æ

THE LERAY-SCHAUDER ALTERNATIVE FOR NONEXPANSIVE MAPS FROM THE BALL CHARACTERIZE HILBERT SPACE

Michael Ireland
Department of Mathematics
The University of Newcastle
Newcastle 2308, NSW, Australia

William A. Kirk *
Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242-1419 ,USA
e-mail: kirk@math.uiowa.edu

and

Brailey Sims
Department of Mathematics
The University of Newcastle
Newcastle 2308, NSW, Australia
e-mail: bsims@maths.newcastle.edu.au

Abstract: *We show that for a nonexpansive map from the unit ball of a Hilbert space into the space the existence of a fixed point and the Leray-Schauder alternative are mutually exclusive alternatives, and that this characterizes Hilbert space. The equivalence of several formulations of the Leray-Schauder alternative is also established.*

1991 Mathematics Subject Classification: 47H09, 47H10.

Key Words and Phrases: Hilbert space, nonexpansive mapping, fixed point, Leray-Schauder alternative principle, extension property.

For a real Banach space X we denote by B_X and S_X the unit ball and unit sphere respectively:

$$B_X := \{x \in X : \|x\| \leq 1\} \quad \text{and} \quad S_X := \{x \in X : \|x\| = 1\} = \text{bdry}(B_X).$$

When the space is a Hilbert space we will denote it by H and the inner-product by $\langle \cdot, \cdot \rangle$.

---

* This work was carried out while the second author was visiting The University of Newcastle in February, 1998 and that author wishes to thank the University for its support and hospitality.

We say that a mapping $T : B_X \longrightarrow X$ satisfies the *Leray-Schauder alternative principle* if either

(i) T has a fixed point in B_X; that is, $\text{Fix}(T) := \{x : Tx = x\} \neq \emptyset$, or

(ii) (*The Leray-Schauder alternative*) there exists an $x_0 \in S_X$ and a scalar $\lambda > 1$ such that $Tx_0 = \lambda x_0$.

As indicated we will refer to the second possibility as the Leray- Schauder alternative for T.

Typically, the Leray-Schauder altrnative principle for a particular type of mapping is established via a homotopy argument. See, for example, Granas [G], where it is shown that if U is a nonempty open subset of a complete metric space (X, d), $T_t : \overline{U} \to X$ for $t \in [0, 1]$ is a homotopic family of maps which are

(a) uniformly contractive; that is, $d(T_t x, T_t y) \leq kd(x, y)$, for all $t \in [0, 1]$ and some $k < 1$,

satisfy

(b) $d(T_t x, T_s x) \leq M|t - s|$ for all $t, s \in [0, 1]$, $x \in \overline{U}$ and some $M > 0$

and for which

(c) $\text{Fix}(T_t) \cap \text{bdry}(U) = \emptyset$, for all $t \in [0, 1]$,

then, if T_0 has a fixed point in U so does T_t for each $t \in (0, 1]$.

Applying this to the homotopic family tT, where $t \in [0, 1]$ and $T : B_X \to X$ is a strict contration, we readily deduce the Leray-Schauder alternative principle for such a T.

Unfortunately, examples of Marlène Frigon [F] show that such a homotopy argument is not possible when T is only required to be *nonexpansive*; that is, $\|Tx - Ty\| \leq \|x - y\|$, even when T maps B_{ℓ_2} into ℓ_2. Never-the-less we shall see that it is relatively straight forward to show that such maps do indeed satisfy the Leray-Schauder alternative principle.

THEOREM 1: *Let C be a nonempty closed bounded convex subset of the Hilbert space H, and let $T : C \longrightarrow H$ be a nonexpansive mapping, then there exists x_0, necessarily in bdry(C), such that the following are equivalent.*

(i) Fix$(T) = \emptyset$.

(ii) $0 < \|Tx_0 - x_0\| = \mathit{dist}(Tx_0, C)$.

(iii) $C \subset \{x \in H : \langle Tx_0 - x_0, x - x_0 \rangle \leq 0\}$.

(iv) $Tx_0 \notin \bigcup_{c \in C} B[c, \|c - x_0\|]$.

Before proving the theorem we note the following two well know lemmas, proofs of which are included only for completeness. In both lemmas, C is a nonempty closed bounded convex subset of a Hilbert space H.

LEMMA 1: *The closest point map $Proj_C$ from H onto C is nonexpansive and characterized by $Proj_C(x) \in C$ and $\langle c - Proj_C(x), x - Proj_C(x) \rangle \leq 0$ for all $x \in H$ and $c \in C$.*

PROOF: The characterization follows from the observation that $\mathrm{Proj}_C(x)$ is the closest point of C to x if and only if $\mathrm{Proj}_C(x) \in C$ and there is a hyperplane through $\mathrm{Proj}_C(x)$ which separates C from $B[x, \|x - \mathrm{Proj}_C(x)\|]$, and that this hyperplane is necessarily the unique hyperplane supporting $B[x, \|x - \mathrm{Proj}_C(x)\|]$ at $\mathrm{Proj}_C(x)$; namely,

$$\{y \in H : \langle x - \mathrm{Proj}_C(x), y \rangle = \langle x - \mathrm{Proj}_C(x), \mathrm{Proj}_C(x) \rangle \}.$$

That Proj_C is nonexpansive now follows from the calculation:

For every $x, y \in H$,

$$\begin{aligned} \|x - y\|^2 = & \|\mathrm{Proj}_C(x) - \mathrm{Proj}_C(y)\|^2 \\ & + \|(I - \mathrm{Proj}_C)x - (I - \mathrm{Proj}_C)y\|^2 \\ & + 2\langle x - \mathrm{Proj}_C(x), \mathrm{Proj}_C(x) - \mathrm{Proj}_C(y) \rangle \\ & + 2\langle y - \mathrm{Proj}_C(y), \mathrm{Proj}_C(y) - \mathrm{Proj}_C(x) \rangle, \end{aligned}$$

and that both the last two terms are positive, so that $\|\mathrm{Proj}_C(x) - \mathrm{Proj}_C(y)\| \leq \|x - y\|$.

∎

The next lemmas follows from more general results due to Browder, Göhde, and Kirk, see the book by Goebel and Kirk [G-K] for more details on this and metric fixed point theory in general. The proof we give essentially relies on Hilbert spaces enjoying the Opial property.

LEMMA 2: *If $T : C \longrightarrow C$ is nonexpansive, then T has a fixed point in C.*

PROOF: Choose $x_0 \in C$, then for each $n \in \mathbf{N}$ the mapping $T_n x := (1 - 1/n)Tx + (1/n)x_0$ is a strict contraction mapping C into C, and so by the Banach contraction mapping principle has a fixed point x_n. This gives a sequence (x_n) with $\|x_n - Tx_n\| \to 0$. By passing to a subsequence if necessary, we may also assume that (x_n) converges weakly to some point $x \in C$.

Now,

$$\begin{aligned} \|x_n - Tx\|^2 &= \langle (x_n - x) + (x - Tx), (x_n - x) + (x - Tx) \rangle \\ &= \|x_n - x\|^2 + \|x - Tx\|^2 + 2\langle x_n - x, x - Tx \rangle, \end{aligned}$$

so,

$$\begin{aligned}\|x - Tx\|^2 &= \|x_n - Tx\|^2 - \|x_n - x\|^2 - 2\langle x_n - x, x - Tx\rangle \\ &\le (\|x_n - Tx_n\| + \|Tx_n - Tx\|)^2 - \|x_n - x\|^2 - 2\langle x_n - x, x - Tx\rangle \\ &\le (\|x_n - Tx_n\| + \|x_n - x\|)^2 - \|x_n - x\|^2 - 2\langle x_n - x, x - Tx\rangle \\ &= \|x_n - Tx_n\|(\|x_n - Tx_n\| + 2\|x_n - x\|) - 2\langle x_n - x, x - Tx\rangle \\ &\longrightarrow 0, \quad \text{as } n \to \infty.\end{aligned}$$

Thus, $Tx = x$, establishing the result.

∎

Proof of theorem 1: To see that (i) implies (ii) we first observe that the mapping $\text{Proj}_C \circ T$ is nonexpansive, by lemma 1., and maps C into C. Thus, by lemma 2., $\text{Proj}_C \circ T$ has a fixed point $x_0 \in C$, with $Tx_0 \notin C$, otherwise we would have $x_0 = \text{Proj}_C \circ Tx_0 = Tx_0$ contradicting (i). It now follows, using the definition of Proj_C, that $0 < \|Tx_0 - x_0\| = \|Tx_0 - \text{Proj}_C \circ Tx_0\| = \text{dist}(Tx_0, C)$, establishing (ii).

That (ii) is equivalent to (iii) follows immediately from lemma 1. Thus, it only remains to prove that (iii) implies (iv) implies (i).

(iii) $\Longrightarrow$ (iv): Suppose (iv) is not true, then there exists $c \in C$ with $Tx_0 \in B[c, \|c - x_0\|]$, so both c and Tx_0 lie on the positive side of the support hyperplane to $B[c, \|c - x_0\|]$ at x_0; namely $\{x \in H : \langle c - x_0, x\rangle = \langle c - x_0, x_0\rangle\}$. That is, $\langle Tx_0 - x_0, c - x_0\rangle \ge 0$, contradicting (iii).

(iv) $\Longrightarrow$ (i): Suppose (i) is not true; that is, there exists $c_0 \in C$ with $Tc_0 = c_0$. Then, $\|Tx_0 - c_0\| = \|Tx_0 - Tc_0\| \le \|x_0 - c_0\|$, so $Tx_0 \in B[c_0, \|c_0 - x_0\|]$, and so certainly $Tx_0 \in \bigcup_{c \in C} B[c, \|c - x_0\|]$, contradicting (iv).

∎

Remark 1: The equivalence of conditions (ii) and (iii) of theorem 1. and their relation to (i) were essentially studied by Williamson [W], where (iii) was introduced as a generalized Leray-Schauder alternative.

Remark 2: Condition (ii) of theorem 1. was considered by Browder and Petryshyn [B-P] and the equivalence of (i) and (iii) represents a Ky Fan [Ky F] type result for nonexpansive maps on non-compact domains.

Remark 3: Condition (iv) of theorem 1. seems new and like (ii) can be formulated in any Banach space where it may play the role of a generalized Leray-Schauder alternative. In particular one is led to ask: in which spaces X are the following two conditions equivalent for a nonexpansive map $T : B_X \to X$?

(a) $\text{Fix}(T) \neq \emptyset$.

(b) For all $x \in B_X$ we have $Tx \in \bigcup_{p \in B_X} B[p, \|p - x\|]$.

Clearly we always have (a) implies (b).

Remark 4: When $C = B_H$ it is clear that (ii) of theorem 1. is equivalent to the Leray-Schauder alternative (the closest point map onto the unit ball is radial retraction). This observation combined with the above theorem yields the following.

Corollary 1: *If $T : B_H \longrightarrow H$ is a nonexpansive mapping, then T satisfies the Leray-Schauder alternative principle and the two alternatives are mutually exclusive.*

We conclude by showing that this dichotomy between the two alternatives of the Leray-Schauder alternative principle for nonexpansive mappings of the unit ball is only possible when the space is a Hilbert space, and so characterizes Hilbert spaces among all Banach spaces.

Theorem 2: *A Banach space X is a Hilbert space if and only if for all nonexpansive mappings $T : B_X \longrightarrow X$ the two possibilities below are mutually exclusive.*

(i) Fix$(T) \neq \emptyset$.

(ii) The Leray-Schauder alternative holds.

Proof: Necessity has been established in corollary 1. Thus, we need only establish sufficiency. To this end, suppose X is not a Hilbert space. Then, there exists points x_0 and p_0 in S_X such that every closest point of the line $\mathbf{R}p_0 := \{\lambda p_0 : \lambda \in \mathbf{R}\}$ to x_0 lies outside B_X. This follows, for example, from characterization (13.8) of Amir's book [A], or see [H].

Let y_0 be a closest point of $\mathbf{R}p_0$ to x_0, then we have, $y_0 = \lambda p_0$ for some λ with $|\lambda| > 1$. Replacing p_0 by $-p_0$ if necessary, we therefore have,

$$y_0 = \lambda p_0, \quad \text{where } \lambda > 1, \text{ and}$$
$$\|x_0 - y_0\| < \|x_0 - p_0\|.$$

Denote by $\mathcal{L}$ the line through x_0 and y_0, which we can identify with a copy of $\mathbf{R}$, and define $T : \{x_0, p_0\} \subset B_X \longrightarrow \mathcal{L}$ by

$$T(x_0) := x_0 \quad \text{and} \quad T(p_0) := y_0.$$

Then, T is nonexpansive and, since $\mathbf{R}$ is an *injective* metric space (see for example [A-P]), T has a nonexpansive extension $\tilde{T}$ from B_X into $\mathcal{L} \subset X$.

Thus, $\tilde{T} : B_X \longrightarrow X$ is a nonexpansive mapping which has a fixed point, $\tilde{T}(x_0) = x_0$, and for which the Leray-Schauder alternative holds, $\tilde{T}(p_0) = y_0 = \lambda p_0$, with $\lambda > 1$. These two conditions are therefore not mutually exclusive in X.

∎

References

[A] Amir, Dan, Characterizations of Inner Product Spaces, *Birkhäuser Verlag*, 1986.

[A-P] Aronszajn, N. and Panitchpakdi, P., *Extensions of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math., **6**(1956), 405–439.

[B-P] Browder, F. E. and Petryshyn, W. V., *Construction of fixed points of nonlinear maps in Hilbert space*, J. Math. Anal. Appl., **20** (1967), 197–228.

[Ky F] Fan, Ky, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z., **112** (1969), 234–240.

[F] Frigon, M., *On continuation methods for contractive and nonexpansince mappings*, Recent Advances on Metric Fixed Point Theory, 19–30, *Ciencias 48* (1996), University of Sevilla, Sevilla.

[G-K] Goebel, K. and Kirk, W. A., Topics in Metric Fixed Point Theory, *Cambridge Univ. Press*, 1990.

[G] Granas, A., *Continuation method for contractive maps*, Top. Method. Nonlinear Analysis, **3** (1994), 375–379.

[H] Hirschfeld, R. A. , *On best approximation in normed vector spaces II*, Nieuw Arch. Wisk., **6** (1958), 99–107.

[W] Williamson, T. E., Jr. *The Leray-Schauder condition is necessary for the existence of solutions*, in Fixed Point Theory, Proceedings, 1980, edited by E. Fadell and G. Fournier, Lecture Notes in Mathematics, 886 (1981), *Springer-Verlag*.