τ AND WEAK DEMICLOSENESS PRINCIPLE AND ASYMPTOTIC BEHAVIOR FOR ASYMPTOTICALLY NONEXPANSIVE TYPE MAPPINGS

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Abstract. The purpose of this paper is to provide the new demicloseness principle– τ (weakly) demicloseness principle. We prove that if X is a Banach space with locally uniformly τ-Opial condition, where τ is a Hausdorff topology on X, C is a nonempty τ compact subset of X, and $T : C \to C$ is a asymptotically nonexpansive type mapping. If $\{x_\alpha\}$ is a net in C which converges to x in the sense of τ topology and if the net $\{x_\alpha - T^m x_\alpha\}$ converges to zero in the sense of τ topology for each $m \in \mathcal{N}$, then $x - Tx = 0$. We also give the weakly demicloseness theorem in a Banach space with Opial property. This result is to be used to study convergence theorem for almost-orbits of asymptotically nonexpansive type mappings in a Banach space.

1.INTRODUCTION

Let C be a nonempty subset of a Banach space X and let $T : C \mapsto C$ be a mapping. Then T is said to be asymptotically nonexpansive [9] if there exists a sequence $\{k_n\}$ of real number with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \text{ for all} x, y \in C \text{ and } n \in \mathcal{N} \tag{1.1}$$

If (1.1) is valid for all $k_n \equiv 1$, T is said to be nonexpansive. More generally T is of asymptotically nonexpansive type [9] if

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0 \text{ for every } x \in C.$$

1991 *Mathematics Subject Classification.* 47H09, 47H10, 47H20.

Key words and phrases. Demiclosedness principle, asymptotic behavior, asymptotically nonexpansive type mappings.

This work is supported by National Natural Science Foundation of China under grant 10571150.

One of the fundamental and celebrated results in the theory of nonexpansive mappings is Browder's demiclosedness principle [1] which states that if X is a uniformly convex Banach space, C a nonempty closed convex subset of X, and if $T : C \mapsto C$ is a nonexpansive mapping, then $I - T$ is demiclosed at each $y \in X$, that is, for any sequence $\{x_n\}$ in C, the conditions $x_n \to x$ weakly and $(I - T)x_n \to y$ strongly imply that $(I - T)x = y$. (Here I is the identity operator of X). The principle is also valid in a space satisfying Opial's condition. It is known that the demiclosedness principle play a key role in studying the asymptotic behavior and ergodic theory of nonexpansive mappings. Recently much effort has been devoted to studying the demiclosedness of $I - T$ at 0 for T of asymptotically nonexpansive (type) mapping either in a Banach spach with (locally) uniformly Opial condition or in a nearly uniformly convex Banach space with Opial's condition; See ([3, 5, 10]). For example, P. K. Lin, K. K. Tan and H. K. Xu [10] proved the following:

Theorem 1 (P. K. Lin, K. K. Tan and H. K. Xu [10]). *Suppose X is a Banach space with locally uniformly Opial condition, C is a nonempty weakly compact convex subset of X, and $T : C \to C$ is a asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero, i.e. if $\{x_n\}$ is sequence in C which converges weakly to x and if the sequence $\{x_n - Tx_n\}$ converges strongly to zero, then $x - Tx = 0$.*

However, the condition $x_n - Tx_n \to 0$ strongly is too restriction in studying the asymptotic behavior of asymptotically noneapansive (type) mappings. It is the objective of the present paper to provide a new demiclosedness principle— τ (weakly) demiclosedness principle. We prove the following theorem in section 3: Suppose X is a Banach space with locally uniformly τ-Opial condition, where τ is a Hausdorff topology on X, C is a nonempty τ compact subset of X, and $T : C \to C$ is a asymptotically nonexpansive type mapping. If $\{x_\alpha\}$ is net in C which converges to x in the sense of τ topology and if the net $\{x_\alpha - T^m x_\alpha\}$ converges to zero in the sense of τ topology for each $m \in \mathcal{N}$, then $x - Tx = 0$. We also give weakly demicloseness theorem in a Banach space with Opial property. This kind of τ (weakly) demiclosedness theorems is extremely useful in studying the asymptotic behavior of asymptotically nonexpansive type mappings. For example , as an immediate consequence, we shows that the open question in [10, p. 941] has an affirmative answer even for non-lipschitzian semigroup and c not convex (See Theorem 4.3 and 4.4 below).

2. PRELIMINARIES

Throughout this paper X denotes a Banach space, τ is a Hausdorff topology on X, C a nonempty τ compact convex subset of X. We denote weak convergence in X by $\rightharpoonup$ and convergence by $\to$ and τ convergence by $\xrightarrow{\tau}$. Recall that X is said to have the Opial property if $\{x_n\}$ converges to x weakly implies

$$\limsup_n \|x_n - x\| < \limsup_n \|x_n - y\|$$

for all $y \neq x$. Given a sequence $\{x_n\}$ in X, let

$$r(c; x_n) := \inf\{\limsup_{n\to\infty} \|x_n + x\| - 1 : \|x\| \geq c\},$$

we say X has the locally uniform Opial condition [10] if

$$r(c; x_n) > 0 \text{ whenever } c > 0, \liminf_{n\to\infty} \|x_n\| \geq 1, \text{ and } x_n \rightharpoonup 0,$$

and the uniform Opial condition [10] if for all $c > 0$,

$$r(c) := \inf\{r(c; x_n) : \limsup_{n\to\infty} \|x_n\| \geq 1 \text{ and } x_n \rightharpoonup 0\} > 0$$

and the uniform Opial condition [10] if for all $c > 0$,

$$r(c) := \inf\{r(c; x_n) : \liminf_{n\to\infty} \|x_n\| \geq 1 \text{ and } x_n \rightharpoonup 0\} > 0.$$

Now we say that x has τ-Opial property if $\{x_\alpha\}$ converges to x in the τ topology implies

$$\limsup_n \|x_\alpha - x\| < \limsup_\alpha \|x_n - y\|$$

for all $y \neq x$. Given a directed net $\{x_\alpha : \alpha \in I\}$ in X, let

$$r(c; x_\alpha) := \inf\{\limsup_{\alpha\in I} \|x_\alpha + x\| - 1 : \|x\| \geq c\},$$

we say X has the locally uniform τ-Opial condition if

$$r(c; x_\alpha) > 0 \text{ whenever } c > 0, \limsup_{\alpha\in I} \|x_\alpha\| \geq 1, \text{ and } x_\alpha \xrightarrow{\tau} 0,$$

A Banach space X is said to have property-P [18] if whenever $\{x_n\}$ is a nonconstant weakly null sequence we have $\liminf_n \|x_n\| < diam\,\{x_n\}$. We say that X has asymptotic (P) if whenever $\{x_n\}$ is a weakly null sequence which is not norm convergent we have $\liminf_n \|x_n\| < diam_a\,\{x_n\}$, here $diam_a\,\{x_n\} = \lim_n diam\,\{x_k : k \geq n\}$ is the asymptotic diameter of the sequence $\{x_n\}$.

According to [8], a Banach space X is said to satisfy the Generalized Gossez-Lami Dozo property (GGLD) if

$$\liminf_n \|x_n\| < \limsup_m \limsup_n \|x_n - x_m\|$$

whenever $\{x_n\}$ is a weakly null sequence which is not norm convergent.

It is shown in [17] that asymptotic (P) and GGLD are equivalent conditions and that both the locally uniform Opial condition and UKK (Uniformly Kadec-Klee, [6]) are stronger properties than asymptotic (P).

In the rest of this paper, let T be an asymptotically nonexpansine type mapping from C into itself such that T^N is continuous for some $n \in \mathcal{N}$. For each x in C, put

$$c_n(x) = \sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} \vee 0,$$

we have

$$\lim_{n\to\infty} c_n(x) = 0, \tag{1.2}$$

for all $x \in C$. Denote by $F(T)$ the set of fixed points of T.

The following lemmas show that in the definitions of Opial property and locally uniformly Opial property the conditions for sequence and for net are equivalent.

Lemma 2.1. *The following conditions are equivalent:*

(i) *X has Opial property*

(ii) *If $\{x_\alpha\}$ converges to x weakly implies $\limsup_{alpha} \|x_\alpha - x\| < \limsup_\alpha \|x_\alpha - y\|$ for all $y \neq x$.*

Proof. We only need to prove that (i) implies (ii). Let $y \neq x$, and

$$\rho_1 = \limsup_\alpha \|x_\alpha - x\|$$

and

$$\rho_2 = \limsup_\alpha \|x_\alpha - y\|$$

First, we choose α_1 such that

$$\rho_1 - 1 \le \|x_{\alpha_1} - x\| \le \rho_1 + 1$$

$$\|x_{\alpha_1} - y\| \le \rho_2 + 1$$

$$|\langle x_{\alpha_1} - x, j(x_{\alpha_1} - x)\rangle| \le 1$$

where $j(x) \in J(x)$. Now we may define inductively sequence $\alpha_n\}$ such that

$$\rho_n - \frac{1}{n} \leq \|x_{\alpha_n} - x\| \leq \rho_n + \frac{1}{n}$$

$$\|x_{\alpha_n} - y\| \leq \rho_2 + \frac{1}{n}$$

$$|\langle x_{\alpha_n} - x, j(x_{\alpha_i} - x)\rangle| \leq \frac{1}{n} \quad \text{for all} \quad 1 \leq i \leq n-1$$

where $j(x_{\alpha_i} - x) \in J(x_{\alpha_i} - x)$. Let z be a weakly cluster point of $\{x_{\alpha_n}\}$, then we have

$$\langle z - x, j(x_{\alpha_n} - x)\rangle = 0$$

It follows that

$$\|x_{\alpha_n} - x\| \leq \|x_{\alpha_n} - z\|$$

By Opial property, we have $\rho_1 < \rho_2$. This completes the proof. □

Lemma 2.2. *The following conditions are equivalent:*

(i) *X has locally uniformly Opial property*
(ii) *If $\{x_\alpha\}$ converges to 0 weakly and $\limsup_{n\to\infty} \|x_\alpha\| = 1$. Then for each $\delta > 0$ there exists $\epsilon > 0$ such that $\limsup_{\alpha} \|x_\alpha - x\| \geq 1 + \epsilon$ for all $\|x\| \geq \delta$.*

Proof. We only need to prove that (i) implies (ii). If (ii) is not true, then there exists $\delta > 0$ and sequence $\{y_n\}$ with $\|y_n\| \geq \delta$ such that

$$\limsup_{\alpha} \|x_\alpha - y_n\| < 1 + \frac{1}{n}$$

Now we may define inductively sequence $\{\alpha_n\}$ such that

$$1 - \frac{1}{n} \leq \|x_{\alpha_n}\| \leq 1 + \frac{1}{n}$$

$$\|x_{\alpha_n} - y_i\| \leq 1 + \frac{1}{i} \quad \text{for all } 1 \leq i \leq n$$

and

$$\langle x_{\alpha_n}, j(x_{\alpha_i})\rangle \leq \frac{1}{n} \quad \text{for all } 1 \leq i \leq n-1$$

where $j(x_{\alpha_i}) \in J(x_{\alpha_i})$. let z be a weakly cluster point of $\{x_{\alpha_n}\}$, then we have

$$\langle z, j(x_{\alpha_n})\rangle = 0$$

It follows that

$$\|x_{\alpha_n}\| \le \|x_{\alpha_n} - z\|$$

By Opial property, we have $z = 0$. Since

$$\|x_{\alpha_n} - y_i\| \le 1 + \frac{1}{i}$$

By locally uniformly Opial property, we have $y_n \to 0$, this is a contraction. The proof is completed. □

We can also prove following lemma from the definition of locally uniformly τ-opial property. (See also [10])

Lemma 2.3. *Let X be a Banach space with locally uniformly convex τ-opial property. Then for any net $\{x_\alpha\}$ in X which converges to x in the sense of τ topology and for any sequence $\{y_n\}$ in X, $\limsup_{n\to\infty} \limsup_\alpha \|x_\alpha - y_n\| \le \limsup_\alpha \|x_\alpha - x\|$ implies $\{y_n\}$ converges to x in norm.*

In the sequel, for any net $\{x_\alpha\}$ in C and $x \in C$, we adopt the following notations:

$$\omega_\tau\{x_\alpha\} = \{\tau \text{ cluster points of } \{x_\alpha\}\}$$

$$\omega_w\{x_\alpha\} = \{\text{weak cluster points of } \{x_\alpha\}\}$$

$$\omega_w(x) = \omega_w(\{T^n x\})$$

$$L(\{x_\alpha\}) = \{p \in C : \lim_\alpha \|x_\alpha - p\| \ \text{ exists }\}$$

$$\Delta_w(T) = \{x \in C : T^n x \rightharpoonup x \ \text{ weakly}\}$$

$$\Delta_s(T) = \{x \in C : T^n x \to x \text{ strongly}\}$$

The following lemma will be useful later.

Lemma 2.4. *Let C be a weakly compact subset of a Banach space X with asymptotic (P) and let T be an asymptotically nonexpansive type mapping on C. Then $\Delta_w(T) = \Delta_s(T)$. Moreover, if T^N is continuous for some $N \in \mathcal{N}$, then $\Delta_w(T) = \Delta_s(T) = F(T)$.*

Proof. It is clear that $\Delta_s(T) \subset \Delta_w(T)$. Now for $p \in \Delta_w(T)$, we first claim that there exists $\lim_{n\to\infty} \|T^n p - p\|$. In fact, the condition $T^n p$ converges to p weakly implies that

$$\begin{aligned} \|T^n p - p\| &\le \liminf_{m\to\infty} \|T^n p - T^{m+n} p\| \\ &\le c_n(p) + \liminf_{m\to\infty} \|T^m p - p\|, \end{aligned}$$

for all $n \in \mathcal{N}$. This implies that $\limsup_{n\to\infty} \|T^n p - p\| \le \liminf_{n\to\infty} \|T^n p - p\|$, and hence $\lim_{n\to\infty} \|T^n p - p\|$ exists. Let $r = \lim_{n\to\infty} \|T^n p - p\|$, for $m \in \mathcal{N}$ we have

$$\begin{aligned} \limsup_{n\to\infty} \|T^n p - T^m p\| &= \limsup_{n\to\infty} \|T^{n+m} p - T^m p\| \\ &\le c_m(p) + \limsup_{n\to\infty} \|T^n p - p\| \\ &= c_m(p) + r \end{aligned}$$

Taking the $\limsup$ for m we have

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \|T^n p - T^m p\| \le r.$$

It then follows from *GGLD* property that $T^n p \to p$ strongly.

Suppose in addition that there is an $N \in \mathcal{N}$ such that T^N is continuous at p, we have $T^{N+n} p \to T^N p$ strongly. This implies $T^N p = p$. Since $Tp = T^{nN+1} p$, for all n, take the limit as $n \to \infty$, we obtain $Tp = p$. This completes the proof.

3. τ- DEMICLOSEDNESS PRINCIPLE

Theorem 3.1. *Let X be a Banach space, τ be a Hausdorff topology on X such that X has locally uniformly τ-Opial property. Let C be a τ-compact convex subset of X and let T be an asymptotically nonexpansive mapping on C such that T^N is continuous for some $N \in \mathcal{N}$. Suppose $\{x_\alpha : \alpha \in I\}$ be a directed net in C, then the condition that $x_\alpha \xrightarrow{\tau} x$ and $T^m x_\alpha \xrightarrow{\tau} x$ for each $m \in \mathcal{N}$ implies $x = Tx$.*

Proof. Let $b_m = \limsup_{\alpha\in I} \|T^m x_\alpha - x\|$, then for $i, j \in \mathcal{N}$,

$$\begin{aligned} b_{i+j} &= \limsup_{\alpha\in I} \|T^{i+j} x_\alpha - x\| \\ &\le \limsup_{\alpha\in I} \|T^{i+j} x_\alpha - T^i x\| \quad \tau\text{-Opial property} \\ &\le c_i(x) + \limsup_{\alpha\in I} \|T^j x_\alpha - x\| \\ &= c_i(x) + b_j \end{aligned}$$

Since $\lim_{i\to\infty} c_i(x) = 0$, we have

$$\lim_{i\to\infty} b_i = b := \inf\{b_i : i \in \mathcal{N}\}.$$

Let $\{O_\alpha : \alpha \in A\}$ be a τ-open base of neighborhood of x. One can define the binary relation "$\geq$" on A by $\alpha_1 \geq \alpha_2$ if and only if $O_{\alpha_1} \subseteq O_{\alpha_2}$. It is easily seen that $(A, \geq)$ is a directed system. Let $\mathcal{N}$ be the set of positive integers and let $B = A \times \mathcal{N} = \{(\alpha, n) : \alpha \in A, n \in \mathcal{N}\}$. For any $\beta = (\alpha, n) \in B$, we write $P_1\beta = \alpha, P_2\beta = n$ and $O_\beta = O_\alpha$. In this case, $(B, \geq)$ is also a directed system when the binary relation "$\geq$" on B is defined by $\beta_1 \geq \beta_2$ if and only if $P_1\beta_1 \geq P_1\beta_2$ and $P_2\beta_1 \geq P_2\beta_2$.

For $\beta \in B$, there exist $n_\beta^{(1)} \in \mathcal{N}$ such that

$$c_n(x) \leq \frac{1}{P_2\beta} \tag{3.1}$$

and

$$b_n = \limsup_{\alpha\in I} \|T^n x_\alpha - x\| \leq b + \frac{1}{P_2\beta} \tag{3.2}$$

for all $n \geq n_\beta^{(1)}$. Let $l_\beta = 2P_2\beta + n_\beta^{(1)}$ and $l_\beta^i = l_\beta - i$ for each $1 \leq i \leq P_2\beta$.

By hypothesis, for $\beta \in B$,

$$\tau - \lim_{\alpha\in I} T^{l_\beta} x_\alpha = x.$$

Select $\alpha_\beta^1 \in I$ such that if $\alpha \geq \alpha_\beta^1$, then

$$T^{l_\beta} x_\alpha \in O_\beta. \tag{3.3}$$

Since $l_\beta^{(1)} \geq n_\beta^{(1)}$ for each $1 \leq i \leq P_2\beta$, it then follows from (3.2) that for each $1 \leq i \leq P_\beta$,

$$b_{l_\beta^{(i)}} = \limsup_{\alpha\in I} \|T^{l_\beta^{(i)}} x_\alpha - x\| \leq b + \frac{1}{P_2\beta}$$

Therefore there exists $\alpha_\beta^2 \in I$ such that

$$\|T^{l_\beta^{(i)}} x_\alpha - x\| \leq b + \frac{2}{P_2\beta} \tag{3.4}$$

for all $1 \le i \le P_\beta$, and $\alpha \ge \alpha_\beta^2$. Now since $b \le b_{l_\beta}$, there exists $\alpha_\beta \in I$ such that $\alpha_\beta \ge \alpha_\beta^1$, α_β^2 and

$$\|T^{l_\beta} x_{\alpha_\beta} - x\| \ge b - \frac{1}{P_2\beta} \tag{3.5}$$

Since $\alpha_\beta \ge \alpha_\beta^1$, α_β^2, it then follows from (3.3) and (3.4) that

$$T^{l_\beta} x_{\alpha_\beta} \in O_\beta \tag{3.6}$$

and

$$\|T^{l_\beta^{(i)}} x_{\alpha_\beta} - x\| \le b + \frac{2}{P_2\beta}, \quad \text{for each} \quad 1 \le i \le P_2\beta \tag{3.7}$$

(3.6) implies that $T^{l_\beta} x_{\alpha_\beta}$ is τ-convergent to x, and (3.4), (3.5) and (3.7) imply that

$$\begin{aligned} \|T^{l_\beta} x_{\alpha_\beta} - T^i x\| &\le \|T^{l_\beta^{(i)}} x_{\alpha_\beta} - x\| + c_i(x) \\ &\le b + c_i(x) + \frac{2}{P_2\beta} \\ &\le \|T^{l_\beta} x_{\alpha_\beta} - x\| + c_i(x) + \frac{3}{P_2\beta} \end{aligned}$$

for each $1 \le i \le P_2\beta$. By Lemma 2.3, $T^i x \to x$, since T^N is continuous at x, we have $T^{N+n}x \to T^N x$. This implies $T^N x = x$. Since $Tx = T^{nN+1}x$, for all n, take the limit as $n \to \infty$, we obtain $Tx = x$. This completes the proof. □

Remark. *In Theorem 3.1, we do not suppose that C is a convex subset. It is not clear that whether the conclusion is still true if we only assume that X has the Opial property. We will discuss it in the following that when C is a weakly compact convex subset of X.*

Theorem 3.2. *Let C be a nonempty weakly compact convex subset of a Banach space X with Opial property, and let $T : C \mapsto C$ be an asymptotically nonexpansive type mapping. For a net $\{x_\alpha : \alpha \subset I\}$ in C, suppose that x_α converges weakly to x and $x_\alpha - T^m x_\alpha$ converges weakly to 0 for all $m \in \mathcal{N}$, then $x \in \Delta_w(T)$, i.e., $T^n x \rightharpoonup y$.*

Proof. Let $b_m = \limsup_{\alpha \in I} \|T^m x_\alpha - x\|$, then

$$\lim_{i \to \infty} b_i = b := \inf\{b_i : i \in \mathcal{N}\}.$$

Let $\{O_\alpha : \alpha \in A\}$ be a weakly open base of neighborhood of x. One can define the binary relation "$\ge$" on A by $\alpha_1 \ge \alpha_2$ if and only if $O_{\alpha_1} \subseteq O_{\alpha_2}$. It is easily seen that $(A, \geq)$ is a directed system. Let $\mathcal{N}$ be the set of positive integers and let $B = A \times \mathcal{N} = \{(\alpha, n) : \alpha \in A, n \in \mathcal{N}\}$. For any $\beta = (\alpha, n) \in B$, we write $P_1\beta = \alpha, P_2\beta = n$ and $O_\beta = O_\alpha$. In this case, $(B, \geq)$ is also a directed system when the binary relation "$\geq$" on B is defined by $\beta_1 \geq \beta_2$ if and only if $P_1\beta_1 \geq P_1\beta_2$ and $P_2\beta_1 \geq P_2\beta_2$.

Now for each $\beta \in B$, there exist $n_\beta^{(1)} \in \mathcal{N}$ such that

$$c_n(x) \leq \frac{1}{P_2\beta} \tag{3.8}$$

and

$$b_n = \limsup_{\alpha \in I} \|T^n x_\alpha - x\| \leq b + \frac{1}{P_2\beta} \tag{3.9}$$

for all $n \geq n_\beta^{(1)}$. Suppose that z is a weak cluster point of $\{T^n x\}$. Then $z \in clco\{T^n x : n \geq n_\beta^{(1)}\}$, it follows that there is an integer $p\,(> 2)$ and nonnegative numbers a_{β_i} $(1 \leq i \leq p)$ with $\sum_{i=1}^{p} a_{\beta_i} = 1$ and $m_{\beta_1}, m_{\beta_2}, \ldots, m_{\beta_p} \in \{n \in \mathcal{N} : n \geq n_\beta^{(1)}\}$ such that

$$\|z - \sum_{i=1}^{p} a_{\beta_i} T^{m_{\beta_i}} x\| < \frac{1}{P_2\beta}. \tag{3.10}$$

Let $l_\beta = \sum_{i=1}^{p} m_{\beta_i}$ and $l_\beta^i = l_\beta - m_{\beta_i}$ for each $1 \leq i \leq p$.

Since for $\beta \in B$,

$$w - \lim_{\alpha \in I} T^{l_\beta} x_\alpha = x,$$

one can choose $n_\beta^{(2)} \geq n_\beta^{(1)}$ such that if $n > n_\beta^{(2)}$ then

$$T^{l_\beta} x_\alpha \in O_\beta \tag{3.11}$$

Since $l_\beta^i \geq n_\beta^{(1)}$ for all $1 \leq i \leq p$, it then follows from (3.9) that

$$b_{l_\beta^i} = \limsup_{n \to \infty} \|T^{l_\beta^i} x_\alpha - x\| \leq b + \frac{1}{P_2\beta},$$

for all $1 \leq i \leq p$. Therefore there exists $n_\beta^{(3)} \geq n_\beta^{(2)}$ such that

$$\|T^{b_{l_\beta^i}} x_\alpha - x\| \leq b + \frac{2}{P_2\beta} \tag{3.12}$$

for all $1 \leq i \leq p$ and $n \geq n_\beta^{(3)}$. Now since $b \leq b_{l_\beta}$, there exists $n_\beta \geq n_\beta^{(3)}$ such that

$$\|T^{l_\beta} x_{n_\beta} - x\| \geq b - \frac{1}{P_2\beta} \tag{3.13}$$

Since $n_\beta \geq n_\beta^{(3)} \geq n_\beta^{(2)} \geq n_\beta^{(1)}$, it then follows from (3.11) and (3.12) that

$$T^{l_\beta} x_{n_\beta} \in O_\beta \tag{3.14}$$

$$\|T^{l_\beta^i} x_{n_\beta} - x\| \leq b + \frac{2}{P_2\beta} \quad \text{for all } 1 \leq i \leq p \tag{3.15}$$

(3.7) implies that $T^{l_\beta} x_{n_\beta}$ is weakly convergent to x, and (3.8), (3.10), (3.13) and (3.14) imply that

$$\begin{aligned}
\|T^{l_\beta} x_{n_\beta} - z\| &\leq \|T^{l_\beta} x_{n_\beta} - \sum_{i=1}^{p} a_{\beta_i} T^{m_{\beta_i}} x\| + \|\sum_{i=1}^{p} a_{\beta_i} T^{m_{\beta_i}} x - z\| \\
&\leq \sum_{i=1}^{p} a_{\beta_i} \|T^{l_\beta} x_{n_\beta} - T^{m_{\beta_i}} x\| + \frac{1}{P_2\beta} \\
&\leq \sum_{i=1}^{p} a_{\beta_i} (c_{m_{\beta_i}}(x) + \|T^{l_\beta^i} x_{n_\beta} - y\|) + \frac{1}{P_2\beta} \\
&\leq b + \frac{4}{P_2\beta} \\
&\leq \|T^{l_\beta} x_{n_\beta} - x\| + \frac{5}{P_2\beta}
\end{aligned}$$

This implies that $\limsup_{\beta \in B} \|T^{l_\beta} x_{n_\beta} - z\| \leq \limsup_{\beta \in B} \|T^{l_\beta} x_{n_\beta} - x\|$. By the Opial property, we have $z = x$. This completes the proof. □

Theorem 3.2 and Lemma 2.4 immediately yield the following weak demiclosedness theorem.

Theorem 3.3. *Let X be a Banach spach with the asymptotic (P) and Opial's property. Let C be a weakly compact convex subset of X and let $T : C \mapsto C$ be an asymptotically nonexpansive type mapping such that T^N is continuous for some $N \in \mathcal{N}$. Then $I - T$ is weak demiclosed at 0. that is , if $\{x_\alpha\}$ is a net in C with $x_\alpha \rightharpoonup x$ and $x_\alpha - T^m x_\alpha \rightharpoonup 0$ for each $m \in \mathcal{N}$, then $Tx = x$.*

Remark. *It is easily seen that if $T : C \mapsto C$ is a uniformly continuous mapping of asymptotically nonexpansive type, then the condition $\|x_n - Tx_n\| \to 0$ implies that $\|x_n - T^m x_n\| \to 0$ for all $m \in \mathcal{N}$. Therefore Theorem 3.3 immediately reduces to Theorem 3.1 and 3.6 of [10] and Theorem 2.4 of [5].*

4. Asymptotic behavior

In this section, we provide τ and weak convergence theorems for almost-orbit of asymptotically nonexpansive type mapping by using the results in section 3.

Let $T : C \mapsto C$ be a mapping of asymptotically nonexpansive type. Recall that a sequence $\{x_n\}$ in C is said to be an almost-orbit of T if

$$\lim_{n\to\infty} \sup_{m\geq 1} \|x_{n+m} - T^m x_n\| = 0.$$

Now we say that $\{x_n\}$ is a weak almost-orbit of T if

$$w - \lim_{n\to\infty} (x_{n+m} - T^m x_n) = 0 \text{ for all } m \in \mathcal{N}.$$

We begin with the following simple lemmas.

Lemma 4.1. *Let C be a τ- compact subset of a Banach space X with tau-Opial property and $\{x_n\}$ be a sequence in C such that $\omega_\tau(\{x_n\}) \subset L(\{x_n\})$, then $\{x_n\}$ is tau-convergence to some point in $L(\{x_n\})$.*

proof. Clearly, $\omega_\tau(\{x_n\})$ is nonempty since C is a τ- compact subset. Let $p_i \in \omega_\tau(\{x_n\})$, $i = 1, 2$, and $p_1 = \tau - \lim_{i\to\infty} x_{n_i}$, $p_2 = \tau - \lim_{i\to\infty} x_{m_i}$. Suppose that $p_1 \neq p_2$. It then follows from $p_1, p_2 \in L(\{x_n\})$ and τ-Opial property that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p_1\| &= \lim_{i\to\infty} \|x_{n_i} - p_1\| \\ &< \lim_{i\to\infty} \|x_{n_i} - p_2\| \\ &= \lim_{n\to\infty} \|x_n - p_2\|.\end{aligned}$$

In the same way we have $\lim_{n\to\infty} \|x_n - p_2\| < \lim_{n\to\infty} \|x_n - p_1\|$. This is a contradiction. Consequently, $p_1 = p_2$. This completes the proof.

Lemma 4.2. *Let C be a nonempty subset of a Banach space X and T be an asymptotically nonexpansive type mapping from C to itself, and let $\{x_n\}$ be an almost-orbit of T. Then $F(T) \subset \Delta_s(T) \subset L(\{x_n\})$.*

Proof. Let $x \in \Delta_s(T)$ and $b_n = \sup_{m\geq 1} \|x_{n+m} - T^m x_n\|$. Since

$$\begin{aligned}\|x_{n+m} - x\| &\leq \|x_{n+m} - T^m x_n\| + \|T^m x_n - T^m x\| + \|T^m x - x\| \\ &\leq b_n + \|T^m x_n - T^m x\| + \|T^m x - x\|\end{aligned}$$

for all $n, m \in \mathcal{N}$. Now for fixed $n \in \mathcal{N}$, letting $m \to \infty$, we get

$$\limsup_{m\to\infty} \|x_m - x\| \leq b_n + \|x_n - x\|$$

This implies that $x \in L(\{x_n\})$. The proof is completed.

Using Theorem 3.1 and Lemma 4.1 we get following *tau* convergent theorem.

Theorem 4.1. *Let C be a nonempty τ- compact subset of a Banach space X with τ locally uniformly Opial property , $T : C \mapsto C$ be an asymptotically nonexpansive type mapping such that T^N is continuous for some $N \in \mathcal{N}$, and let $\{x_n\}$ be a almost-orbit of T. Then $\{x_n\}$ is τ convergent to some fixed point of $F(T)$ if and only if it is τ asymptotically regular, i.e. $\{x_{n+1} - x_n\}$ τ-converges to* 0.

Remark. *Theorem 4.1 give an affirmative answer to a question (remark 4.1) of [10].*

Theorem 4.2. *Let C be a nonempty weakly compact convex subset of a Banach space X with Opial's prorperty , $T : C \mapsto C$ be an asymptotically nonexpansive type mapping, and let $\{x_n\}$ be a weak almost-orbit of T. Suppose that $\Delta_w(T) \subset L(\{x_n\})$. Then $\{x_n\}$ is weakly convergent if and only if it is weakly asymptotically regular (i.e., $\{x_{n+1} - x_n\}$ weakly converges to 0).*

Under the additional assumption of asymptotic (P), we have $\Delta_w(T) = \Delta_s(T)$ from Lemma 2.4. It is not clear that $\Delta_s(T) \subset L(\{x_n\})$ is true for weak almost-orbit of T. However, we know from Lemma 4.2 that it is always true for almost-orbits of T. Therefore we have following weak convergent theorem for almost-orbits of asymptotically nonexpansive type mapping.

Theorem 4.3. *Let C be a nonempty weakly compact convex subset of a Banach space X with asymptotic (P) and Opial property , $T : C \mapsto C$ be an asymptotically nonexpansive type mapping, and let $\{x_n\}$ be an almost-orbit of T. Then $\{x_n\}$ is weakly convergent (to some point $x \in \Delta_s(T)$) if and only if it is weakly asymptotically regular (i.e., $\{x_{n+1} - x_n\}$ c converges to 0 weakly). Moreover, if T^N is continuous at x for some $N \in \mathcal{N}$, then $x \in F(T)$.*

Remark. *(a) Note that both the locally uniform Opial condition and UKK are stronger properties than asymptotic (P). Therefore Theorem 4.4 immediately reduces to Theorem 4.1 of [10] in the case of asymptotically nonexpansive mappings and the Banach space with uniformly Opial's property and to main Theorem of [11] in the case of Banach space with UKK norm and Opial's property and to Theorem 3.1 of [5] in the case of asymptotically regular (that is $T^{n+1}x - T^n x \to 0$ strongly).*

(b) Theorem 4.4 give an affirmative answer to a question (remark 4.1) of [10].

(c) It is known that asymptotic (P) (GGLD) lie between weak normal structure and weak uniform normal structure. However, we do not know whether the conclusion is still true if we only assume that X has the Opial property.

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Acknowledgements

This work was conducted while the first author was visiting The University of Newcastle. He thanks professor B. Sims and school of Mathematical and Physical Sciences for their kind hospitality during his visit.