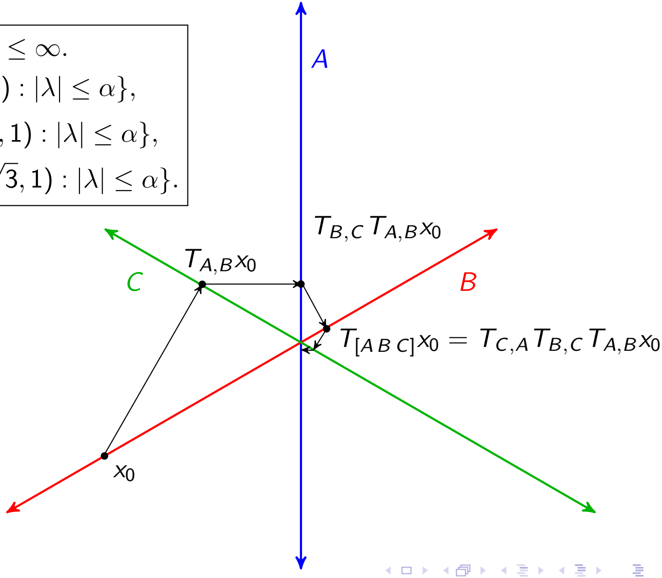

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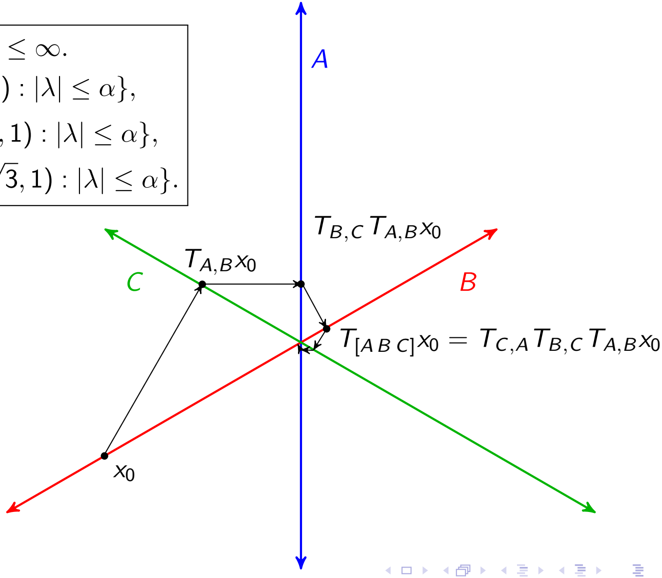

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A natural question

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Failure of Norm Convergence (Hundal, 2004)

Let $\mathcal{H} = \ell_2$ and $\{e_i\}$ denote the standard basis vectors. Define

$$C_1 = \{x \in \mathcal{H} : \langle e_1, x\rangle \leq 0\}, \quad C_2 = \text{an “unnatural” cone}.$$

Then $C_1 \cap C_2 = \{0\}$. If $x_0 = \exp(-100)e_1 + e_3$, then

$$\lim_{n\to\infty} \|(P_{C_2}P_{C_1})^n x_0\| > 0.$$

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Under these assumptions, the previous theorem remains true.

These are many other applicable variants. E.g. Krasnoselski–Mann iterations:

$$x_{n+1} = x_n + \lambda_n (Tx_n - x_n),$$

where $\lambda_n \in [0,1]$ such that $\sum_{i=1}^{\infty} \lambda_n (1 - \lambda_n) = +\infty$.

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Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex sets with a nonempty intersection. For any $x_0 \in \mathcal{H}$, set

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Proof.

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Proof. For $x_0 \in \mathcal{H}$, set $\mathbf{x}_0 = (x_0, \dots, x_0)$. Now consider the (product space) iteration

$$\mathbf{x}_{n+1} = P_D\left(\prod_{i=1}^{N} T_{C_i,C_{i+1}}\right)\mathbf{x}_n.$$

Numerical Experiments

Here we shall consider the N-set feasibly problem:

$$\text{Find } x \in \bigcap_{i=1}^{N} C_i, \text{ where } C_i = y_i + r_i \mathcal{B}_{\mathbb{R}^n} := \{x \in \mathbb{R}^n : \|x - y_i\| \leq r_i\}.$$

We have also consider the same problem replacing the ball constraints with (non-convex) spheres, and certain types of (convex) ellipsoids. Results are similar.

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- Initialise: Random $x_0 \in [-5, 5]^n$.
- Termination criterion:
  - $\|x_{n+1} - x_n\| < \epsilon$ where $\epsilon = 10^{-3}, 10^{-6}$.
  - Maximum of 1000 iterations.
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- We compared:
  - Cyclic Douglas–Rachford, applied directly to the problem.
  - The classical Douglas–Rachford, in the product formulation.

Numerical Results

Table 1. Mean (Max) results for N ball constraints in $\mathbb{R}^n$ with $\epsilon = 10^{-3}$.

n	N	Iterations cycDR	Iterations DR	Time (s) cycDR	Time (s) DR	Error cycDR	Error DR
100	10	4.6 (5)	22.9 (45)	0.004 (0.005)	0.022 (0.041)	0 (0)	7.91e-34 (1.65e-33)
100	20	3.4 (4)	42.4 (113)	0.006 (0.007)	0.071 (0.183)	0 (0)	1.59e-33 (6.11e-33)
100	50	2.3 (3)	75.3 (241)	0.008 (0.011)	0.288 (0.907)	2.03e-14 (2.02e-13)	6.37e-08 (6.37e-07)
100	100	2.1 (3)	97.9 (151)	0.014 (0.019)	0.717 (1.096)	0 (0)	5.51e-33 (3.85e-32)
100	200	2.0 (2)	186.2 (329)	0.025 (0.025)	2.655 (4.656)	9.68e-15 (9.68e-14)	2.17e-08 (2.17e-07)
100	500	2.0 (2)	284.2 (372)	0.059 (0.060)	9.968 (12.989)	0 (0)	2.70e-07 (9.51e-07)
100	1000	2.0 (2)	383.0 (507)	0.118 (0.119)	26.656 (35.120)	0 (0)	4.30e-07 (9.42e-07)
100	1100	2.0 (2)	380.7 (471)	0.129 (0.130)	29.160 (36.001)	0 (0)	8.35e-07 (1.79e-06)
100	1200	2.0 (2)	372.3 (537)	0.141 (0.144)	31.140 (44.886)	0 (0)	8.08e-07 (1.79e-06)
100	1500	2.0 (2)	466.0 (631)	0.178 (0.181)	49.282 (66.533)	0 (0)	5.38e-05 (5.34e-04)
100	2000	2.0 (2)	529.3 (725)	0.232 (0.234)	74.878 (102.148)	9.31e-19 (5.29e-18)	4.79e-06 (4.00e-05)
1000	10	15.0 (16)	12.4 (26)	0.024 (0.026)	0.023 (0.048)	2.12e-19 (2.12e-18)	1.24e-32 (3.34e-32)
1000	20	8.2 (9)	20.4 (71)	0.024 (0.027)	0.069 (0.237)	0 (0)	3.02e-32 (6.98e-32)
1000	50	4.3 (5)	38.8 (112)	0.028 (0.031)	0.311 (0.884)	2.67e-19 (2.67e-18)	1.24e-31 (5.29e-31)
1000	100	3.3 (4)	80.8 (222)	0.037 (0.042)	1.260 (3.436)	0 (0)	2.15e-31 (6.84e-31)
1000	200	2.4 (3)	138.5 (270)	0.048 (0.058)	4.730 (9.446)	0 (0)	6.50e-31 (2.52e-30)
1000	500	2.0 (2)	201.3 (313)	0.085 (0.086)	20.356 (31.166)	3.90e-20 (3.90e-19)	2.10e-30 (6.11e-30)
1000	1000	2.0 (2)	348.8 (518)	0.162 (0.164)	73.420 (108.493)	0 (0)	1.36e-06 (1.20e-05)
1000	1100	2.1 (3)	334.4 (550)	0.183 (0.260)	77.174 (126.896)	0 (0)	1.10e-07 (7.62e-07)
1000	1200	2.0 (2)	353.8 (518)	0.190 (0.193)	89.153 (128.683)	0 (0)	1.74e-07 (9.63e-07)
1000	1500	2.1 (3)	403.9 (607)	0.245 (0.346)	126.707 (189.011)	1.33e-19 (1.33e-18)	3.17e-07 (8.94e-07)
1000	2000	2.0 (2)	487.0 (593)	0.307 (0.312)	239.210 (374.390)	0 (0)	3.58e-07 (1.11e-06)

Numerical Results

Table 2. Mean (Max) results for N ball constraints in $\mathbb{R}^n$ with $\epsilon = 10^{-6}$.

n	N	Iterations cycDR	Iterations DR	Time (s) cycDR	Time (s) DR	Error cycDR	Error DR
100	10	4.7 (6)	22.9 (45)	0.005 (0.005)	0.023 (0.044)	0 (0)	7.91e-34 (1.65e-33)
100	20	3.6 (5)	42.4 (113)	0.006 (0.008)	0.077 (0.199)	0 (0)	1.59e-33 (6.11e-33)
100	50	2.6 (4)	77.4 (262)	0.010 (0.014)	0.320 (1.068)	0 (0)	1.24e-32 (5.96e-32)
100	100	2.1 (3)	97.9 (151)	0.015 (0.020)	0.781 (1.195)	0 (0)	5.51e-33 (3.85e-32)
100	200	2.3 (3)	187.1 (329)	0.029 (0.038)	2.909 (5.077)	0 (0)	5.89e-33 (2.30e-32)
100	500	2.3 (3)	329.6 (661)	0.071 (0.093)	12.554 (24.975)	0 (0)	1.81e-32 (6.37e-32)
100	1000	2.3 (3)	427.4 (635)	0.141 (0.184)	32.431 (47.903)	0 (0)	2.21e-32 (8.10e-32)
100	1100	2.3 (3)	467.4 (714)	0.153 (0.199)	38.936 (59.259)	0 (0)	3.92e-32 (3.17e-31)
100	1200	2.1 (3)	451.8 (698)	0.154 (0.218)	41.059 (63.259)	0 (0)	1.12e-31 (8.08e-31)
100	1500	2.1 (3)	507.2 (712)	0.193 (0.277)	58.578 (81.907)	0 (0)	2.66e-31 (8.15e-31)
100	2000	2.3 (3)	627.8 (808)	0.276 (0.361)	96.554 (124.880)	0 (0)	1.50e-31 (7.53e-31)
1000	10	15.1 (17)	12.4 (26)	0.024 (0.027)	0.030 (0.063)	0 (0)	1.24e-32 (3.34e-32)
1000	20	8.2 (9)	20.4 (71)	0.025 (0.027)	0.095 (0.330)	0 (0)	3.02e-32 (6.98e-32)
1000	50	4.5 (6)	38.8 (112)	0.029 (0.035)	0.434 (1.249)	0 (0)	1.24e-31 (5.29e-31)
1000	100	3.3 (4)	80.8 (222)	0.038 (0.043)	1.761 (4.730)	0 (0)	2.15e-31 (6.84e-31)
1000	200	2.5 (3)	138.5 (270)	0.051 (0.059)	6.224 (12.089)	0 (0)	6.50e-31 (2.52e-30)
1000	500	2.3 (3)	201.3 (313)	0.099 (0.125)	26.108 (40.534)	0 (0)	2.10e-30 (6.11e-30)
1000	1000	2.1 (3)	388.7 (905)	0.174 (0.241)	103.839 (243.085)	0 (0)	2.17e-30 (1.79e-29)
1000	1100	2.3 (3)	354.4 (660)	0.205 (0.264)	120.706 (220.612)	0 (0)	2.26e-30 (9.82e-30)
1000	1200	2.3 (3)	376.3 (620)	0.223 (0.288)	161.133 (260.857)	0 (0)	1.61e-30 (1.26e-29)
1000	1500	2.2 (3)	526.0 (1000)	0.265 (0.358)	276.095 (541.502)	2.68e-22 (2.68e-21)	1.08e-09 (5.98e-09)
1000	2000	2.1 (3)	595.0 (894)	0.332 (0.469)	427.933 (646.182)	0 (0)	4.48e-31 (1.97e-30)

Interactive Geometry and Visualisations

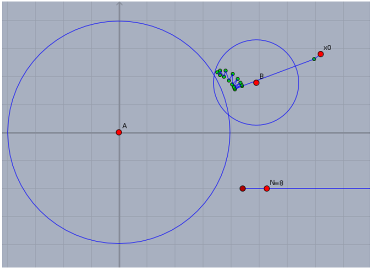

Figure 1. Two circle constraints in $\mathbb{R}^2$, drawn in *Cinderella*.

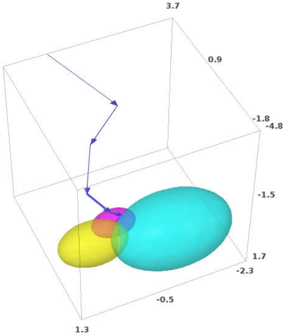

Figure 2. Three ball constraints in $\mathbb{R}^3$, drawn in *Sage*.

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- Further numerical experiments. e.g. arbitrary ellipsoids.

References

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Many resources available at:

http://carma.newcastle.edu.au/DRmethods