Biorthogonal System in Approximation Theory

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Orthogonal System

Let $\mathbb{M} \subset \mathbb{N}$ be an index set, $\{p_n\}_{n\in\mathbb{M}}$ be a subset of an inner product space H equipped with the inner product $<,\cdot,>$. This subset is called an orthogonal system if

$$< p_n, p_m >= c_n \delta_{mn},$$

where c_n is a non-zero constant and δ_{mn} is a Kronecker symbol

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{else.} \end{cases}$$

Examples: trigonometric functions, orthogonal wavelets and polynomials, etc.

Biorthogonal System

Let $\{p_n\}_{n\in\mathbb{M}}$ and $\{q_n\}_{n\in\mathbb{M}}$ be two subsets of an inner product space H, where H is equipped with the inner product $<,\cdot,>$. These two subsets are said to form a biorthogonal system if

$$< p_n, q_m >= c_n\delta_{mn},$$

where c_n is a non-zero constant and δ_{mn} is a Kronecker symbol.
Examples: biorthogonal polynomials, biorthogonal wavelets, etc.

Biorthogonal System

Let $\{p_n\}_{n\in\mathbb{M}}$ and $\{q_n\}_{n\in\mathbb{M}}$ be two subsets of an inner product space H, which is equipped with the inner product $<,\cdot,>$. Let $\{p_n\}_{n\in\mathbb{M}}$ and $\{q_n\}_{n\in\mathbb{M}}$ form a biorthogonal system. Then if

$$f = \sum_{n\in\mathbb{M}} a_n p_n,$$

$$a_n = \frac{1}{c_n} < f, q_n > .$$

Solving a linear system can be reduced to finding a biorthogonal system [Brezinski, 93].

Finite Element Method

The finite element method is the most popular method for solving partial differential equations. Finite elements are special kinds of splines.

- Consider a variational problem: find $u \in V$ such that

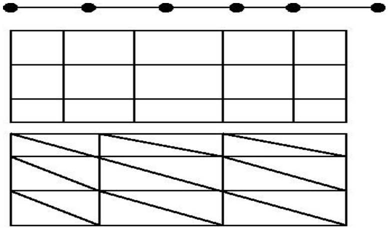
$$a(u, v) = f(v) \quad \text{for all} \quad v \in V,$$

  where V is a subspace of a Hilbert space, and $a(\cdot, \cdot)$ is a bilinear form and f is a linear form.
- The finite element method for this problem is obtained by replacing the infinite dimensional space V by a finite dimensional one.
- The finite dimensional space V_h is constructed by using a triangulation of the given domain, where we want to solve our problem.

Finite Element Method

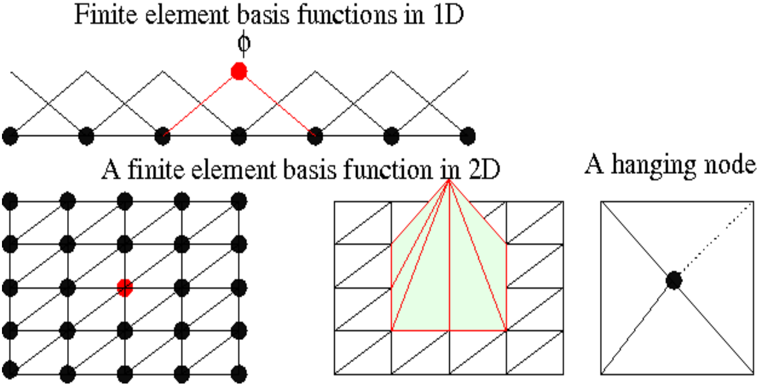
Let $\Omega \subset \mathbb{R}^d$ be a domain (closed and bounded region). Let $\mathcal{T}_h$ be a partition of Ω into smaller subdomains (intervals, triangles, quadrilaterals, tetrahedra, hexahedra, etc.).
The finite element method is characterized by defining a set of basis functions on $\mathcal{T}_h$:

- Each basis function is associated with a point in the domain.
- The size of support of each basis function is of order of the size of a typical subdomain.
- Thus the finite support size is a distinguishing feature of the finite element approach.



Finite Element Method

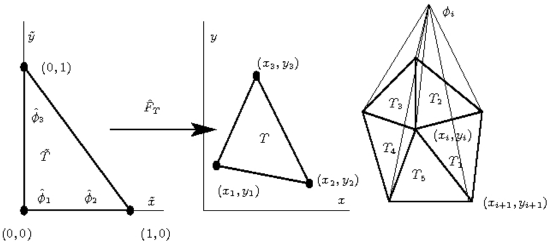
Let $\{\phi_1, \cdots, \phi_n\}$ be the set of finite element basis functions on the mesh $\mathcal{T}_h$ and $\mathcal{G}$ be the set of points in Ω where these basis functions are associated. A finite element basis function is called nodal if its value is one at its associated point and zero at other points in $\mathcal{G}$.



Finite Element Space

The global finite element space is formed by the following process:

- A set of local basis functions are defined on a reference element
- A mapping is computed which maps the reference element to the subdomain
- The basis functions on the reference element are mapped by this mapping to compute the basis functions on the subdomain
- Then global basis functions are computed by glueing these mapped basis function together

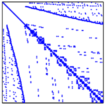
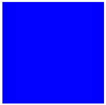


Weak Constraint and its Algebraic From

In many problems, we have to project a quantity of interest onto a continuous finite element space. Examples are gradient reconstruction, mortar finite elements, mixed formulation of biharmonic, Darcy and elasticity equations. The projection of σ_h onto S_h can be expressed as the weak constraint:

$$\int_\Omega u_h \mu_h \, dx = \int_\Omega \sigma_h \mu_h \, dx, \quad u_h \in S_h, \; \mu_h \in M_h$$

Algebraic constraint (abusing the notation): $u_h = M^{-1}\sigma_h$, M is a Gram matrix
Orthogonal projection is obtained by sing the same discrete space for u_h and μ_h

$u_h = \Big($  $\Big)^{-1} \sigma_h =$ $\Big($  $\Big) \sigma_h$

Weak Constraint and its Algebraic From

- The space for u_h is H^1-conforming, but it suffices to have L^2-conforming space for μ_h.
- If S_h contains the piecewise polynomial space of degree p, it is enough that M_h spans the piecewise polynomial space of degree $p-1$.
- We want to utilize these two properties to construct a space M_h so that basis functions for S_h and M_h form a biorthogonal system.
- We get an oblique projection.

Biorthogonality in Finite Elements

S_h is a finite element space, and we call M_h the biorthogonal (or dual) space
Biorthogonal space $M_h \iff M$ is diagonal
If M is diagonal:

- The projection is easy
- Static condensation $\Longrightarrow$ positive definite system
- Modification of nodal basis and nested spaces $\Longrightarrow$ $\mathcal{V}$- or $\mathcal{W}$-cycle multigrid
- Nonlinear contact problems (variational inequality) $\Longrightarrow$ Non-penetration can be realized pointwise

Some Notations

- V_h^p: H^1-conforming finite element space of degree p on a line
- $\mathbf{\Phi}_p := \{\varphi_1^p, \ldots \varphi_{p+1}^p\}$: Set of local finite element basis functions of degree p on the reference edge $I = [-1, 1]$ using lexicographical ordering

- M_h^p: Dual space spanned by biorthogonal basis functions of degree p
- $\mathbf{\Psi}_p := \{\psi_1^p, \ldots \psi_{p+1}^p\}$: Set of local biorthogonal basis functions of degree p

$$\int_I \psi_i^p(s)\varphi_j^p(s)\, ds = \delta_{ij} \int_I \varphi_j^p(s)\, ds$$

Special interest for mortar, Darcy, biharmonic and elasticity mixed finite elements:

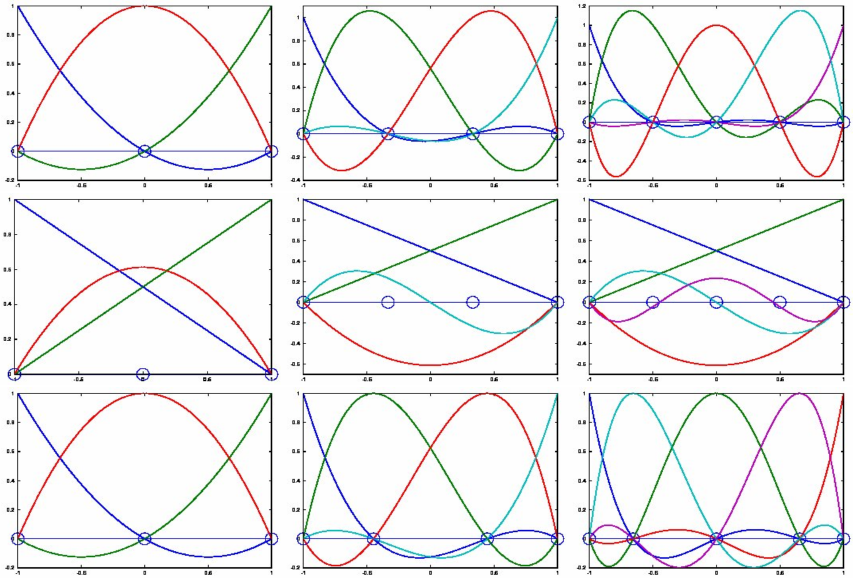
$$V_h^{p-1} \subset M_h^p$$

Biorthogonality in Finite Elements

- First approach: Lagrange nodal FE. Optimal a priori estimates only for $p = 1$ and $p = 2$.
- Second approach: Lagrange hierarchical FE. No nodal property. Existence of optimal biorthogonal base. BUT [Oswald et al. 01] larger support (≥ 3 edges).
- Third approach: Gauss–Lobatto nodal FE. Optimal biorthogonal spaces for a finite element space of any order with equal support.

Next slide: examples of these three types of basis functions $\{\phi_1^p, \cdots, \phi_m^p\}$ for $p = 2, 3, 4$. Here $m = p + 1$.

Finite Element Basis Functions on the Reference Edge

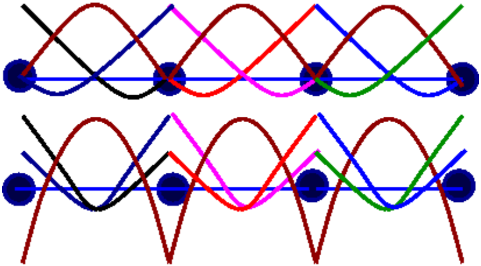


Finite Element Basis Functions on the Reference Edge

There are two types of basis functions in one dimension.

- Two basis functions associated with the vertices
- $p-1$ inner basis functions

The glueing condition does not affect the inner basis functions. It only affects the two vertex basis functions.



Algebraic Condition

$\mathbf{\Psi}_p$ and $\mathbf{\Phi}_p$ span the space of polynomials of degree p, say $\mathcal{P}_p(I)$.
Let us regard $\mathbf{\Psi}_p$ and $\mathbf{\Phi}_p$ as column vectors with an abuse of notation.

$$\mathbf{\Phi}_p = [\phi_1^p, \cdots, \phi_{p+1}^p]^T, \quad \mathbf{\Psi}_p = [\psi_1^p, \cdots, \psi_{p+1}^p]^T.$$

Since $\mathbf{\Psi}_p = \{\psi_1^p, \cdots, \psi_{p+1}^p\}$ also spans a polynomial space of degree p, there exists a matrix N^p with

$$N^p \in \mathbb{R}^{p \times p+1}$$

such that

$$\mathbf{\Phi}_{p-1} = N^p \mathbf{\Psi}_p.$$

Local space $\mathbf{\Psi}_p$ contains the polynomial space of degree p, but the global space may not contain even a piecewise polynomial space of degree $p-1$.

Algebraic Condition

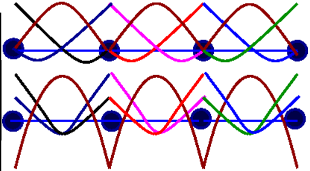
Lemma

$V_h^{p-1} \subset M_h^p$ *if and only if*

$$\begin{aligned} n^p_{1,1} &= n^p_{p,p+1} \quad \textit{and} \quad n^p_{p,1} = n^p_{1,p+1} = 0, \\ n^p_{i,1} &= n^p_{i,p+1} = 0 \quad \textit{for all} \quad 2 \le i \le p-1, \end{aligned}$$

where $n^p_{i,j}$ *is the* (i,j)*-th entry of the matrix* N^p*.*

$$N^p = \begin{bmatrix} * & ** & \cdots & 0 \\ 0 & ** & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & ** & \cdots & 0 \\ 0 & ** & \cdots & * \end{bmatrix}$$



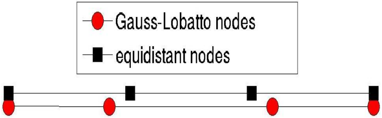
Analytic Condition

- If the nodal points $x_1^p, \dots, x_{p+1}^p$ are symmetric, these conditions reduce to

$$\varphi_1^p \in \text{span}\{\varphi_2^{p-1}, \dots, \varphi_p^{p-1}\}^\perp \text{ and } \varphi_{p+1}^p \in \text{span}\{\varphi_1^{p-1}, \dots, \varphi_{p-1}^{p-1}\}^\perp.$$

- If we define $\varphi_1^p = c_1(1-x)L_p'(x)$, and $\varphi_{p+1}^p = c_2(1+x)L_p'(x)$, then the above conditions are satisfied (L_p is the Legendre polynomial of degree p).
- If $\mathcal{S}_p := \{-1 =: x_1^p < x_2^p < \cdots < x_{n+1}^p =: 1\}$ be the zeros of polynomial $(1-x^2)L_p'(x)$, then $\mathcal{S}_p$ is the set of Gauss–Lobatto nodes of order p.

$p = 3$



Example: $p = 3$

$$N^3_{\text{Gauss–Lobatto}} = \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{10} & \frac{1-\sqrt{5}}{10} & 0 \\ 0 & \frac{4}{5} & \frac{4}{5} & 0 \\ 0 & \frac{1-\sqrt{5}}{10} & \frac{1+\sqrt{5}}{10} & 1 \end{bmatrix}, \quad N^3_{\text{Lagrange}} = \begin{bmatrix} \frac{11}{15} & \frac{2}{5} & -\frac{1}{5} & 0 \\ \frac{4}{15} & \frac{4}{5} & \frac{4}{5} & \frac{4}{15} \\ 0 & -\frac{1}{5} & \frac{2}{5} & \frac{11}{15} \end{bmatrix}.$$

$\Longrightarrow$ biorthogonal basis (equidistant nodes): $V_h^2 \not\subset M_h^3$
$\Longrightarrow$ biorthogonal basis (Gauss-Lobatto nodes): $V_h^2 \subset M_h^3$

Analytic Condition

Gauss–Lobatto nodes $\Longrightarrow$ there exists a Quadrature formula exact for all polynomials of degree $\leq 2p-1$

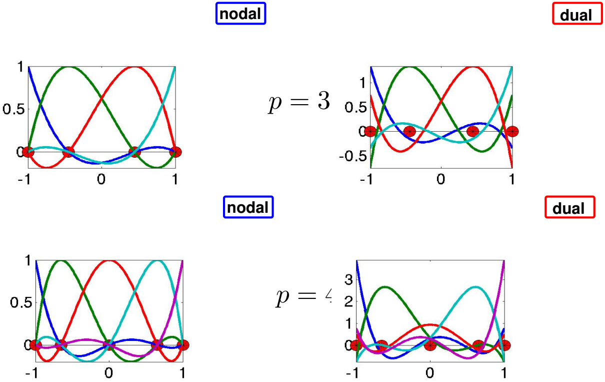
$$\int_I \varphi_l^p(\hat{s})\varphi_i^{p-1}(\hat{s})\,d\hat{s} = \sum_{j=1}^{p+1} w_j^p \varphi_l^p(x_j^p)\varphi_i^{p-1}(x_j^p) = 0, \begin{cases} l = 1, 2 \leq i \leq p \\ l = p+1, 1 \leq i \leq p-1 \end{cases}$$

Theorem

$V_h^{p-1} \subset M_h^p$ if and only if the finite element basis of V_h^p which defines M_h^p is based on the Gauss–Lobatto points.

$\Longrightarrow$ Optimal a priori estimates for mortar finite elements, biharmonic, Darcy and elasticity equations.

Biorthogonal basis functions for cubic and quartic finite element spaces



Extension to Higher Dimension

- If a finite element space has a tensor product structure, the biorthogonal basis functions can be constructed by using the tensor product construction. This includes meshes of d-parallelotopes.
- In simplicial meshes, the lowest order case is straightforward. The biorthogonal basis with such optimal approximation property does not exist for the quadratic case. Relax the notion and use quasi-biorthogonality.
- The situation for serendipity elements is similar.

Numerical Results for Biharmonic Equation

We want to find $u \in H_0^2(\Omega)$ such that $\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f \, v \, dx, \quad v \in H_0^2(\Omega)$ in $\Omega := (0,1)^2$. Here we put $\phi = \Delta u$, and get the weak form using the clamped boundary condition

$$\int_\Omega \phi \psi \, dx = \int_\Omega \Delta u \psi \, dx = -\int_\Omega \nabla u \cdot \nabla \psi \, dx.$$

Table: Discretization errors in different norms for the clamped boundary condition

level	# elem.	$\|u-u_h\|_{0,\Omega}$		$\|u-u_h\|_{1,\Omega}$		$\|\Delta u-\phi_h\|_{0,\Omega}$	
0	32	5.34290e-01		6.32693e-01		6.32041e-01	
1	128	3.26972e-01	0.71	4.01635e-01	0.66	5.16879e-01	0.29
2	512	1.30302e-01	1.33	1.89139e-01	1.09	3.34937e-01	0.63
3	2048	3.99107e-02	1.71	8.32646e-02	1.18	1.88319e-01	0.83
4	8192	1.08809e-02	1.87	3.88438e-02	1.10	9.92016e-02	0.93
5	32768	2.82773e-03	1.94	1.89646e-02	1.03	5.08074e-02	0.97
6	131072	7.19891e-04	1.97	9.41839e-03	1.01	2.56967e-02	0.98
7	524288	1.81559e-04	1.99	4.70081e-03	1.00	1.29204e-02	0.99

Conclusion and Future Work

Conclusion:

- The importance of biorthogonality is highlighted
- The biorthogonal system using nodal finite element space of degree p is constructed
- The approximation property of the biorthogonal system is analyzed

Future work:

- Extend the idea to other splines: e.g., splines with higher smoothness
- Quasi-biorthogonality may be a key where biorthogonality is not possible

Thank you