Here is a problem which is similar but less complex:
Suppose we have three cardboard squares and we paint the edges red, blue or green respectively as shown in Fig. 4.

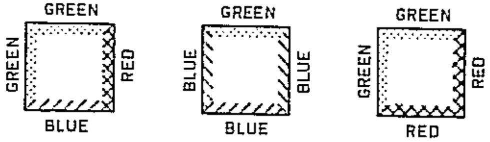

FIG. 4

Problem: Align the squares to form a 3×1 rectangle so that no two edges of like colour adjoin, and with three different colours on each long side. There are 768 possibilities, but the solution is not unique.

D. S. THORPE

341. *Dispersion of a wave packet*

The group velocity of a wave packet consisting of two sinusoidal components of equal amplitude can be computed by elementary means [1]. Most wave packets of practical importance are however composed of a large number, often an infinite number, of components. The calculation of group velocity for a wave packet of this latter type is usually deferred until a more complete treatment of the subject, employing Fourier integrals, can be undertaken.

With the inclusion in the N.S.W. 6th Form science syllabus of Rayleigh's equation for the group velocity of a wave packet it seems desirable to have a simple proof of this equation without a restriction on the number of components. It is the purpose of this note to indicate such a proof.

A Sinusoidal (harmonic) wave is of the type $y = A \sin 2\pi(kx - nt)$, where A is the amplitude of the wave, $k = 1/\lambda$ is the wave number, where λ is the wavelength, and n is the frequency of the wave.
Such a wave progresses in the x-direction with a velocity $c = n/k$.

A *wave packet* consists of a number, N, of sinusoidal waves of equal amplitude and various frequencies superimposed on one another. *Dispersion* of the wave packet takes place if the frequency is a function of the wave number, i.e. $n = n(k)$, in which case the wave packet will progress with a velocity U called the *group velocity*, while as time increases the wave packet will spread longitudinally.

In nature wave packets usually occur with the wavelengths of the sinusoidal components "peaked" about a particular wavelength λ_0 (denote by k_0 the corresponding wave number) and consequently their frequencies cluster around $n_0 = n(k_0)$.

For such a case we may approximate the relationship between frequency and wave number by the Taylor expansion of n about k_0.

$$n(k) = n_0 + n_1(k-k_0) + n_2(k-k_0)^2 + \ldots \qquad (1)$$

which for a sharply peaked wave packet we may take as

$$n(k) = n_0 + n_1(k-k_0). \qquad (2)$$

If we differentiate (1) with respect to k and evaluate the result at k_0 we see that

$$\{dn(k)/dk\}_{k_0} = n_1 \qquad (3)$$

which we will need later.

If we denote the sinusoidal component waves of the wave packet by

$$y_i = A \sin 2\pi(k_i x - n_i t) \quad \text{where } i = 1, 2, \ldots, N,$$

then the resultant wave packet will be represented by

$$y = y_1+y_2+\dots+y_N = \sum_{i=1}^{N} A \sin 2\pi(k_i x - n_i t). \quad (4)$$

Since $n_i = n(k_i)$, if we substitute (2) into (4) we have

$$y = A \sum_i \sin 2\pi(k_i x - n_1 k_i t + (n_1 k_0 - n_0)t).$$

Let $\theta = 2\pi(n_1 k_0 - n_0)t$, which is independent of k_i, and expand sine in the above sum. We then obtain

$$y = A \cos\theta \sum_i \sin 2\pi(k_i x - n_1 k_i t) + A \sin\theta \sum_i \cos 2\pi(k_i x - n_1 k_i t). \quad (5)$$

Thus we see that the resultant wave packet is represented as a sum of waves of the type

$$\sin 2\pi(k_i x - n_1 k_i t) \text{ or } \cos 2\pi(k_i x - n_1 k_i t)$$

whose amplitudes are modulated by $A \cos\theta$ or $A \sin\theta$ respectively. All of these waves and hence their resultant, the wave packet, progress with a velocity $n_1 k_i / k_i = n_1$ thus we conclude that the group velocity is $U = n_1$.

Therefore from (3) we have

$$U = \{dn(k)/dk\}_{k_0}. \quad (6)$$

But $n(k) = c(k)k$ so
$$dn(k)/dk = d(c(k)k)/dk = c(k) + k(dc(k)/dk)$$

and since $k = 1/\lambda$ we have that
$$kd(\cdot)/dk = 1/\lambda \, d\lambda/dk \, d(\cdot)/d\lambda = -\lambda d(\cdot)/d\lambda.$$

Substituting this into (6) we obtain Rayleigh's equation

$$U = c_0 - \lambda_0 \{dc/d\lambda\}_0$$

where the subscript 0 indicates evaluation at the average or 'peak' value for the sinusoidal components.

In the case where the sinusoidal components have unequal amplitudes we must replace the A in (4) by A_i which will remain inside the summation in (5). The argument and the conclusion however remains unaltered.

If the number of sinusoidal components were infinite the summation in (4) would be replaced by integration. Hence the wave packet would be represented by

$$y = \int_{-\infty}^{\infty} A(k) \sin 2\pi(kx - n(k)t) dk$$

the same argument then applies to the integral to give the group velocity as in (6).

Reference

[1] Jenkins and White, *Fundamentals of Optics*, McGraw-Hill, 1957, p. 225.

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