Algorithms for Utility Function Approximation

(A variational approach to the problem of revealed preferences)

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Summary

- Consumer Preference Theory
- Normal Cones and Cycles
- A Satisfactory Class of Utilities
- The Strong Axiom (SARP)
- Concave Utilities and Finite Data
- Approximations via Afriat Utilities
- Main Convergence Result
- A Best Fit Problem and Sampling Errors
- Some Unsolved issues with Errors
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Consumer Preference Theory

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- All economic models have to build some model of how consumer (and producers) tend to react to market pricing of commodities.
- Let $\mathbf{R}^n_+ := \{x \in \mathbf{R}^n \mid x^i \geq 0$ for all $i\}$ denote the set of commodity bundles.
- A *Utility* function $u : \mathbf{R}^n_+ \to \underline{\mathbf{R}} := \mathbf{R} \cup \{-\infty\}$ reflects the preference structure with respect to possible consumption of n commodities $x \in \mathbf{R}^n_+$.

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- Depending on the expected behaviour of the consumer, economic theory has developed expectations on what the functional structure of such functions should be and a "toolbox" of apriori determined functional forms
- It is natural to assume u is non-decreasing and so $u(x_1) \leq u(x_2)$ when $x_1 \leq x_2$ in the order defined by the positive cone $\mathbf{R}^n_+$ ("more is not worse").

- In reality one does not have direct access to such information but only the responses a consumer makes to offer of a commodity bundle at a given price structure $p \in \mathbf{R}_+^n$.

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- Thus one might observe that a consumer bought a certain bundle at a given price in preference to another bundle that might have also been "within budget".
- We refer to this as a revealed preference.
- In actual fact we are making observations of a consumption relation

$$x \in X(p) := \{\text{the commodities } x \text{ preferred at price } p\}$$

- The problem of revealed preferences asks the following question: Given the ability to take any finite sample $x_i \in X(p_i)$ for $i = 1, \ldots, m$ can one claim the actions of the consumer are governed by a preference order derived from a utility function u?

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- Part of revealed preference theory concerns itself with the properties u must possess to define a valid preference relation $y\mathcal{R}x$ (y is preferred to x) via a utility using $y \in S_{-u}(x) := \left\{z \in \mathbf{R}^n_+ \mid -u(z) \leq -u(x)\right\}$.

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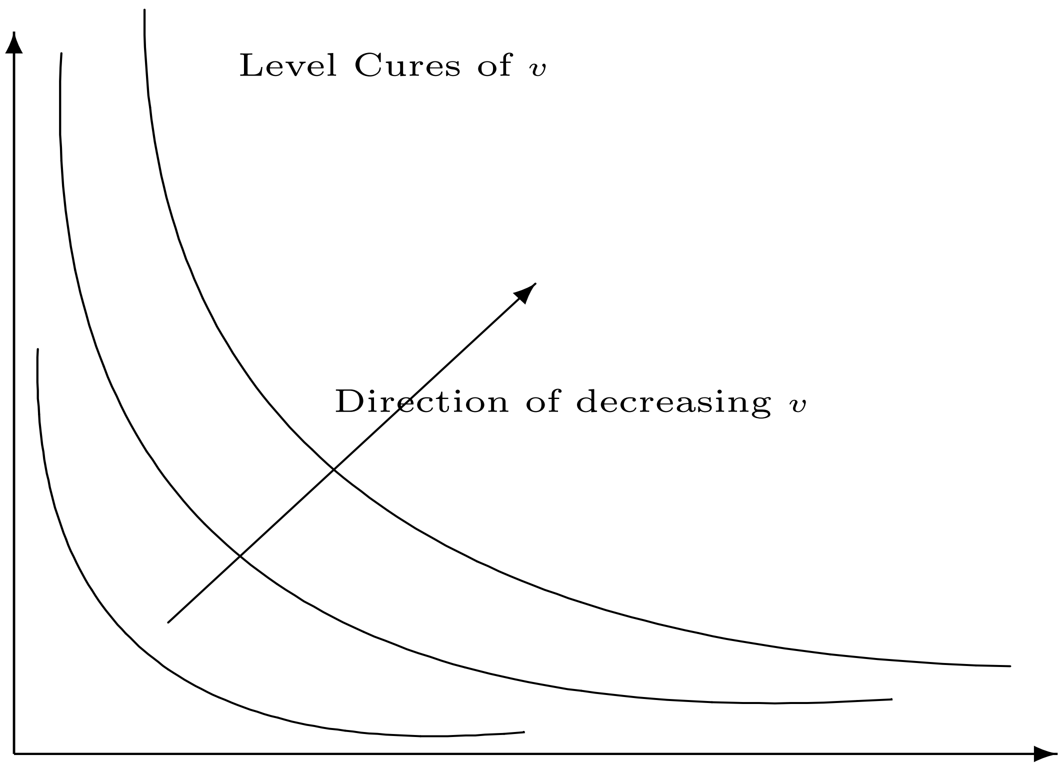
- One basic proper it that $S_{-u}(x)$ must be convex for each x, a property forcing $-u$ to be quasi-convex.

- Another property that is assumed is the nonsatiation assumption which amounts to saying that in every neighbourhood of U of x there exists a y preferred to x i.e. $u(y) > u(x)$ (i.e. "no flats").



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  that is at least weakly preferred to all elements in $BG(p, w)$.
- As $BG(p, w) = B(\lambda p, \lambda w)$ for all $\lambda > 0$ we may as well assume that $w = 1$ (unit wealth) and denote $BG(p, 1) = BG(p)$.

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- We say that x is a revealed preference to y and denote this by $x \succeq_{X_{\mathcal{R}}} y$ when $\langle p, x - y \rangle \geq 0$. That is y was in budget as $1 = \langle p, x \rangle \geq \langle p, y \rangle$ but as $(x, p) \in X_{\mathcal{R}}$ we have x chosen instead of y.

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- The transitive closure of $\succeq_{X_{\mathcal{R}}}$ gives a partial order $\succeq_R$ that denotes $x \succeq_R y$ when there exists $x = x_0$, x_1, $\ldots, x_n = y$ with $x_{i+1} \succeq_{X_{\mathcal{R}}} x_i$ for all i.

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- Similarly we denote $x \succ_R y$ when $x \succeq_R y$ and there exists i with $x_{i+1} \succ_{X_{\mathcal{R}}} x_i$ or $\langle p_{i+1}, x_{i+1} - x_i\rangle > 0$ for $(x_i, p_i) \in X_{\mathcal{R}}$.

- The *generalised axiom of revealed preference* (GARP) says that there can not exists a cycle $\{(x_i, p_i) \mid i = 0, \ldots, m\}$ (with $x_0 = x_{m+1}$) such that all

$$\langle p_{i+1}, x_{i+1} - x_i \rangle \geq 0$$

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- That is $x_{i+1} \succeq_{X_{\mathcal{R}}} x_i$ for $i = 0, \ldots, m$ with $x_0 = x_{m+1}$ implies the transitive closure satisfies $x_0 \succeq_R x_0$.

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- Now we cannot have some $\langle p_{i+1}, x_{i+1} - x_i \rangle > 0$ because we obtain the contradiction $x_0 \succ_R x_0$ and so $\langle p_{i+1}, x_{i+1} - x_i \rangle = 0$ for all i.

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- When the order relation is induce by a utility then the demand relation can be written in terms of an optimization problem

$$
\begin{aligned}
X_u(p) &:= \left\{x \in \mathbf{R}^n_+ \mid \; u(x) \geq u(y) \text{ for all } y \text{ s.t. } \langle p, y\rangle \leq 1\right\} \\
&= \{x \in BG(p) \mid u(x) = v(p)\}
\end{aligned}
$$

$$
\text{where } v(p) := \sup\{u(y) \mid \langle y, p\rangle \leq 1\} \qquad (2)
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- The indirect utility function $v(p)$ assigns to any price vector the greatest utility the consumer may achieve when he is constrained to spend no more than one unit of money (and *must be quasi–convex non–increasing*).

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- When v is associated with u via (1) then (under minimal assumptions, presuming the original quasi–concavity of u) one may recover u via the duality formula

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$$u(x) = \inf \{v(p) \mid \langle x, p\rangle \leq 1\}. \qquad (5)$$

- Thus one only needs to construct the indirect v in order to effective obtain the utility u.

Normal Cones and Cycles

- As the "non-satiation" assumption implies any optimal value satisfies $\langle x, p \rangle = 1$. We have p attains the infimum in

$$u(x) = \inf \{v(p) \mid \langle x, p \rangle \leq 1\}. \quad (6)$$

when $u(x) = v(p)$ or $x \in X_u(p)$ which implies

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- Thus we may write

$$X_u(p) =$$

$$\left\{x \in \mathbf{R}^n_+ \mid \langle x, p\rangle = 1 \text{ and } \langle p' - p, x\rangle \leq 0 \text{ implies } v(p') \geq v(p)\right\}$$

- Under the "non-satiation" assumption we can also say that when $x \in X(p)$, $\langle x, p \rangle = 1$ and $\langle p' - p, x \rangle < 0$ implies $v(p') > v(p)$. This is because $\langle p', x \rangle < 1$ and so is strictly in budget. Thus it is possible to improve the utility obtained from x at price p' (via the "non-satiation" assumption).

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- Thus when duality and non-satiation applies we have $x \in X_u(p)$ corresponds to the statement that for

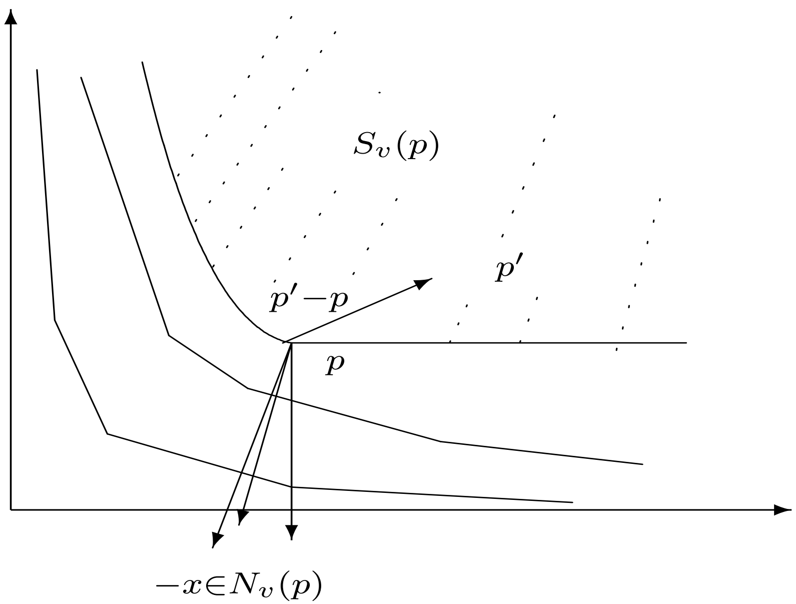
$$S_v(p) := \left\{p' \in \mathbf{R}^n_+ \mid v(p') \leq v(p)\right\} \text{ then}$$

$$\forall p' \in S_v(p) \Longrightarrow \langle p' - p, -x\rangle \leq 0$$

$$\text{or } x \in [-N_v(p)] \cap \{x \mid \langle x, p\rangle = 1\}. \tag{9}$$

- The normal cone to $S_v(p)$ at p used before is denoted by

$$N_v(p) := \left\{ y \mid \langle p' - p, y \rangle \leq 0 \quad \text{for all } p' \in S_v(p) \right\}.$$



The normal cone to the level set $S(p)$ at p.

- It is well known when v is convex

$$N_v(p) = \operatorname{cone} \partial v\,(p) := \cup_{\lambda \geq 0} \lambda \partial v\,(p)$$

and so our symmetric duality and nonsatiation assumption holds then for all $p \in \mathbf{R}^n{}_+$ we have

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- Here the convex subdifferential of v is given by

$$\partial v\,(p) := \{x \mid v(q) - v(p) \geq \langle x, q - p \rangle \text{ for all } q\}.$$

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- A functions $v : \mathbf{R}^n{}_+ \rightarrow \overline{\mathbf{R}}$ whose closure of the (convex) strict level sets $\tilde{S}_v$ satisfies

$$\overline{\tilde{S}_v(p)} = \overline{S_v(p)}$$

$$\text{where} \quad \tilde{S}_v(p) = \left\{p' \in \mathbf{R}^n{}_+ \mid v\left(p'\right) < v\left(p\right)\right\}. \tag{11}$$

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  is said to be in the class Π (or graphically pseudo-convex).
- When $\operatorname{int} \tilde{S}_v(p) \neq \emptyset$ for all $v\left(p\right) > \inf v$ we say v is solid.

- Suppose the indirect utility $v : \mathbf{R}^n_+ \to \overline{\mathbf{R}}$ is a proper, solid function in the class Π that admits the duality formula (3). Then the utility $u : \mathbf{R}^n_+ \to \overline{\mathbf{R}}$ is a proper, solid and $-u \in \Pi$. In particular we must have u non-satiated.

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- Suppose the direct utility $u : \mathbf{R}^n_+ \to \underline{\mathbf{R}}$ is a proper, solid and $-u \in \Pi$. Then the indirect utility $v : \mathbf{R}^n_+ \to \overline{\mathbf{R}}$ is a proper, solid and in the class Π. In particular we must have v non-satiated.

The Strong Axiom (SARP)

Suppose $p_0, \ldots, p_q \in \mathbf{R}^n_+$ such that there exist $x_i \in X_u(p_i)$ with

- $\langle p_i, x_{i+1} \rangle \leq \langle p_i, x_i \rangle$ then x_{i+1} could have been purchased at price p_i as it is in budget. Since $x_i \in X_u\left(p_i\right)$ we have chosen x_i instead of x_{i+1}. Thus x_{i+1} is not strictly preferred to x_i or $u(x_i) \geq u\left(x_{i+1}\right)$ and

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- Iteration of this gives $u(x_0) \geq u(x_1) \geq \cdots \geq u(x_q)$.

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- Iteration of this gives $u(x_0) \geq u(x_1) \geq \cdots \geq u(x_q)$.
- If $\langle p_q, x_q \rangle > \langle p_q, x_0 \rangle$ then by non-satiation a ξ close to x_0 exists with $\langle p_q, x_q \rangle > \langle p_q, \xi \rangle$ with $u(\xi) > u(x_0) \geq u(x_q)$ violating the assumption that x_p solves the maximum utility problem at price p_q.

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- Iteration of this gives $u\left(x_0\right) \geq u\left(x_1\right) \geq \cdots \geq u\left(x_q\right)$.
- If $\langle p_q, x_q \rangle > \langle p_q, x_0 \rangle$ then by non-satiation a ξ close to x_0 exists with $\langle p_q, x_q \rangle > \langle p_q, \xi \rangle$ with $u(\xi) > u\left(x_0\right) \geq u\left(x_q\right)$ violating the assumption that x_p solves the maximum utility problem at price p_q.
- That is

$$\langle p_q, x_q - x_0 \rangle \leq 0 \text{ for all } x_q \in X_u(p_q) \backslash \left\{0\right\}.$$

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$$\langle p_i, x_{i+1} - x_i \rangle \geq 0 \quad \text{implies} \quad \langle p_q, x_0 - x_q \rangle \leq 0. \tag{14}$$

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$$\langle p_i, x_{i+1} - x_i \rangle \geq 0 \quad \text{implies} \quad \langle p_q, x_0 - x_q \rangle \leq 0. \tag{15}$$

- We say that Γ is called cyclically pseudo–monotone if in addition to (13) we also have for all $p_q \in \Gamma(x_q) \setminus \{0\}$

$$\exists\, i \text{ such that } \langle p_i, x_{i+1} - x_i \rangle > 0 \text{ then } \langle p_q, x_0 - x_q \rangle < 0.$$

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- The first part is just the SARP for the multi-function $\Gamma = -X_u$.

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- A multifunction $\Gamma : S \rightrightarrows \mathbf{R}^n$ is cyclically pseudo–monotone if and only if we have for all $i = 1, \ldots, q$ and $p_i \in \Gamma(x_i)$

$$\forall i \quad \langle p_i, x_{i+1} - x_i \rangle \geq 0 \quad \Rightarrow \quad \langle p_i, x_{i+1} - x_i \rangle = 0 \text{ for all } i. \tag{18}$$

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- Factoring the minus sign in we have shown the surprising results that GARP holds iff SARP holds (in the strengthened form of pseudo-monotonicity of $-X_u$ or equivalently of N_v).

Concave Utilities and Finite Data

Placing $I = \{1, \ldots, m\}$ let

$$a_{ij} := \langle p_i, x_j - x_i \rangle \quad \text{for } i, j \in I \quad \text{and}$$

$$b_{ij} := \langle x_i, p_j - p_i \rangle \quad \text{for } i, j \in I$$

We refer to the following inequalities as the direct Afriat inequalities

$$\phi_j \leq \phi_i + \lambda_i a_{ij} \quad \text{for } i, j \in I.$$

We refer to the following inequalities as the indirect Afriat inequalities

$$\psi_j \geq \psi_i - \mu_i b_{ij} \quad \text{for } i, j \in I. \tag{20}$$

- We note that the following are equivalent:
- SARP$\equiv$GARP holds for X_u
- there is a feasible solution to the direct Afriat inequalities (in (ϕ_i, λ_i) for $i, j \in I$)
- there is a feasible solution to the indirect Afriat inequalities (in (ψ_i, μ_i) for $i, j \in I$).
- As λ_i simply imposes a scaling of the function values we can demand that $\lambda_i \geq 1$ and can fit an Afriat utility.
- As μ_i simply imposes a scaling of the function values we can demand that $\mu_i \geq 1$ and can fit an indirect Afriat utility.

- Given a set of data $(\{x_i, p_i\})_{i\in I}$ and a set of direct parameters $\{(\phi_i, \lambda_i)\}_{i\in I}$ we define the indirect Afriat utility as:

$$v_m(p) := \max\{\psi_1 - \mu_1\langle x_1, p - p_1\rangle, \ldots, \psi_m - \mu_m\langle x_m, p - p_m\rangle\}$$

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- Thus any such finite sample from the demand function generated by such an indirect utility must satisfy GARP because N_v is cyclically pseudo-monotone.

Approximations via Afriat Utilities

- Roughly speaking, a sequence of extended-real-valued functions $\{f_m\}_{m=0}^{\infty}$ epi-converges to an extended-real-valued function f if their level sets $S_{f_m}(x)$ converges as sets to $S_f(x)$ for all $x \notin \arg\min f$.

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- As level sets correspond to indifference curves this is exactly the behaviour we seek from an approximation.
- Any epi-convergent family must converge to a lower semi-continuous function. In general as lower semi-continuity is not considered a fundamental notion when studying quasi-convex functions.

Approximations via Afriat Utilities (cont.)

Definition 1 *Given a family of extended-real valued, quasi-convex function* $\{g^v\}_{v\in N}$ *we say this family essentially epi-converges to* g *as* $v \to w$ *iff for all* λ *we have*

- *There exists a* $\lambda_v \to \lambda$ *such that we have* $\tilde{S}_\lambda(g) \subseteq \liminf_v \tilde{S}_{\lambda_v}(g^v)$;
- *For all* $\lambda_v \to \lambda$ *we have* $\limsup_v S_{\lambda_v}(g^v) \subseteq S_\lambda(g)$.

This concept appears to be weaker than epi-convergences as it does not require lsc of g.

Approximations via Afriat Utilities (cont.)

- Denote $\bar{f}(x) := \inf \left\{ \lambda \mid x \in \overline{S_\lambda (f)} \right\}$ and indeed f is lsc at a if and only if $f(a) = \bar{f}(a)$.

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Proposition 3 *If we have a family of extended-real valued, functions $\{g^v\}_{v \in N}$ that essentially epi-converges to g as $v \to w$ then $\{g^v\}_{v \in N}$ actually epi-converges to $\overline{g}$ (and hence also essentially epi-converges to $\overline{g}$ as well).*

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- There exists theorems that link epi–convergence of a sequence of convex functions $\{f_m\}_{m=1}^{\infty}$ to f and graphical convergence of the subdifferential of f i.e.

$$\partial f(x) = g\text{-}\lim_{m \to \infty} \partial f_m(x).$$

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- We then renormalise our Afriat utilities

$$\hat{v}_m(p) := -k_m^{-1}(v_m(p))$$

which is the composition of a convex function on $\mathbf{R}^n$ and a concave increasing mapping on $\mathbf{R}$.

- Now p_1 lies on the level curve $\{p \mid \hat{v}_m(p) = -1\}$ for each m and also $\tau \mapsto \hat{v}_m(\tau p_1) = -\tau$ is finite.

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$$N_m(\bar{p}) = \operatorname{cone} \partial \hat{v}_m(\bar{p}) . \tag{23}$$

- We may make the following change of origin and basis of the local coordinate system around p_1. Consider the direction $d = p_1 / \|p_1\|$ of strict monotonicity of $\hat{v}_m$ to be the nth vector in the canonical basis and p_1 the origin.

- Now a neighbourhood of $p_1$may be taken to have the form $V = Y \times T$ where Y and T are closed convex neighbourhoods of the origin in $\mathbf{R}^{n-1}$ and $\mathbf{R}$ respectively and the resultant function we will denote by $t \mapsto f_m(y,t)$ is decreasing and lower semi-continuous.

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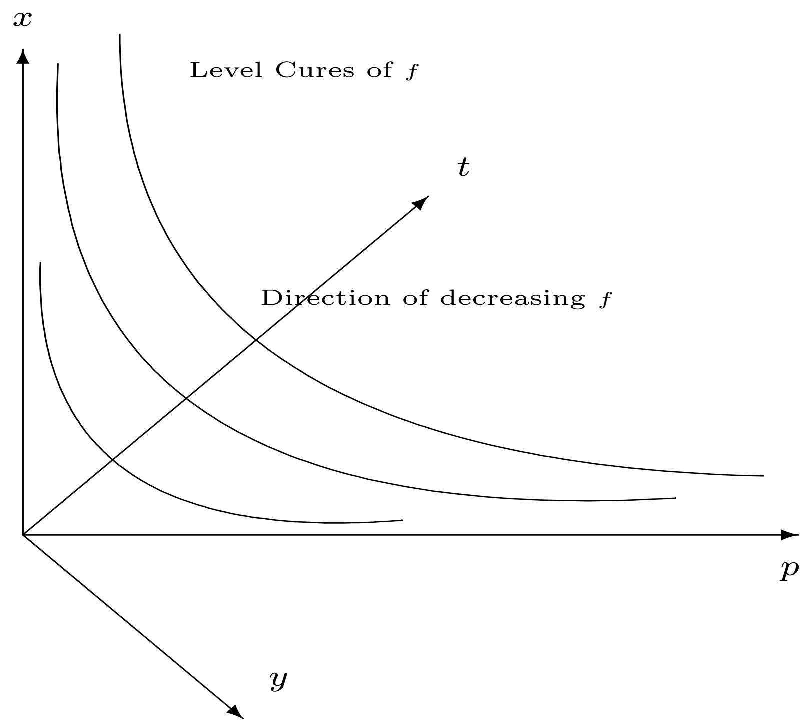
- Define the indifference curves (continuous in λ) as

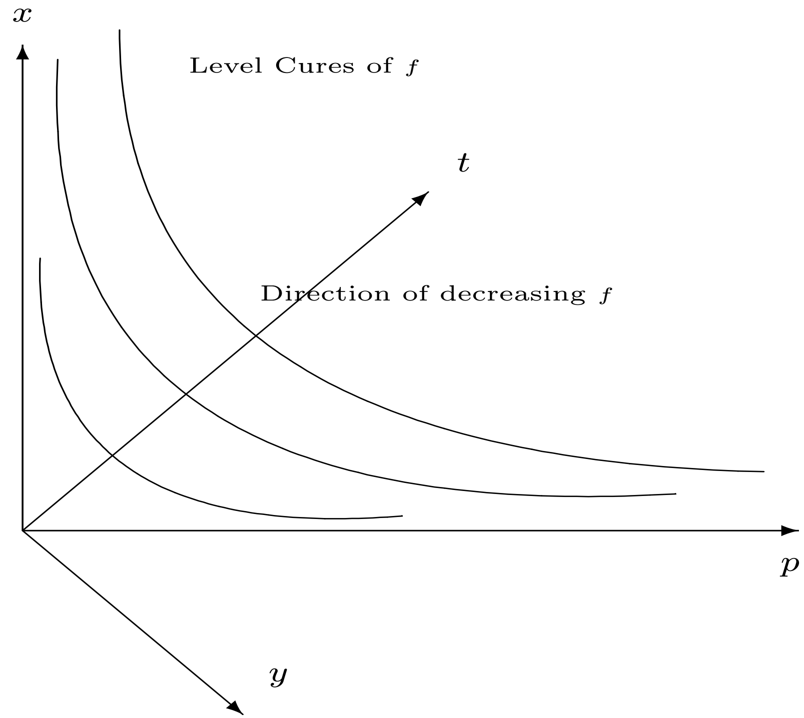
$$g_m(y,\lambda) = \inf\{t \mid f_m(y,t) \le \lambda\}, \qquad \lambda \in (\lambda_0, +\infty) \quad \text{and}$$

$$N_{f_m}(y,t) = \operatorname{cone}\{(z,-1) \mid z \in \partial_y g_m(y,\lambda) \text{ for } \lambda = f_m(y,t)\}$$

Then

$$f_m(y,t) = \sup\{\lambda \mid g_m(y,\lambda) > t\} \quad \text{for } (y,t) \in Y \times T \quad (*) \tag{25}$$


x
Level Cures of f
t
Direction of decreasing f
p
y



The monotonic decreasing property in λ and a continuity property of $\lambda \mapsto g(y, \lambda)$ for any such family of proper, convex level set functions $\{g(\cdot, \lambda)\}_{\lambda \in \Lambda}$ correspond directly to a solid, pseudo–convex function f, as defined via the transformation (*), being strictly decreasing in t.

- Now suppose we have the epi-convergence of the convex functions $\{g_m(\cdot,\lambda)\}_{\lambda\in\Lambda}$. As $\operatorname{epi} g_m(\cdot,\lambda)$ corresponds to the indifference curve at level $\lambda = f_m(0,t) = -t$, convergence of $\operatorname{epi} g_m(\cdot,\lambda)$ corresponds to convergence of level curves, precisely the epi-convergence of $\{f_m\}_{m=1}^{\infty}$!

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- Epi–convergence satisfies a compactness property: From any sequence of functions $\{g_m\}_{m=1}^{\infty}$ we may extract an epi-convergent subsequence and in this manner we may extract an epi–convergent subsequence from $\{f_m\}_{m=1}^{\infty}$.

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- Now we may use graphical convergence of subdifferentials.

Main Convergence Result

Theorem *Suppose we have an underlying preference relation $\succeq$ and define*

$$\Gamma\left(p\right) := \left\{-x \mid x \succeq y \text{ whenever } \langle y, p\rangle \leq \langle x, p\rangle\right\}$$

with the demand correspondence $X\left(p\right) = -\Gamma\left(p\right) \cap \left\{x \mid \langle x, p\rangle = 1\right\}$.

1. *Suppose* $\Gamma : D \rightrightarrows \mathbf{R}^n$ *is cyclically pseudo–monotone (i.e. SARP holds for* X*);*
2. *has closed graph and convex, conic images on a closed, bounded set* $D \subseteq \operatorname{dom}\Gamma$ *such that* $\overline{\operatorname{int} D} = D$ *and*
3. *there exists a* $d \in \mathbf{R}^n$ *such that* $\langle x, d\rangle < 0$ *for all* $x \in \Gamma(p)\backslash\{0\}$ *and* $p \in D$.

Then there exists a solid, pseudo-convex indirect utility function $v : D \rightarrow \overline{\mathbf{R}}$ *such that* $p_i \in \arg\min\left\{v(p) \mid \langle x_i, p\rangle \leq 1\right\}$ *for all* i *and*

$$X\left(p\right) = -N_v\left(p\right) \cap \left\{x \mid \langle x, p\rangle \leq 1\right\} \quad \text{for all } p \in \operatorname{int} D.$$

- The proof is constructive in the sense that we approximate v via a subsequence of renormalised Afriat indirect utilities $\{\hat{v}_{m_k}\}_{k=1}^{\infty}$ and show that we have epi-convergence of a subsequence.

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- In particular if we only have access to a countably dense set of values $\mathcal{X} := \{(x_i, p_i)\}_{i=1}^{\infty} \subseteq \mathrm{Graph} X$ then we may write

$$X(p) = \left[\limsup_{\delta \downarrow 0} \operatorname{cone} \operatorname{co} X \left(B_\delta(p) \cap \{p_i\}_{i=1}^{\infty} \right) \right] \cap \{x \mid \langle x, p \rangle = 1\}$$

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- That is the demand correspondence can be recovered from only a dense selection.

A Best Fit Problem and Sampling Errors

- In this section we assume we have access to a set of raw data $\{(x_i, p_i) \mid i \in I\}$, $I := \{0, \ldots, m\}$ where x_i is "not far" from $X_u(p_i)$.

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- here we enforce $s_0 = 0$ so as to not disturb the nominal state of the economy, but this is optional.

We formulate the least squares best fit problem as:

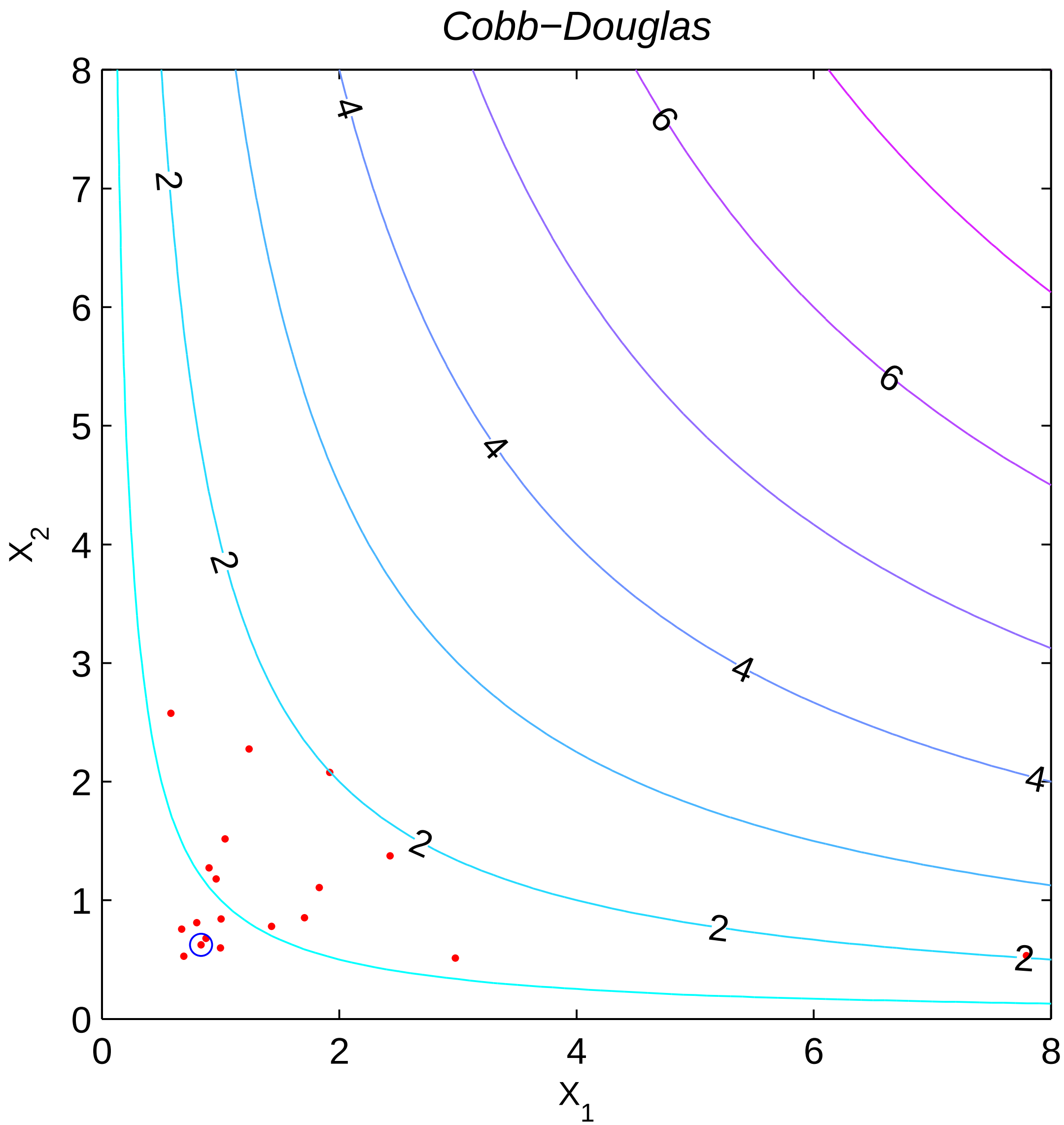
$$\min_{(\phi,\lambda,s)} \sum_{i\in I,\, i\neq 0} s_i^2 + \sum_{i\in I} \lambda_i$$

subject to

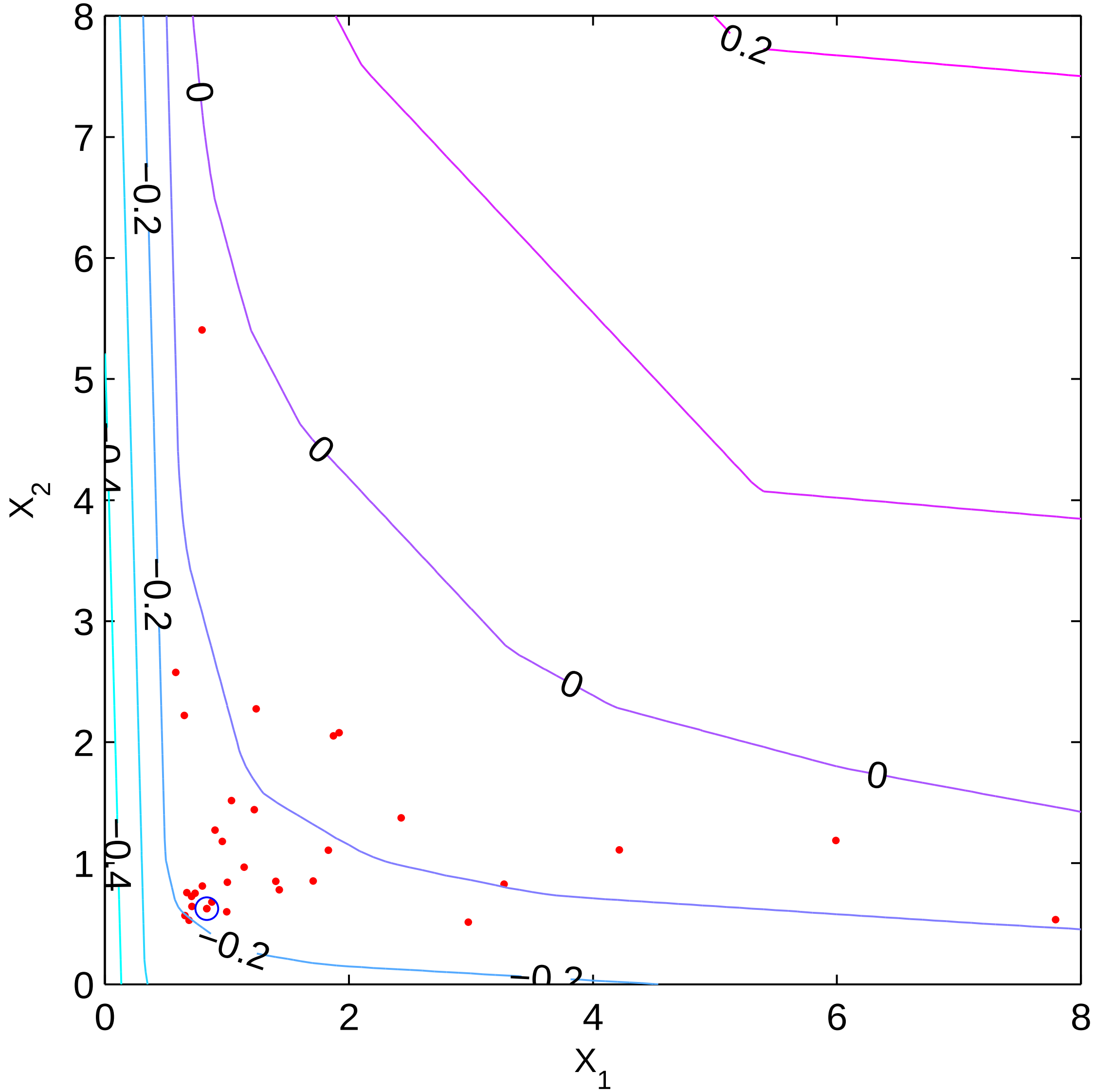
$$\phi_j - \phi_i \leq \lambda_i \left[\langle p_i, x_j - x_i\rangle + \langle p_i, s_j - s_i\rangle\right] \quad \text{for } i, j \in I,\ i \neq j$$

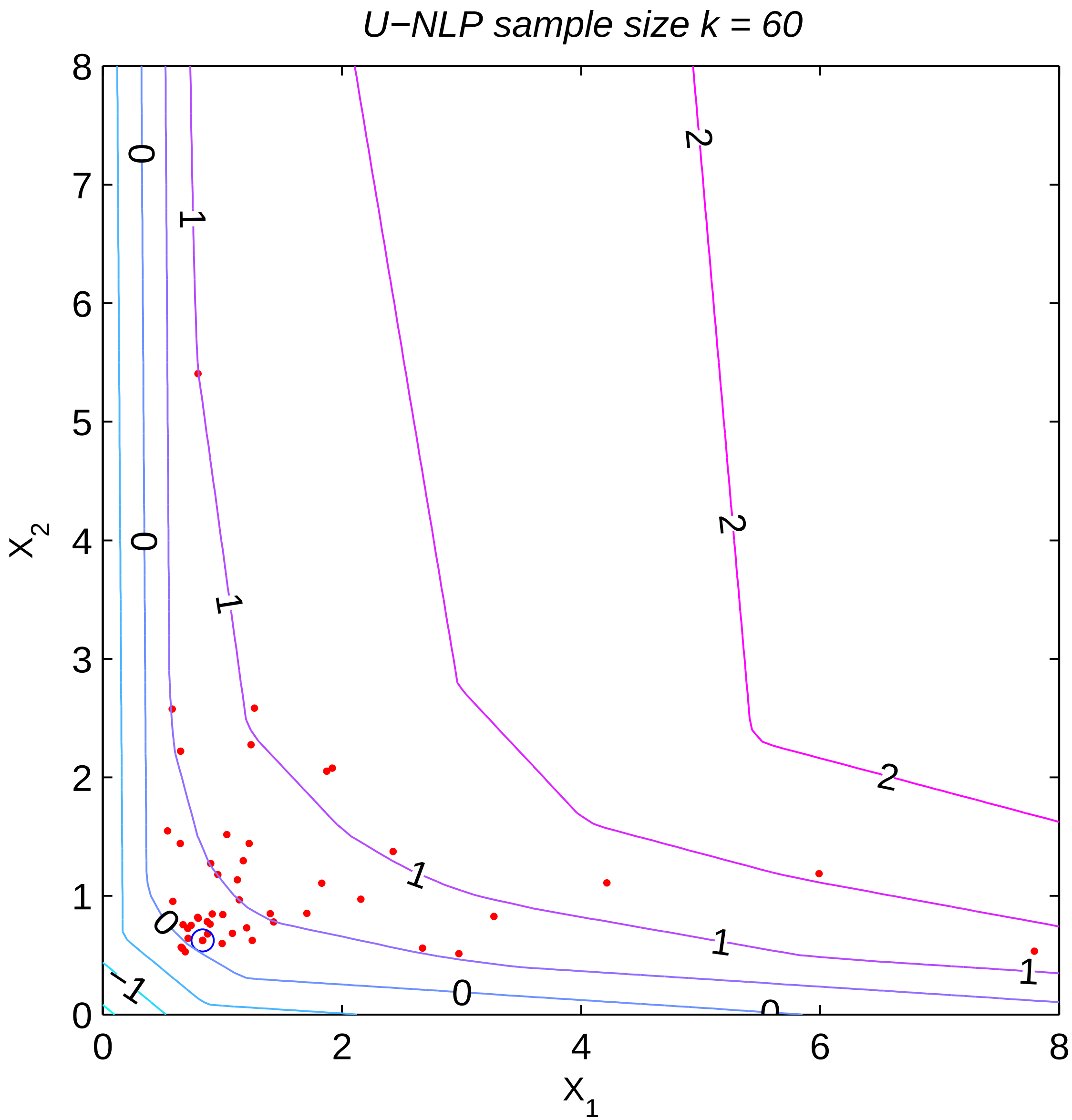
$$\langle p_i, s_i\rangle = 0,\ \lambda_i \geq 1 \text{ and } \quad x_i + s_i \geq 0, \qquad \text{(NLPA}^+\text{)}$$

Then we may place $u(x) = \min\{\phi_i + \lambda_i\langle p_i, x - x_i - s_i\rangle\}$ for all x.


Cobb−Douglas
X2
X1
0
1
2
3
4
5
6
7
8
0
2
4
6
8
2
2
2
2
2
4
4
4
4
6
6

U−NLP sample size k = 40




U−NLP sample size k = 60
X2
X1
0
1
2
3
4
5
6
7
8
0
2
4
6
8
0
1
2
−1

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- We answer these questions by performing a sensitivity analysis on the slack values s_i by introducing errors to data that initially satisfied GARP. By randomly generating price data and the data is now of the form $(x_i + \bar{s_i}, p_i)$. We Run (NLPA$^+$) and compare the shifts in the slacks s_i for different sizes of the variance in $\bar{s_i}$.

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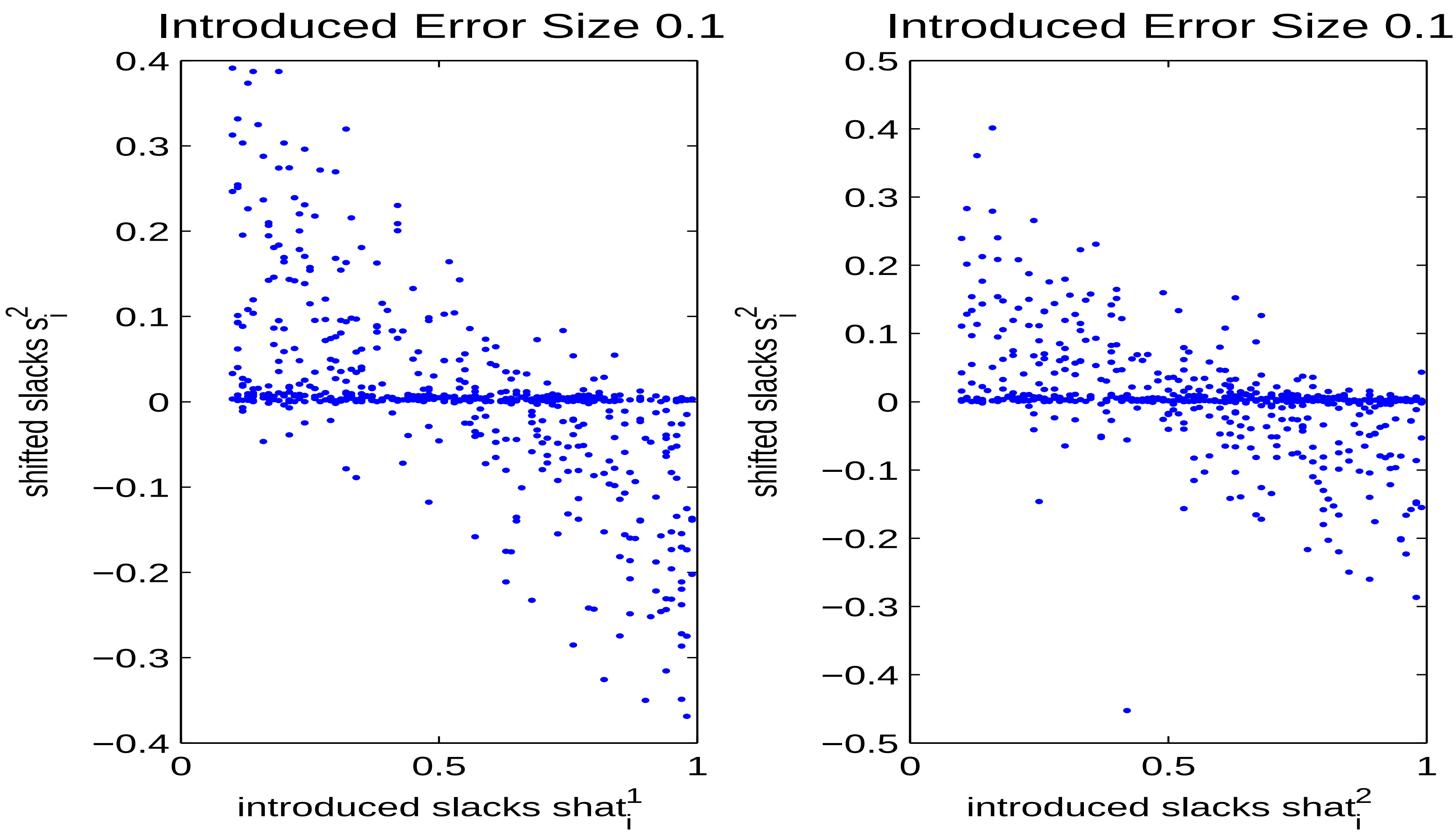
Are the real errors corrected for a dense set?



Scatter plots of the two componets s_i^j against $\bar{s}_i^j$ for $j = 1, 2$.

Estimating Elasticities from the Utility

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- Denote input prices by p and changed prices as P (similarly X for a commodity bundle changed from x).
- We take $X = x_0$ and $P = p_0$ as the base point for this calculation. That is $X_l = x_{0l}$ and $P_k = p_{0k}$. We need to estimate ΔX_l and ΔP_k.

Calculating Compensated Elasticity

- Approximate the compensated elasticity by:

$$e^c_{ij} = \frac{p_j}{x_i}\left(\frac{\partial x_i}{\partial p_j}\right)_{dU=0} \simeq \frac{p_j}{x_i}\frac{\Delta x_i}{\Delta p_j}$$

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- The elasticity we want to compute is a little bit complex because we are interested price elasticity with respect to demand subject to utility remaining constant.

- Consider the parametric optimization problem

$$\min_{X} \langle P,\, X \rangle$$

$$\text{Subject to} \quad X \geq 0$$

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- Since u is piecewise linear and concave, this problem can be written as a parametric linear program,

$$\begin{array}{lccl} \min & \langle P,\, X\rangle & & \\ \text{Subject to} & X & \geq & 0 \\ & \phi_0 & \leq & \phi_i + \lambda_i \langle p_i,\, X - x_i\rangle, \quad \forall i = 0, \ldots, N. \end{array}$$

(LP(P))

- For $P = p_0$, by looking at the constraint on ϕ_0 corresponding to $i = 0$, we see that any feasible X satisfies $\langle p_0, X\rangle \geq \langle p_0, x_0\rangle$. Thus x_0 solves (LP(p_0)).

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- The compensated elasticity is directly related to the change in the solution of this LP under changes in price p_0.
- This is a standard sensitivity problem for an LP!

Calculating Uncompensated Elasticity

(Engel Aggregation)

- The uncompensated elasticity E_i allows the consumer to maximise their utility subject to a budget constraint Y whilst holding commodity prices fixed.

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- The uncompensated elasticity E_i allows the consumer to maximise their utility subject to a budget constraint Y whilst holding commodity prices fixed.
- Hence we consider the problem of optimizing the utility subject to a given budget Y:

$$\begin{aligned} &\max_{(X,r)} && r \\ \text{subject to} && & \langle X, P\rangle \leq Y \\ && & X \geq 0 \\ && & r \leq \phi_i + \lambda_i \langle P, X - x_i\rangle \quad \forall \quad i = 1, \ldots, k \end{aligned} \tag{LP(Y)}$$

- In this problem we determine an upper and lower bound on the budget y so that the solution to ($LP\,(Y)$) remains optimal for X.

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- The Engel aggregation (uncompensated elasticity) is now defined as

$$E_i = \frac{y}{x_i}\frac{\partial x_i}{\partial y} \simeq \frac{Y}{x_i}\frac{X_i^+ - X_i^-}{Y^+ - Y^-} \forall i = 1, \dots, L \qquad (28)$$

Hicks-Slutsky Partition

The Hicks-Slutsky Partition allow us to find the elasticities from e^c_{ij} and E_i:

$$e_{ij} = e^c_{ij} - \alpha_j E_i. \tag{29}$$

Where

$$\alpha_j = \frac{x_j p_j}{y}, \quad e_{ij} = \frac{p_j}{x_i}\left(\frac{\partial x_i}{\partial p_j}\right) \tag{30}$$

Tea vs Coffee (ABS statistics)

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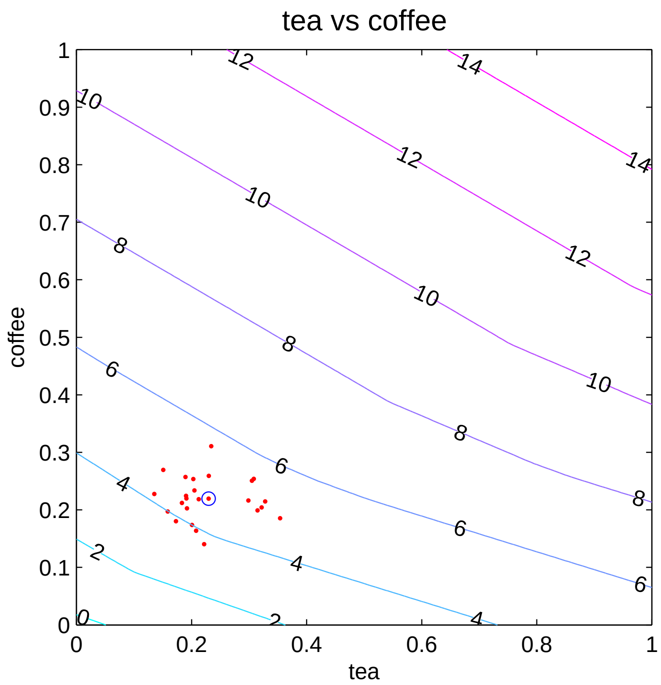
- Now the calculated elasticities of demand are,

$$\begin{pmatrix} e^c_{11} - \alpha_1 E_1 & e^c_{12} - \alpha_2 E_1 \\ e^c_{21} - \alpha_1 E_2 & e^c_{22} - \alpha_2 E_2 \end{pmatrix} = \begin{pmatrix} -2.3106 & 0.4923 \\ 0.8488 & -1.2306 \end{pmatrix}.$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0.3547 \\ 0.6453 \end{pmatrix}.$$

Tea vs Coffee (ABS statistics)



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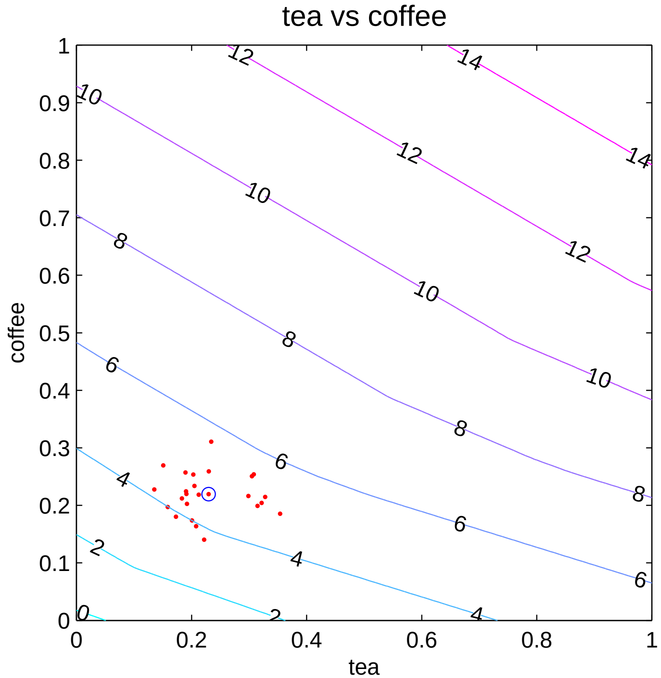

The Afriat Utility has fit parallel lines to the data which are slightly skewed to the right. Tea and coffee are still considered to be perfect substitutes with coffee favoured slightly more than tea as one would give up less coffee to gain more tea.