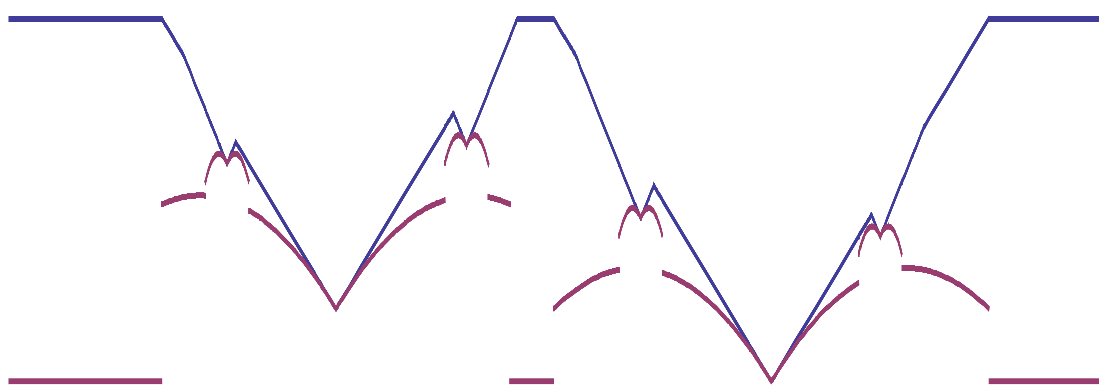
An explicit non-expansive function whose subdifferential is the entire dual ball

or

"DIMPLES"

Speaker: *Scott Sciffer*

Co-author: *Jonathan Borwein*



Introduction

A function $f : X \to \mathbf{R}$ is *non-expansive* (or *1-Lipschitz*) if

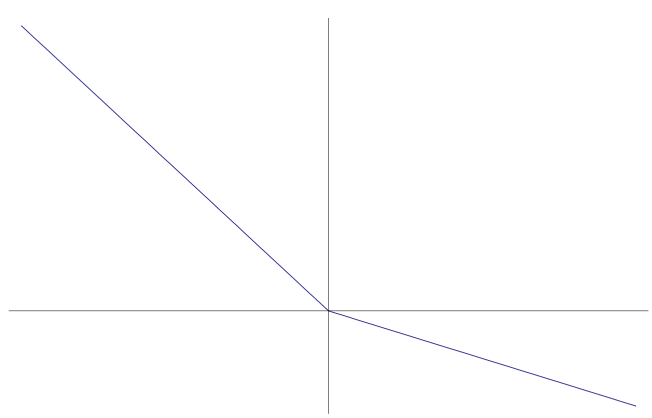
$$|f(x) - f(y)| \le \|x - y\| \quad \forall x, y \in X.$$

(X will be an arbitrary real Banach space throughout, however we may imagine $X = \mathbf{R}$ without compromising anything.)

The *Clarke derivative* of f at x in direction v is

$$f^\circ(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

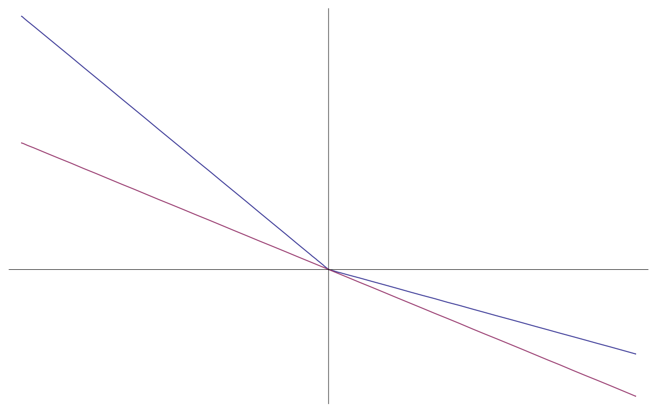
For each $x \in X$, $f(x; v)$ is *sublinear* in v.



The *Clarke subdifferential* $\partial_c f$ is

$\partial_c f(x) := \{x^* \in X^* |\ \langle x^*, v\rangle \leq f^\circ(x; v) \text{ for all } v \in X\}$,

and each $x^* \in \partial_c f(x)$ is a *subgradient* of f at x.



Example: If $f(x) = |x|$, then

$$\partial_c f(x) = \begin{cases} \{-1\} & x < 0 \\ [-1, 1] & x = 0 \\ \{1\} & x > 0 \end{cases}$$

Here the Clarke subdifferential is identically the dual ball at $x = 0$.

Borwein, Moors and Wang (2000, 2001) —
'almost all' non-expansive functions on 'lots of' Banach spaces have Clarke subdifferential identically equal to the dual unit ball.

Compare and contrast this result with —
'almost all' continuous functions are nowhere differentiable.

For non-expansive functions we know they are densely differentiable (at least on many Banach spaces; Preiss' Theorem), but the result of BMW tells us the derivatives are almost always as badly discontinuous as they could possibly be!

Explicit constructions of such functions unknown even in 2-dimensions.

Our aim today is to provide an explicit recipe for constructing such a function on an arbitrary real Banach space.

A measure theoretic construction on $\mathbf{R}$

A set $E \subset \mathbf{R}$ is called *ubiquitous* if both E and E^C have positive measure in every interval. It is easy to construct such a set.

Define

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ \\ -1 & \text{if } x \in E^C \end{cases}$$

Define

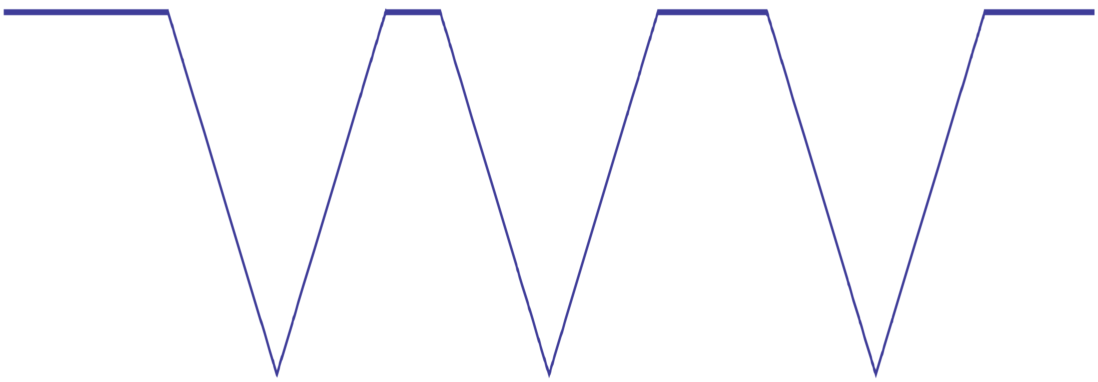
$$f(x) = \int_0^x \chi_E(t)dt.$$

Then f is non-expansive, and differentiable almost everywhere with derivative χ_E. Hence there is a dense set of points where $f' = 1$ and a dense set of points where $f' = -1$. It follows that $\partial_c f$ is identically the dual unit ball.

Unfortunately there is no obvious way to extend this simple argument to higher dimensions.

“DIMPLES” — a first attempt

Imagine building a function by putting little dints — or *dimples* — into a flat surface.

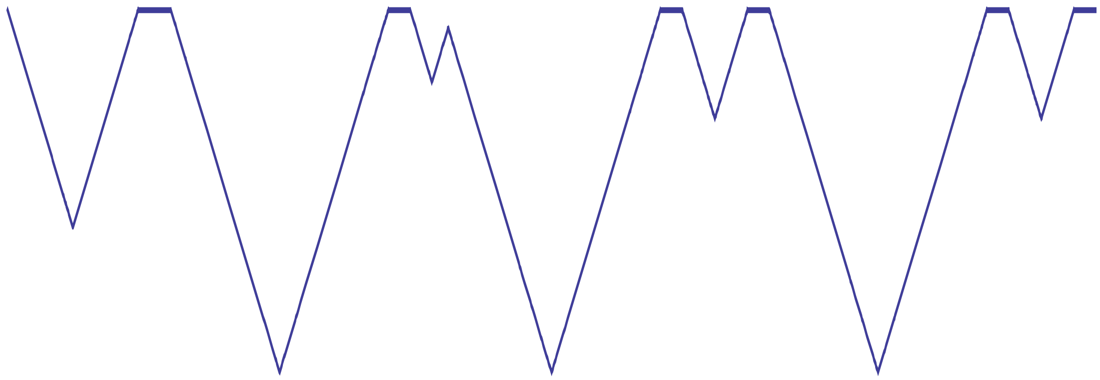


If each ‘dimple’ is made using the norm, then

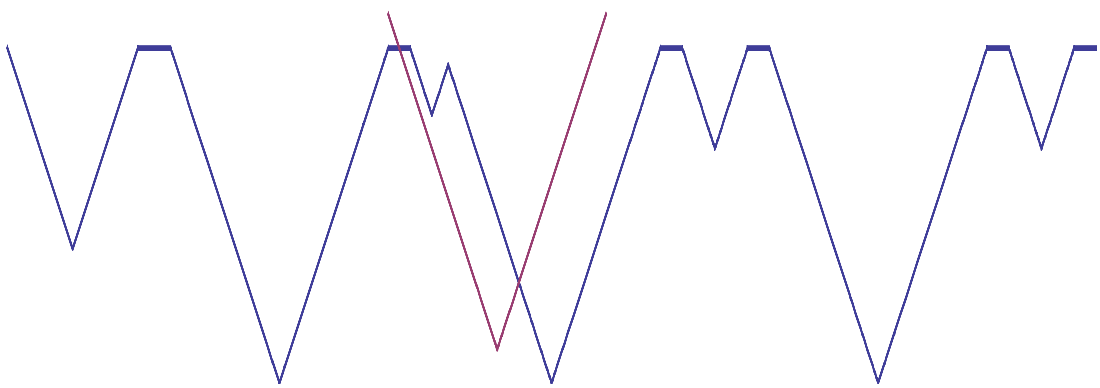
$$\partial_c f = B(X^*)$$

at each local minimum.

Now just add more dimples!



If we just keep doing that, until the dimples become dense, then we are done, aren't we?



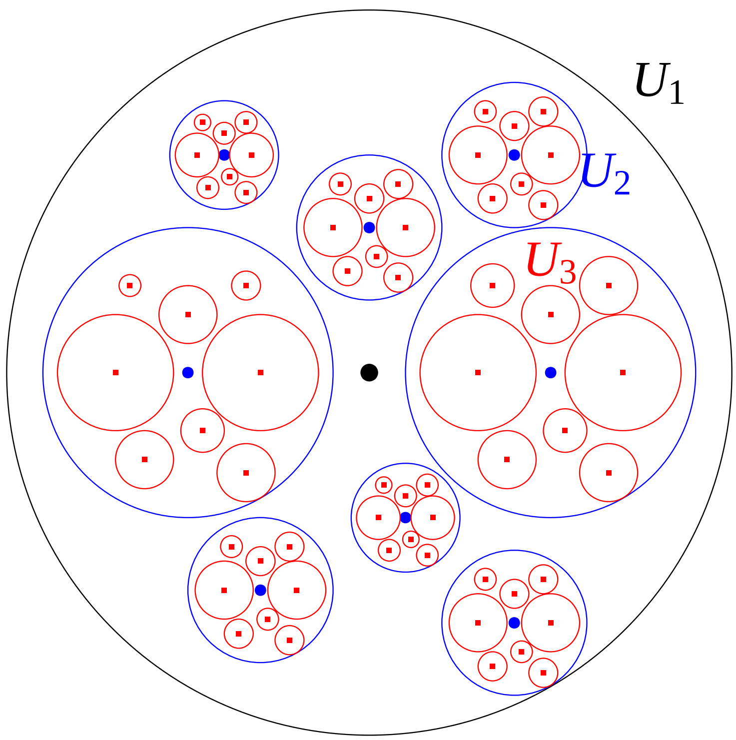
A problem! Since our dimples will have to start overlapping, later dimples can eliminate earlier ones.

“DIMPLES” — try again!

Choose (Zorn’s Lemma) sets U_n such that

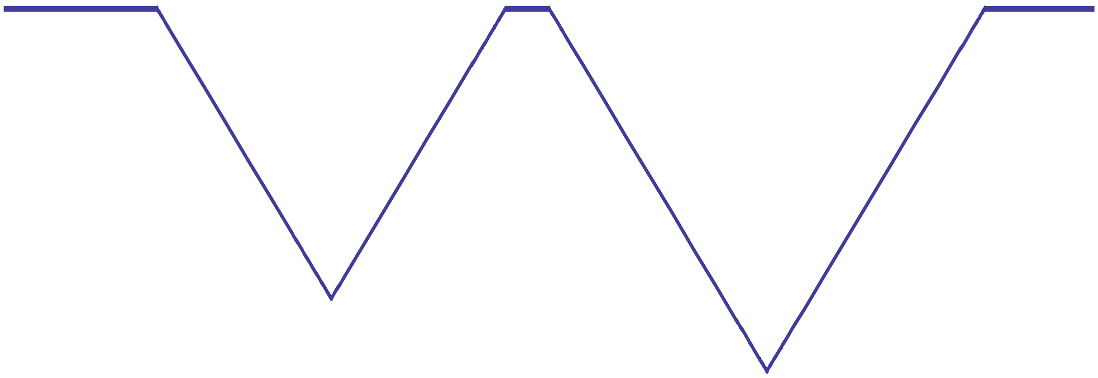
(i) U_n is a dense union of open balls, radius $< \frac{1}{n}$
(ii) U_n is a nested family
(iii) if x is a centre of a ball of U_n then $x \notin U_{n+1}$

This may be easiest to visualise in 2-dimensions.

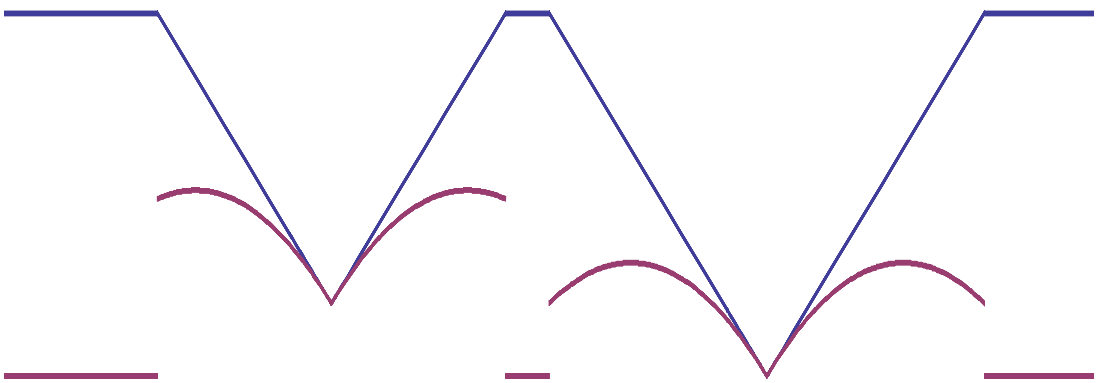


Each ball is a “region of influence” of a dimple.

Use U_1 to add dimples *of slope* $\frac{1}{2}$ to create f_1.

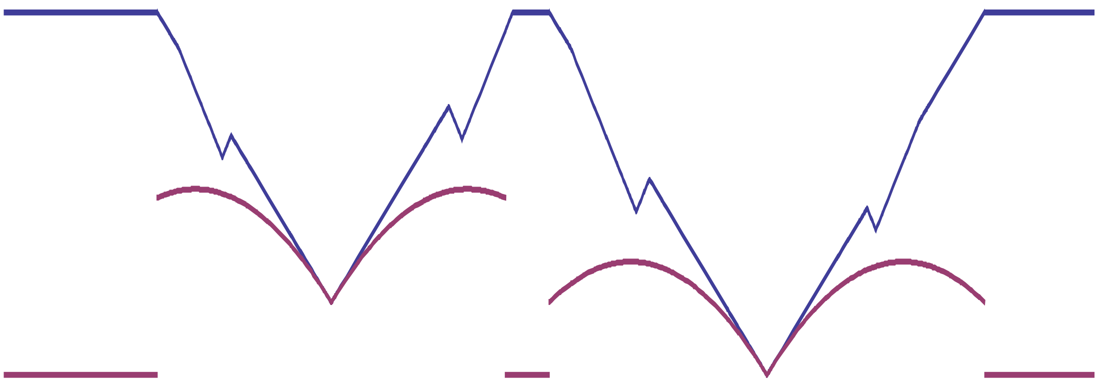


Add lower bounding function, g_1.



Notice, on each "region of influence", $g_1 \leq f_1$ with equality only at the ball centres. At these points the directional derivatives of f_1 and g_1 agree, and equal $\frac{1}{2}$ in all directions.

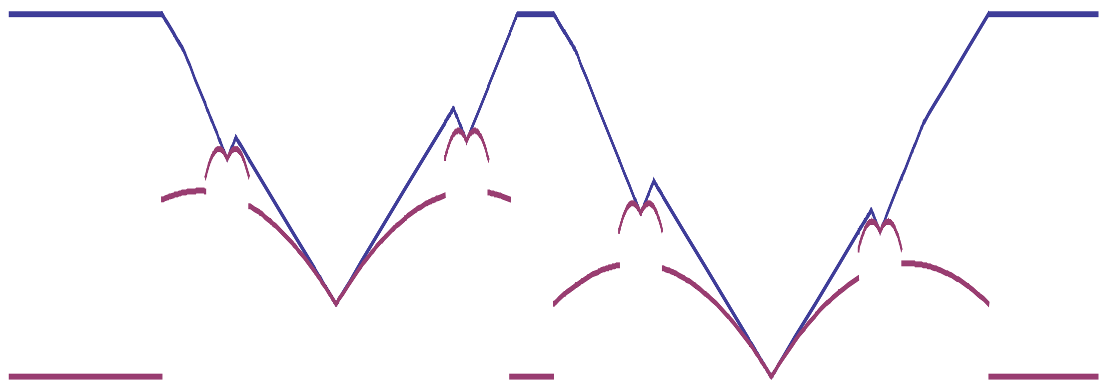
Use U_2 to add dimples *of slope* $\frac{3}{4}$ to create f_2.



Notice that since U_n is a nested family, we are adding (smaller and steeper) dimples to our existing dimples.

We can ensure they do not penetrate the lower bound g_1, and we can ensure they do not extend outside their preset “region of influence”.

And adjust the lower bound to create g_2.



Notice, on each "region of influence" from U_2, $g_2 \leq f_2$ with equality only at the ball centres. At these points the directional derivatives of f_2 and g_2 agree, and equal $\frac{3}{4}$ in all directions. This has been achieved without altering the directional derivatives at the ball centres from U_1.

Now use the most powerful word in all mathematics....

etc.

The functions f_n converge to a function f which is non-expansive, so

$$\partial_c f(x) \subseteq B[X^*] \quad \forall x \in X.$$

So if x_n is a ball centre of U_n then

$$(1 - \frac{1}{2^n})B[X^*] \subseteq \partial_c f(x_n) \subseteq B[X^*].$$

However, given any $x \in X$ we can choose $x_n \to x$ where x_n is a ball centre from U_n. It follows that

$$\partial_c f(x) = B[X^*] \quad \forall x \in X.$$

Closing Remarks

The existence, indeed abundance, of such non-expansive functions has long been known (BMW); but required geometric conditions on the Banach space, or rotundity properties of the norm, and no explicit construction was available even in 2-dimensions.

Here we have produced an explicit construction (once the initial family of open sets U_n is chosen) requiring nothing but elementary real analysis. It therefore lifts effortlessly from the 1-dimensional case to an arbitrary Banach space without the need for any geometric conditions on the space.

In fact we have done more. The Clarke subdifferential is the 'bulkiest' of all the subdifferentials used to study the differentiability of Lipschitz functions. The 'leanest' such subdifferential is called the *approximate subdifferential* (definition omitted), and the construction also serves to make this identically equal to the dual unit ball.

Finally, it should be clear that by altering the shape of the dimples, we can make a function whose subdifferential is identically anything we like (within reason).

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