On the topology of pointwise convergence on the boundaries of L_1-preduals

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Dedicated to *John R. Giles*

Introduction

We shall say that a Banach space $(X, \|\cdot\|)$ is an L_1-predual if X^* is isometric to $L_1(\mu)$ for some suitable measure μ. Some examples of L_1-preduals include $(C(K), \|\cdot\|_\infty)$, and more generally, the space of continuous affine functions on a Choquet simplex endowed with the supremum norm. The other notion we shall consider in this talk is that of a boundary. Specifically, for a non-trivial Banach space X over $\mathbb{R}$ we say that a subset B of B_{X^*}, the closed unit ball of X^*, is a boundary, if for each $x \in X$ there exists a $b^* \in B$ such that $b^*(x) = \|x\|$. The prototypical example of a boundary is Ext(B_{X^*}) - the set of all extreme points of B_{X^*}, but there are many other interesting examples. In a recent paper by Moors and Reznichenko the authors investigated the topology on a Banach space X that is generated by $\mathrm{Ext}(B_{X^*})$ and, more generally, the topology on X generated by an arbitrary boundary of X. In this talk we continue this study.

To be more precise we must first introduce some notation. For a nonempty subset Y of the dual of a Banach space X we shall denote by $\sigma(X, Y)$ the weakest linear topology on X that makes all the functionals from Y continuous. In the paper by MR they showed that "for any compact Hausdorff space K, any countable subset $\{x_n : n \in \mathbb{N}\}$ of $C(K)$ and any boundary B of $(C(K), \|\cdot\|_\infty)$, the closure of $\{x_n : n \in \mathbb{N}\}$ with respect to the $\sigma(C(K), B)$ topology is separable with respect to the topology generated by the norm".

In this talk we extend this result by showing that if $(X, \|\cdot\|)$ is an L_1-predual, B is any boundary of X and $\{x_n : n \in \mathbb{N}\}$ is any subset of X then the closure of $\{x_n : n \in \mathbb{N}\}$ in the $\sigma(X, B)$ topology is separable with respect to the topology generated by the norm whenever $\mathrm{Ext}(B_{X^*})$ is weak* Lindelöf.

Preliminary Results

Let X be a topological space and let $\mathcal{F}$ be a family of nonempty, closed and separable subsets of X. Then $\mathcal{F}$ is rich if the following two conditions are fulfilled:

(i) for every separable subspace Y of X, there exists a $Z \in \mathcal{F}$ such that $Y \subseteq Z$;

(ii) for every increasing sequence $(Z_n : n \in \mathbb{N})$ in $\mathcal{F}$, $\overline{\bigcup_{n\in\mathbb{N}} Z_n} \in \mathcal{F}$.

For any topological space X, the collection of all rich families of subsets forms a partially ordered set, under the binary relation of set inclusion. This partially ordered set has a greatest element, namely,

$$\mathcal{G}_X := \{S \in 2^X : S \text{ is a closed and separable subset of } X\}.$$

On the other hand, if X is a separable space, then the partially ordered set has a least element, namely, $\mathcal{G}_\varnothing := \{X\}$.

The raison d'être for rich families is revealed next.

Proposition 1 *Suppose that X is a topological space. If $\{\mathcal{F}_n : n \in \mathbb{N}\}$ are rich families of X then so is $\bigcap_{n\in\mathbb{N}} \mathcal{F}_n$.*

Throughout this talk we will be primarily working with Banach spaces and so a natural class of rich families, given a Banach space X, is the family of all closed separable linear subspaces of X, which we denote by $\mathcal{S}_X$. There are however many other interesting examples of rich families.

Theorem 1 *Let X be an L_1-predual. Then the set of all closed separable linear subspaces of X that are themselves L_1-preduals forms a rich family.*

Lemma 1 *Let Y be a closed separable linear subspace of a Banach space X and suppose that $L \subseteq \mathrm{Ext}(B_{X^*})$ is weak* Lindelöf. Then there exists a closed separable linear subspace Z of X, containing Y, such that for any $l^* \in L$ and any x^*, $y^* \in B_{Z^*}$ if $l^*|_Z = \frac{1}{2}(x^* + y^*)$ then $x^*|_Y = y^*|_Y$.*

Using this Lemma we can obtain the following theorem.

Theorem 2 *Let X be a Banach space and let $L \subseteq \mathrm{Ext}(B_{X^*})$ be a weak* Lindelöf subset. Then the set of all Z in $\mathcal{S}_X$ such that $\{l^*|_Z : l^* \in L\} \subseteq \mathrm{Ext}(B_{Z^*})$ forms a rich family.*

Proof: Let $\mathscr{L}$ denote the family of all closed separable linear subspaces Z of X such that $\{l^*|_Z : l^* \in L\} \subseteq \mathrm{Ext}(B_{Z^*})$. We shall verify that $\mathscr{L}$ is a rich family of closed separable linear subspaces of X. So first let us consider an arbitrary closed separable linear subspace Y of X, with the aim of showing that there exists a subspace $Z \in \mathscr{L}$ such that $Y \subseteq Z$. We

begin by inductively applying Lemma 1 to obtain an increasing sequence $(Z_n : n \in \mathbb{N})$ of closed separable linear subspaces of X such that: $Y \subseteq Z_1$ and for any $l^* \in L$ and any $x^*, y^* \in B_{Z^*_{n+1}}$ if $l^*|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*)$ then $x^*|_{Z_n} = y^*|_{Z_n}$.
We now claim that if $Z := \overline{\bigcup_{n\in\mathbb{N}} Z_n}$ then $l^*|_Z \in \mathrm{Ext}(B_{Z^*})$ for each $l^* \in L$. To this end, suppose that $l^* \in L$ and $l^*|_Z = \frac{1}{2}(x^* + y^*)$ for some $x^*, y^* \in B_{Z^*}$.
Then for each $n \in \mathbb{N}$,

$$l^*|_{Z_{n+1}} = (l^*|_Z)|_{Z_{n+1}} = \frac{1}{2}(x^*+y^*)|_{Z_{n+1}} = \frac{1}{2}(x^*|_{Z_{n+1}}+y^*|_{Z_{n+1}})$$

and $x^*|_{Z_{n+1}}, y^*|_{Z_{n+1}} \in B_{Z^*_{n+1}}$ Therefore, by construction $x^*|_{Z_n} = y^*|_{Z_n}$. Now since $\bigcup_{n\in\mathbb{N}} Z_n$ is dense in Z and both x^* and y^* are continuous we may deduce that $x^* = y^*$; which in turn implies that $l^*|_Z \in \mathrm{Ext}(B_{Z^*})$. This shows that $Y \subseteq Z$ and $Z \in \mathscr{L}$.

To complete this proof we must verify that for each increasing sequence of closed separable subspaces $(Z_n : n \in \mathbb{N})$ in

$\mathscr{L}$, $\overline{\bigcup_{n\in\mathbb{N}} Z_n} \in \mathscr{L}$. This however, follows easily from the definition of the family $\mathscr{L}$. ☺

Let X be a normed linear space. Then we say that an element $x^* \in B_{X^*}$ is weak* exposed if there exists an element $x \in X$ such that $y^*(x) < x^*(x)$ for all $y^* \in B_{X^*} \setminus \{x^*\}$. It is not difficult to show that if $\mathrm{Exp}(B_{X^*})$ denotes the set of all weak* exposed points of B_{X^*} then $\mathrm{Exp}(B_{X^*}) \subseteq \mathrm{Ext}(B_{X^*})$. However, if X is a separable L_1-predual then the relationship between $\mathrm{Exp}(B_{X^*})$ and $\mathrm{Ext}(B_{X^*})$ is much closer.

Lemma 2 *If X is a separable L_1-predual, then $\mathrm{Exp}(B_{X^*}) = \mathrm{Ext}(B_{X^*})$.*

Let us also pause for a moment to recall that if B is any boundary of a Banach space X then

$$\mathrm{Exp}(B_{X^*}) \subseteq B \cap \mathrm{Ext}(B_{X^*}) \subseteq \mathrm{Ext}(B_{X^*}) \subseteq \overline{B}^{\mathrm{weak}^*}.$$

The fact that $\mathrm{Ext}(B_{X^*}) \subseteq \overline{B}^{\mathrm{weak}^*}$ follows from Milman's theorem and the fact that $B_{X^*} = \overline{\mathrm{co}}^{\mathrm{weak}^*}(B)$; which in turn follows from a separation argument.

Let us also take this opportunity to observe that if B_X denotes the closed unit ball in X then B_X is closed in the $\sigma(X, B)$ topology for any boundary B of X.

Finally, let us end this part of the talk with one more simple observation that will turn out to be useful in our later endeavours.

Proposition 2 *Suppose that Y is a linear subspace of a Banach space $(X, \|\cdot\|)$ and B is any boundary for X. Then for each $e^* \in \mathrm{Exp}(B_{Y^*})$ there exists $b^* \in B$ such that $e^* = b^*|_Y$.*

Proof: Suppose that $e^* \in \mathrm{Exp}(B_{Y^*})$ then there exists an $x \in Y$ such that $y^*(x) < e^*(x)$ for each $y^* \in B_{Y^*} \setminus \{e^*\}$. By the fact that B is a boundary of $(X, \|\cdot\|)$ there exists a

$b^* \in B$ such that $b^*(x) = \|x\| \neq 0$. Then for any $y^* \in B_{Y^*}$ we have

$$y^*(x) \leq |y^*(x)| \leq \|y^*\|\|x\| \leq \|x\| = b^*(x) = (b^*|_Y)(x).$$

In particular, $e^*(x) \leq b^*|_Y(x)$. Since $b^*|_Y \in B_{Y^*}$ and $y^*(x) < e^*(x)$ for all $y^* \in B_{Y^*} \setminus \{e^*\}$, it must be the case that $e^* = b^*|_Y$. ☺

The Main Results

Theorem 3 *Let B be any boundary for a Banach space X that is an L_1-predual and suppose that $\{x_n : n \in \mathbb{N}\} \subseteq X$, then $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \subseteq \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$.*

Proof: In order to obtain a contradiction let us suppose that

$$\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \not\subseteq \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}.$$

Choose $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \setminus \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$.

Then there exists a finite set $\{e_1^*, e_2^*, \ldots, e_m^*\} \subseteq \mathrm{Ext}(B_{X^*})$

and $\varepsilon > 0$ so that

$$\bigcap_{1\leq k\leq m} \{y \in X : |e_k^*(x) - e_k^*(y)| < \varepsilon\} \cap \{x_n : n \in \mathbb{N}\} = \varnothing.$$

Let $Y := \overline{\mathrm{span}}(\{x_n : n \in \mathbb{N}\} \cup \{x\})$, let $\mathcal{F}_1$ be any rich family of L_1-preduals; whose existence is guaranteed by Theorem 1, and let $\mathcal{F}_2$ be any rich family such that for every $Z \in \mathcal{F}_2$ and every $1 \leq k \leq m$, $e_k^*|_Z \in \mathrm{Ext}(B_{Z^*})$; whose existence is guaranteed by Theorem 2. Next, let us choose $Z \in \mathcal{F}_1 \cap \mathcal{F}_2$ so that $Y \subseteq Z$. Recall that this is possible because, by Proposition 1, $\mathcal{F}_1 \cap \mathcal{F}_2$ is a rich family. Since Z is a separable L_1-predual we have by Lemma 2 that $e_k^*|_Z \in \mathrm{Exp}(B_{Z^*})$ for each $1 \leq k \leq m$. Now, by Proposition 2 for each $1 \leq k \leq m$ there exists a $b_k^* \in B$ such that $e_k^*|_Z = b_k^*|_Z$. Therefore,

$$\begin{aligned} |b_k^*(x) - b_k^*(x_j)| &= |(b_k^*|_Z)(x) - (b_k^*|_Z)(x_j)| \\ &= |(e_k^*|_Z)(x) - (e_k^*|_Z)(x_j)| \\ &= |e_k^*(x) - e_k^*(x_j)|. \end{aligned}$$

for all $j \in \mathbb{N}$ and all $1 \leq k \leq m$. Thus,

$$\bigcap_{1\leq k\leq m} \{y \in X : |b_k^*(x) - b_k^*(y)| < \varepsilon\} \cap \{x_n : n \in \mathbb{N}\} = \varnothing.$$

This contradicts the fact that $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$; which completes the proof. ☺

Corollary 1 *Let B be any boundary for a Banach space X that is an L_1-predual. Then every relatively countably $\sigma(X, B)$-compact subset is relatively countably $\sigma(X, \mathrm{Ext}(B_{X^*}))$-compact. In particular, every norm bounded, relatively countably $\sigma(X, B)$-compact subset is relatively weakly compact.*

Proof: Suppose that a nonempty set $C \subseteq X$ is relatively countably $\sigma(X, B)$-compact. Let $\{c_n : n \in \mathbb{N}\}$ be any sequence in C then by Theorem 3

$$\varnothing \neq \bigcap_{n\in\mathbb{N}} \overline{\{c_k : k \geq n\}}^{\sigma(X,B)} \subseteq \bigcap_{n\in\mathbb{N}} \overline{\{c_k : k \geq n\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$$

Hence C is relatively countably $\sigma(X, \mathrm{Ext}(B_{X^*}))$-compact. In the case when C is also norm bounded the result follows from an earlier result of Kharana. ☺

Recall that a network for a topological space X is a family $\mathscr{N}$ of subsets of X such that for any point $x \in X$ and any open neighbourhood U of x there is an $N \in \mathscr{N}$ such that $x \in N \subseteq U$, and a topological space X is said to be $\aleph_0$-monolithic if the closure of every countable set has a countable network.

Corollary 2 *Let B be any boundary for a Banach space X that is an L_1-predual and suppose that $\{x_n : n \in \mathbb{N}\} \subseteq X$. Then $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$ is norm separable whenever X is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology. In particular, $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$ is norm separable whenever $\mathrm{Ext}(B_{X^*})$ is weak* Lindelöf.*

Proposition 3 *Let B be any boundary for a Banach space X that is an L_1-predual and suppose that A is a separable Baire space. If X is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology then for each continuous mapping $f : A \to (X, \sigma(X, B))$ there exists a dense subset D of A such that f is continuous with respect to the norm topology on X at each point of D.*

Proof: Fix $\varepsilon > 0$ and consider the open set:

$$O_\varepsilon := \bigcup \{U \subseteq A : U \text{ is open and } \|\cdot\| - \text{diam}[f(U)] \leq 2\varepsilon\}.$$

We shall show that O_ε is dense in A. To this end, let W be a nonempty open subset of A and let $\{a_n : n \in \mathbb{N}\}$ be a countable dense subset of W. Then by continuity

$$f(W) \subseteq \overline{\{f(a_n) : n \in \mathbb{N}\}}^{\sigma(X,B)};$$

which is norm separable by Corollary 2. Therefore there exists a countable set $\{x_n : n \in \mathbb{N}\}$ in X such that $f(W) \subseteq$

$\bigcup_{n\in\mathbb{N}}(x_n+\varepsilon B_X)$. For each $n \in \mathbb{N}$, let $C_n := f^{-1}(x_n+\varepsilon B_X)$. Since each $x_n + \varepsilon B_X$ is closed in the $\sigma(X, B)$ topology each set C_n is closed in A and moreover, $W \subseteq \bigcup_{n\in\mathbb{N}} C_n$. Since W is of the second Baire category in A there exist a nonempty open set $U \subseteq W$ and a $k \in \mathbb{N}$ such that $U \subseteq C_k$. Then $U \subseteq O_\varepsilon \cap W$ and so O_ε is indeed dense in A. Clearly, f is $\|\cdot\|$-continuous at each point of $\bigcap_{n\in\mathbb{N}} O_{1/n}$. ☺

Theorem 4 *Suppose that A is a topological space with countable tightness that possesses a rich family $\mathcal{F}$ of Baire subspaces and suppose that X is an L_1-predual. Then for any boundary B of X and any continuous function $f : A \to (X, \sigma(X, B))$ there exists a dense subset D of A such that f is continuous with respect to the norm topology on X at each point of D provided X is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology.*

Proof: In order to obtain a contradiction let us suppose that f does not have a dense set of points of continuity with respect to the norm topology on X. Since A is a Baire space this implies that for some $\varepsilon > 0$ the open set:

$$O_\varepsilon := \bigcup \{U \subseteq A : \ U \text{ is open and } \|\cdot\|\text{-diam}[f(U)] \leq 2\varepsilon\}$$

is not dense in A. That is, there exists a nonempty open subset W of A such that $W \cap O_\varepsilon = \varnothing$. For each $x \in A$, let

$$F_x := \{y \in A : \|f(y) - f(x)\| > \varepsilon\}.$$

Then $x \in \overline{F_x}$ for each $x \in W$. Moreover, since A has countable tightness, for each $x \in W$, there exists a countable subset C_x of F_x such that $x \in \overline{C_x}$.

Next, we inductively define an increasing sequence of separable subspaces $(F_n : n \in \mathbb{N})$ of A and countable sets $(D_n : n \in \mathbb{N})$ in A such that:

(i) $W \cap F_1 \neq \varnothing$;

(ii) $\bigcup\{C_x : x \in D_n \cap W\} \cup F_n \subseteq F_{n+1} \in \mathcal{F}$ for all $n \in \mathbb{N}$, where D_n is any countable dense subset of F_n.

Note that since the family $\mathcal{F}$ is rich this construction is possible.

Let $F := \overline{\bigcup_{n\in\mathbb{N}} F_n}$ and $D := \bigcup_{n\in\mathbb{N}} D_n$. Then $\overline{D} = F \in \mathcal{F}$ and $\|\cdot\|$-diam$[f(U)] \geq \varepsilon$ for every nonempty open subset U of $F \cap W$. Therefore, $f|_F$ has no points of continuity in $F \cap W$ with respect to the $\|\cdot\|$-topology. This however, contradicts Proposition 3. ☺

Corollary 3 *Suppose that A is a topological space with countable tightness that possesses a rich family of Baire subspaces and suppose that K is a compact Hausdorff space. Then for any boundary of $(C(K), \|\cdot\|_\infty)$ and any continuous function $f : A \to (C(K), \sigma(C(K), B))$ there exists a dense subset D of A such that f is continuous with respect to the $\|\cdot\|_\infty$-topology at each point of D.*

——————————— The End ———————————