Fixed point theorems for mappings of asymptotically nonexpansive type

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Abstract

The purpose of this paper is to provide fixed point theorems for asymptotically nonexpansive type mappings in a Banach space with uniform normal structure.

Key words: Fixed point, asymptotically nonexpansive type mapping, uniform normal structure

1 Introduction

Let C be a nonempty subset of a Banach space X and let $T : C \to C$ be a mapping. Then T is said to be asymptotically nonexpansive [6] if there exists a sequence (k_n) of real numbers with $\lim\limits_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \quad \text{for } x, y \text{ in } C \text{ and } n = 1, 2, \ldots$$

If this is valid for $n = 1$ with $k_1 = 1$ (and hence $k_n = 1$ for all n) then T is said to be nonexpansive. If for each x in C, we have

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0,$$

then T is said to be of asymptotically nonexpansive type [8].

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In 1965, Kirk [7] proved that if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping T of C has a fixed point. A nonempty convex subset C of a normed linear space is said to have normal structure if each bounded convex subset K of C consisting of more than one point contains a nondiametral point. That is, a point $x \in K$ such that $\sup\{\|x-y\| : y \in K\} < \sup\{\|u-v\| : u, v \in K\} = \mathrm{diam} K$. Seven years later, in 1972, Goebel and Kirk [6] proved that if the space X is assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping T of C has a fixed point. This was extended to mappings of asymptotically nonexpansive type by Kirk in [8]. More recently these results have been extended to wider classes of spaces, see for example [2], [5], [9], [11] and [12]. In particular, Lim and Xu [12] and Kim and Xu [9] have demonstrated the existence of fixed points for asymptotically nonexpansive mappings in Banach spaces with uniform normal structure, see also [4] for some related results. However, whether normal structure implies the existence of fixed points for mappings of asymptotically nonexpansive type is a natural question that remains open.

The present paper answers a question raised by Kim and Xu in [9]. It extends results in their paper and [12] to mappings of asymptotically nonexpansive type and so represents a further step toward a resolution of the question raised above.

2 Main theorems

In this section, let X be a Banach space, let C be a nonempty bounded subset of X and let $T : C \to C$ be a mapping of asymptotically nonexpansive type. For each $x \in C$ and $n \geq 1$, put

$$r_n(x) = \sup_{y \in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0$$

Then for each $x \in C$,

$$\lim_{n \to \infty} r_n(x) = 0 \tag{1}$$

Let E be a nonempty bounded closed convex subset of a Banach space X and let $d(E) = \sup\{\|x-y\| : x, y \in E\}$ be the diameter of E. For each $x \in E$, let $r(x, E) = \sup\{\|x-y\| : y \in E\}$ and let $r(E) = \inf\{r(x, E) : x \in E\}$, the Chebyshev radius of E relative to itself. The normal structure coefficient of X is defined to be

$$\widetilde{N}(X) = \sup\{\frac{r(E)}{d(E)} : E \text{ is a bounded closed convex subset of } X \text{ with } d(E) > 0\}.$$

Note, the normal structure coefficient $\widetilde{N}(X)$, introduced by Maluta [13], is the reciprocal of $N(X)$ defined by Bynum in [3]. A space X for which $\widetilde{N}(X) < 1$ is said to have uniform normal structure. It is know that a space with uniform normal structure is reflexive and that all uniformly convex or uniformly smooth Banach spaces have uniform normal structure.

Theorem 2.1 *Suppose X is a Banach space with uniform normal structure, C is a nonempty bounded subset of X, and $T : C \to C$ is an asymptotically nonexpansive type mapping such that T is continuous on C. Further, suppose that there exists a nonempty closed convex subset E of C with the following property (P):*

$$x \in E \qquad \text{implies } \omega_w(x) \subset E,$$

where $\omega_w(x)$ is the weak ω-limit set of T at x; that is, the set

$$\{y \in X : y = weak - \lim_i T^{n_i}x \text{ for some } n_i \uparrow \infty\}.$$

Then T has a fixed point in E.

To prove the theorem we use the following lemma from [14].

Lemma 1 *Let C be a nonempty subset of a Banach space X and let T be a mapping of asymptotically nonexpansive type C. Suppose there exists a nonempty bounded closed convex subset E of C with the property (P). Then there is a closed convex nonempty subset K of C and a $\rho \geq 0$ such that:*

(i) if $x \in K$, then every weak limit point of $(T^n x)$is contained in K;
(ii) $\rho_x(y) = \rho$ for all $x, y \in K$, where ρ_x is the functional defined by

$$\rho_x(y) = \limsup_{n\to\infty} \|T^n x - y\|, \quad y \in X.$$

Proof of Theorem 1: Let K, ρ_x and ρ be as in lemma 1. Let x be any element in K and let G be a sub-semigroup of $\mathbb{N}$. That is, $G = \{in_0 : i \in \mathbb{N}\}$ for some $n_0 \in \mathbb{N}$. For each $i \in G$, consider the sequence $(T^j x)_{i\leq j\in G}$. From the definition of $\widetilde{N}(X)$, we have a $y_i \in \overline{co}\{T^j x : i \leq j \in G\}$ (here, $\overline{co}$ denotes the closed convex hull) such that

$$\limsup_{j\in G} \|T^j x - y_i\| \leq \widetilde{N}(X)A((T^j x)_{i\leq j\in G}), \tag{2}$$

where $A(z_n)$ is the asymptotic diameter of the sequence (z_n); that is, the number

$$\lim_n (sup\{\|z_i - z_j\| : i, j \geq n\}).$$

Since X is reflexive, (y_i) admits a subsequence $(y_{i'})$ converging weakly to some $x^* \in X$. From 2 and the w-l.s.c. of the functional $\limsup_{j\in G} \|T^j x - y\|$, it follows that

$$\limsup_{j\in G} \|T^j x - x^*\| \leq \widetilde{N}(X)A((T^j x)_{j\in G}). \tag{3}$$

It is easily seen that $x^* \in \cap_{i\in G}\overline{co}\{T^j x : i \le j \in G\}$ and that

$$\|z - x^*\| \le \limsup_{j\in G} \|z - T^j x\| \text{ for all } z \in X. \tag{4}$$

Using property (P) and the fact that $\cap_{i\in G}\overline{co}\{T^j x : i \le j \in G\} = \overline{co}\,\omega_w\{T^j x : j \in G\}$, which is easy to prove by using the Separation Theorem (cf. [1]), we get that x^* actually lies in K. We claim that:

there exists $x \in K$ such that $\omega(x) \ne \emptyset$, where $\omega(x)$ is the strong ω-limit set of T at x, and
$\rho = 0$.

To derive a contradiction, we suppose that (1) is not true. In particular then, for any sub-semigroup G of $\mathbb{N}$ and for any $x, y \in K$, we have that $D = \limsup_{j\in G} \|T^j x - y\|$ is strictly greater than zero. Let r_0 be a positive number chosen so that $r = (2r_0 + 1)\widetilde{N}(X) < 1$, this is possible since by assumption $\widetilde{N}(X) < 1$.

Now, take any x_0 in K and put $G_0 = \mathbb{N}$, then from 3 and 4 there exists $x_1 \in K$ with

$$0 < D_0 = \limsup_{j\in G_0} \|T^j x_0 - x_1\| \le \widetilde{N}(X) A((T^j x_0)_{j\in G_0})$$

and

$$\|z - x_1\| \le \limsup_{j\in G_0} \|z - T^j x_0\|, \quad \text{for all } z \in X.$$

It then follows from 1 that there exists $n_0 \in \mathbb{N}$ such that

$$r_n(x_1) < r_0 D_0, \quad \text{for all } n \ge n_0.$$

Put $G_1 = \{in_0 : i \in \mathbb{N}\}$, it is a sub-semigroup of $\mathbb{N}$. It follows that there exists $x_2 \in K$ such that

$$0 < D_1 = \limsup_{j\in G_1} \|T^j x_1 - x_2\| \le \widetilde{N}(X) A((T^j x_1)_{j\in G_1})$$

and

$$\|z - x_2\| \le \limsup_{j\in G_1} \|z - T^j x_1\|, \text{ for all } z \in X.$$

By 1 again, there exists $n_1 \in G_1$ such that

$$r_n(x_2) < r_0 D_1, \quad \text{for all } n \ge n_1.$$

Put $G_2 = \{in_1 : i \in \mathbb{N}\}$, it is a sub-semigroup of G_1. It follows that there exists $x_3 \in K$ such that

$$0 < D_2 = \limsup_{j\in G_2} \|T^j x_2 - x_3\| \le \widetilde{N}(X) A((T^j x_2)_{j\in G_2})$$

and

$$\|z - x_3\| \le \limsup_{j\in G_2} \|z - T^j x_2\|, \text{ for all } z \in X.$$

We can repeat the above process to obtain a sequence $(x_n)_{n=1}^\infty$ in K and a series of semigroups $\{G_n\}_1^\infty$ with the properties:

(i) $\mathbb{N} = G_0 \sqsupset G_1 \sqsupset G_2 \sqsupset \dots$;
(ii) $D_n = \limsup_{i\in G_n} \|T^i x_n - x_{n+1}\| \le \widetilde{N}(X)A((T^i x_n)_{i\in G_n})$;
(iii) $\|z - x_{n+1}\| \le \limsup_{i\in G_n} \|z - T^i x_n\|,\quad$ for all $z \in X$;
(iv) $r_i(x_{n+1}) \le r_0 D_n,\qquad$ for all $i \in G_{n+1}$.

Now for $i, j \in G_n$ with $i > j$, we have that $i - j \in G_n \subset G_{n-1}$ and from $(iii) - (iv)$ that

$$\begin{aligned}\|T^i x_n - T^j x_n\| &\le r_j(x_n) + \|T^{i-j}x_n - x_n\| \\ &\le r_j(x_n) + \limsup_{m\in G_{n-1}} \|T^{i-j}x_n - T^m x_{n-1}\| \\ &\le r_j(x_n) + r_{i-j}(x_n) + \limsup_{m\in G_{n-1}} \|x_n - T^m x_{n-1}\| \\ &\le (2r_0 + 1)D_{n-1}.\end{aligned}$$

It follows from (ii) that

$$D_n \le \widetilde{N}(X)(2r_0 + 1)D_{n-1} = rD_{n-1} \le \ldots \le r^{n-1}D_1.$$

Therefore, for each $i \in G_n$ and $n \ge 2$, we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \|x_{n+1} - T^i x_n\| + \|T^i x_n - x_n\| \\ &\le \|x_{n+1} - T^i x_n\| + \limsup_{m\in G_{n-1}} \|T^i x_n - T^m x_{n-1}\| \\ &\le \|x_{n+1} - T^i x_n\| + r_i(x_n) + \limsup_{m\in G_{n-1}} \|x_n - T^m x_{n-1}\|.\end{aligned}$$

Consequently,

$$\|x_{n+1} - x_n\| \le D_n + D_{n-1} \le (r^{n-1} + r^{n-2})D_1.$$

That is, (x_n) is a Cauchy sequence and there is $x \in K$ such that $x_n \to x$ strongly as $n \to \infty$. Since

$$\begin{aligned}\|T^j x - x\| &\le \|T^j x - T^j x_n\| + \|T^j x_n - x_{n+1}\| + \|x_{n+1} - x\| \\ &\le r_j(x) + \|x - x_n\| + \|x - x_{n+1}\| + \|T^j x_n - x_{n+1}\|,\end{aligned}$$

we have

$$\begin{aligned}\liminf_{j\to\infty} \|T^j x - x\| &\le \|x - x_n\| + \|x - x_{n+1}\| + \liminf_{j\to\infty} \|T^j x_n - x_{n+1}\| \\ &\le \|x - x_n\| + \|x - x_{n+1}\| + \limsup_{j\in G_n} \|T^j x_n - x_{n+1}\| \\ &= \|x - x_n\| + \|x - x_{n+1}\| + D_n.\end{aligned}$$

Taking $n \to \infty$, we get
$$\liminf_{j\to\infty} \|T^j x - x\| = 0.$$
This is a contradiction.

To prove (2), by Lemma 1, it is enough to show that if $\rho > 0$ then there exist $z, y \in K$ such that $\rho_y(z) = \limsup_{n\to\infty} \|z - T^n y\| < \rho$. To this end, by (1), there exists $x \in K$ such that $\omega(x) \neq \emptyset$. Let $y = \lim_i T^{n_i} x$ for some $n_i \uparrow \infty$. It is easily seen that $\{T^n y : n \geq 1\} \subset K$. Put

$$\rho_0 = \mathrm{diam}(\overline{co}\{T^n y : n \geq 1\}) = \mathrm{diam}(\{T^n y : n \geq 1\}).$$

Since

$$\begin{aligned}\|T^n y - T^m y\| &= \lim_{i\to\infty} \|T^{n+n_i} x - T^m y\| \\ &\leq \limsup_{i\to\infty} \|T^i x - T^m y\| \\ &= \rho,\end{aligned}$$

we have $\rho_0 \leq \rho$. Since K has normal structure, there exists $z \in \overline{co}\{T^n y : n \geq 1\}$ such that

$$\sup_{n\geq 1} \|z - T^n y\| < \mathrm{diam}(\overline{co}\{T^n y : n \geq 1\}) \leq \rho.$$

This proves (2).

By (2), $K = \{x\}$ and $T^n x \to x$ strongly as $n \to \infty$. Therefore, $Tx = x$ by the continuity of T.

Corollary 1 *Let C and X be as in Theorem 1 and let $T : C \to C$ be an asymptotically nonexpansive mappings. Suppose there exists a nonempty bounded closed convex subset E of C with the property (P). Then T has a fixed point.*

Proof: This follows since an asymptotically nonexpansive mapping is of asymptotically nonexpansive type.

From Theorem 1 we readily capture the following result announced by Taehwa Kim, who also gives an alternative proof [10].

Corollary 2 *Let X be a Banach space with uniform normal structure, let C be a bounded closed convex subset of X, and suppose $T : C \to C$ is a continuous mapping of asymptotically nonexpansive type. Then T has a fixed point.*

We conclude the paper by stating the semigroup version of Theorem 1. The proof is similar to that of Theorem 1 and is therefore omitted.

Theorem 2.2 *Suppose X is a Banach space with uniform normal structure, C is a nonempty bounded subset of X, and $\Im = \{T(t) : t \geq 0\}$ is a semigroup of asymptotically nonexpansive type mappings on C such that $T(t)$ is continuous* *on C for each $t \geq 0$. Suppose also that there exists a nonempty bounded closed convex subset E of C with the following property (P):*

$$x \in E \qquad \textit{implies } \omega_w(x) \subset E,$$

where $\omega_w(x)$ is the weak ω-limit set of $\{T(t)x\}$, i.e. the set

$$\{y \in X : y = weak - \lim_i T(t_i)x \textit{ for some } t_i \uparrow \infty\}.$$

Then $\Im$ has a common fixed point in E, i.e. there exists a $z \in E$ for which $T(t)z = z$ for all $t \geq 0$.

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