Banach Space Geometry and Fixed Point Theory

Brailey Sims

School of Mathematical and Physical Sciences

The University of Newcastle

For John R. Giles

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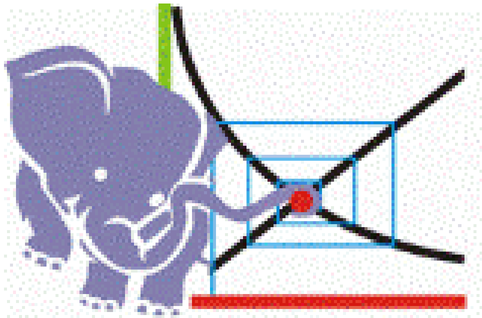


We examine the rich and symbiotic interaction between the geometry of Banach spaces and developments in metric fixed point theory.

Terminology and notations

We say a (nonlinear) mapping $T : A \subseteq X \to B \subseteq Y$ between subsets of normed spaces X and Y is *nonexpansive* if,

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in A$

For example, isometries – such as the right shift operator, strict contractions and resolvents of accretive mappings are all nonexpansive.

Besides representing a border case between the Banach contraction mapping principle and Brouwer's / Schauder's theorem, the fixed point theory of nonexpansive maps is significant because of its close connections with the theory of accretive/monotone operators, variational inequalities and hence optimization and nonlinear analysis in general.

Let **τ** be a linear topolgy on a Banach space X and let C be a nonempty norm closed and bounded convex subset.

Typically **τ** is the weak (**ω**) topolgy on X, or when X is the dual of a given Banach space, $X = Y^*$, the weak* [**ω* = σ**(X,Y)] topology, or in the case of many function spaces the topology of local convergence in measure.

We say C has the:

- *fixed point property* *(for nonexpansive maps)*, ***fpp*** for short, if every nonexpansive self mapping of C has a fixed point.

- *hereditary fpp* if every nonempty closed convex subset of C has the fpp.

We say the space X has the:

- *fpp* if every nonempty norm closed and bounded convex subset C has the fpp.

- *τ - fpp* if every nonempty **τ** – relatively compact, norm closed and bounded, convex subset C has the fpp.

The current state of metric fixed point theory

Building from the initial independent results of Browder and Göhde (uniformly convex spaces have the fpp) and Kirk (reflexive spaces with normal structure have the fpp) in the mid 1960's we now have a rich, though still far from complete, theory of nonexpansive (and related types of mappings) with closed convex domains in Banach spaces.

In briefly surveying this I will only focus on what I consider to be core aspects of the theory.

The *classical theory* (mid sixties to the early eighties) produced a plethora of geometric/topological properties of Banach spaces which were sufficient to ensure the space had **ω**-normal structure and hence the **ω**-fpp

Recall a space has normal structure (τ-normal structure) if it contains no non trivial (diameter > 0) closed bounded (τ-compact) convex *diametral* subsets.

A subset D is diametral if,

$$\inf_{x\in D}\sup_{y\in D}\|x - y\| < \operatorname{diam}(D) := \sup_{x.y\in D}\|x - y\|$$

The positive \`infant' of the unit ball of c_0 has this property.

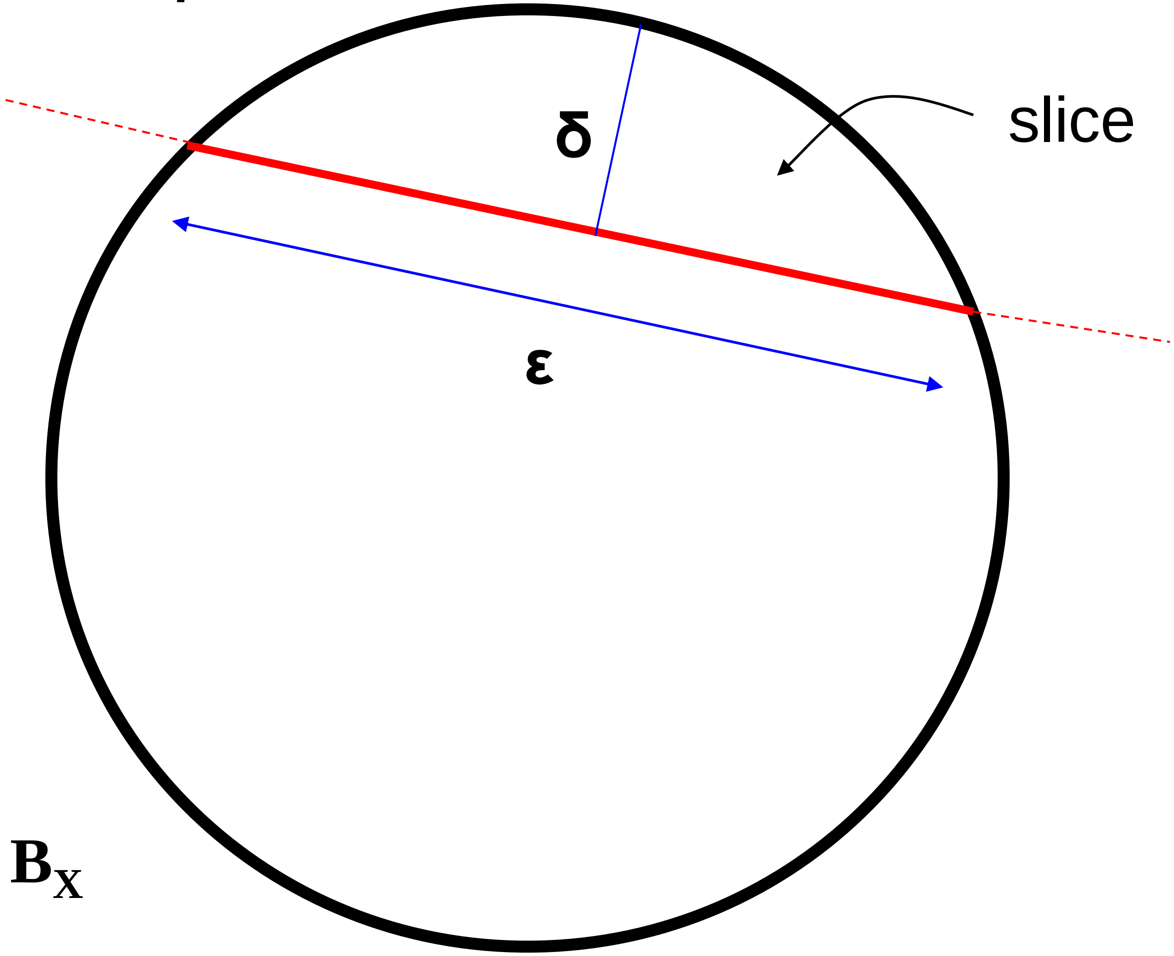
Some of the properties identified as sufficient for **ω**-normal structure are:

- Uniform convexity (UC)
- Opial's property [Gossez-Lami Dozo, 69],
- ε_0 – UCED, for $\varepsilon_0 < 1$ [Day-James - Swaminathan, 71],
- Uniform Smoothness [Turett, 81],
- ε_0 - UKK , for $\varepsilon_0 < 1$ [van Dulst – S, 82],
- $\varepsilon_0(X) < 1$, so Uniform Convexity [Edelstein 68, Prus and others late 80's],
- GGLD or Asymptotic P [Jimenez-Melado, 92],
- Property P [Tan - Xu, 91, Smyth – S, 95].

The weakest of these being Property P: *For every non-constant weak null sequence, (x_n), we have,*

$$\liminf_n \|x_n\| < \mathrm{diam}(x_n)$$

- **UC, UCED, Inquadrate**



UKK

Opial's property: For any weakly null sequence (x_n) and $x \neq 0$ we have,

$$\liminf_n \|x_n\| < \liminf_n \|x + x_n\|$$

A type of orthogonality (in the sense of James)

Starting with the results of Maurey (in particular that c_0 has the w-fpp), since the early eighties the focus shifted to the study of the **ω**- fpp in space where **ω**- normal structure may not be present.

None-the-less, Alspach's 1980 seminal example of a fixed point free isometry on a **ω-** compact convex subset of $L_1[0,1]$; the baker transform on the order interval $[0 \leq f(x) \leq 1]$, remains effectively the only known instance of a failure of the **ω**-fpp.

On the positive side, however, the last two decades have seen the development of widely applicable, easily verifiable criteria for a space to have the **ω**- fpp. Criteria which allow us to deal with most spaces of interest. ☺

Numbered among my favourite sufficient conditions for the **ω**- fpp are:

- X is a weakly orthogonal Banach lattice [Borwein – S, 82 & S, 85]

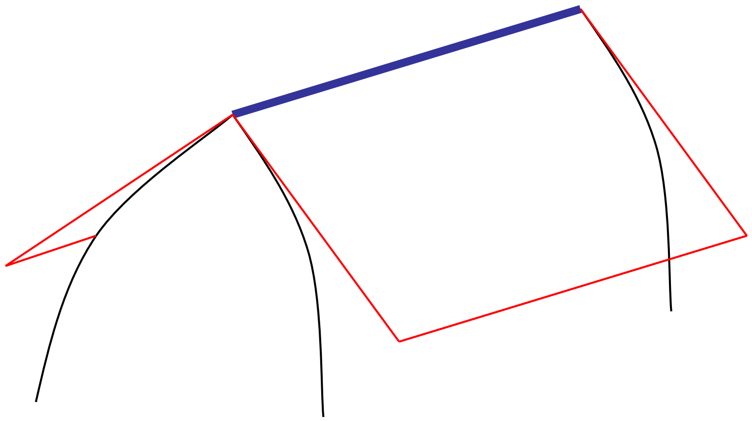
- X has property M [Garcia Falset-S, 96]; weak null types are constant on spheres, and its subsequent generalizations.

- R(X) < 2 [Garcia Falset, 98], here R(X) is the 'smallest' number such that
  $$\limsup \|x - x_n\| \leq R(X)\,(\|x\| \vee \limsup \|x_n\|)$$
  for all x and all **ω**-null sequences (x_n).
  $1 \leq R(X) \leq 2$, $R(c_0) = 1$, $R(L_1[0,1]) = 2$.

- X is ε_0 – uniformly non-creased [Prus, 95]

- X is uniformly non-square (inquadrate); that is, $\varepsilon_0(X) < 2$ [Mazcuñán Navarro 2003]
- X is reflexive with WORTH [Fetter – Gamboa de Buen, 2009]

Proofs for the majority of these results rely on arguments carried out in an appropriate ultra-power of the space.



Uniformly Creased

UNC:

- Super-property!
- Self dual!
- Implies superreflexivity!
- But does not entail normal structure!

Compactly UNCreased ?

WORTH: For any weakly null sequence (x_n) and $x \neq 0$ we have,

$$\limsup_n \left| \, \|x + x_n\| - \|x - x_n\| \, \right| = 0$$

A type of orthogonality (in the sense of Birkhoff)

Spaces with the fpp

Until recently, outside of reflexive spaces there were no known examples of a space with the fpp. This combined with other results:

- A closed subspace of $L_1[0,1]$ has the fpp iff it is reflexive (= superreflexive in this instance) [Maurey, 1981],
- If an ultra-power of the space has fpp (i.e. the spaces has super – fpp) then the space is superreflexive [van Dulst and Pach, 1980].
- A space, X, fails to have fpp if it contains an *asymptotically isometric copy of* ℓ_1; that is, there exists a real null sequence (ε_n) and a norm one sequence (x_n) in X with

$$\Sigma_{n=1}^{\infty}(1-\epsilon_n)|t_n| \leq \|\Sigma_{n=1}^{\infty} t_n x_n\|$$

for all (t_n) in ℓ_1 .

And, many spaces have been shown to contain such a copy of ℓ_1 [Lennard *et al, mid to late 1990's*] ,

- An analogous situation for spaces asymptotically isometrically containing a copy of c_0 [Lennard *et al*]
- A closed, bounded, convex subset of c_0 has the fpp iff it is ω-compact [Llorens Fuster - S, Lennard -Dowling],

Prompted the widely held

conjecture: fpp implies reflexivity.

However, in 2006 Lin established that the (non-reflexive) space ℓ_1 under the equivalent renorming, $|||x||| := \sup_n \gamma_n \|Q_n(x)\|_1$

has the fpp, where $0 \leq \gamma_n \uparrow 1$, and Q_n is the tail projection;

$$Q_n(x) = (0,\ldots,0, x_n, x_{n+1},\ldots).$$

This norm had previously been introduced [Dowling, Lennard, Turett and Johnson] as an example of a non-reflexive space not containing an asymptotically isometric copy of ℓ_1 .

Whether c_0 with the analogous pre-dual renorming has the fpp remains open.

•Lin's result has been abstracted by Japón Pineda and Fetter-Castillo Santos-S.

Suppose τ is a topology on X satisfying

- norm bounded sets are τ- sequentially precompact,
- the τ- closure of a norm bounded set is norm bounded,
- if (x_n) is a τ- null sequence then

$$\limsup_n \|x + x_n\| = \|x\| + \limsup_n \|x_n\|$$

and let $0 \leq \gamma_n \uparrow 1$ and (P_n) be a family of seminorms on X with:

- $P_1(x) = \gamma_1 \|x\|$ and $P_n(x) \leq \gamma_n \|x\|$, for $n = 2,3,\ldots$
- $\lim_n P_n(x) = 0$, for all x,
- if (x_n) is a τ- null sequence then for all k,

$$\limsup_n P_k(x + x_n) = P_k(x) + \limsup_n P_k(x_n)$$

then $|||x||| := \sup_n \ P_n(x)$ defines an equivalent norm on X for which every norm closed and bounded, convex subset of X enjoys the fpp.

Conditions for a norm to have the fpp have been identified by Castillo Santos-S.

Property A: If for every $\boldsymbol{\varepsilon}$ in (0,1) there exists an n such that for all x,y in B_X, n << x < y, then:

$$|x| + |y| - \boldsymbol{\varepsilon} \leq |x + y|.$$

Property Alpha: There exists an $\boldsymbol{\alpha}$> 4, and a decreasing sequence ($\boldsymbol{\alpha}_n$) of positive numbers such that, for every natural number n, and for every norm one $\boldsymbol{\tau}$- null sequence (x_n) we have,

x << n and $||x|| < \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}_n$, implies

$$\limsup || x_n + x || \leq 1 + ||x|| / \boldsymbol{\alpha}$$

Then ♫; If X is a Banach space with a Schauder basis. which has property A and property Alpha, then X has the fixed point property.

THANK YOU

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DISCUSSION
or
QUESTIONS

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