Multiple Dedekind-Rademacher sums in function fields

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Classical results (1. Reciprocity law)

For relatively prime integers $c > 0$, a, the inhomogeneous Dedekind sum is defined by

$$s(a,c) = \frac{1}{4c}\sum_{k=1}^{c-1}\cot\left(\frac{\pi k a}{c}\right)\cot\left(\frac{\pi k}{c}\right).$$

This satisfies the reciprocity law

$$s(a,c) + s(c,a) = \frac{a^2 + c^2 + 1 - 3ac}{12ac}$$

if $a, c > 0$ are coprime.

For $a, b \in \mathbb{Z}$ relatively prime to an integer $c > 0$, H. Rademacher defined the homogeneous Dedekind sum by

$$s(c; a, b) = \frac{1}{4c} \sum_{k=1}^{c-1} \cot\left(\frac{\pi k a}{c}\right) \cot\left(\frac{\pi k b}{c}\right).$$

which satisfies the reciprocity law

$$s(c; a, b) + s(b; a, c) + s(a; a, b) = \frac{a^2 + b^2 + c^2 - 3abc}{12abc}$$

if a, b, c are pairwise coprime.

For $a_1, \ldots, a_d \in \mathbb{Z}$ relatively prime to an integer $a_0 > 0$, D. Zagier defined the higher dimensional Dedekind sum by

$$d(a_0; a_1, \ldots, a_d) = (-1)^{d/2} \frac{1}{a_0} \sum_{k=1}^{a_0-1} \cot\left(\frac{\pi k a_1}{a_0}\right) \cdots \cot\left(\frac{\pi k a_d}{a_0}\right).$$

If $a_0, a_1, \ldots, a_d$ are pairwise coprime, It satisfies the reciprocity law

$$\sum_{j=0}^{d} d(a_j; a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) = 1 - \frac{l_d(a_0, \ldots, a_d)}{a_0 \cdots a_d},$$

where $l_d(a_0, \ldots, a_d)$ are polynomials in $a_0, \ldots, a_d$.

$\cot^{(m)}(z)$: the mth derivative of $\cot(z)$
Let $a_1 \dots, a_d \in \mathbb{Z}$ be relatively prime to $a_0 \in \mathbb{N}$, let $m_0, \dots, m_d \geq 0$. A. Bayad-A. Raouj investigated the multiple Dedekind-Rademacher sum by

$$C\left(\begin{array}{c|ccc} a_0 & a_1, & \dots, & a_d \\ m_0 & m_1, & \dots, & m_d \end{array}\right) = \frac{1}{a_0^{m_0+1}} \sum_{k=1}^{a_0-1} \cot^{(m_1)}\left(\frac{\pi k a_1}{a_0}\right) \cdots \cot^{(m_d)}\left(\frac{\pi k a_d}{a_0}\right).$$

It satisfies the reciprocity law.

Classical results (2. Petersson-Knopp identity)

M.I. Knopp proved the identity

$$\sum_{d|n} \sum_{r=1}^{d} s\left(\frac{n}{d}a + rc, dc\right) = \sigma(n)s(a, c),$$

where
$\sigma(n)$ =the sum of the positive divisors of n. His identity is called the Petersson-Knopp identity.

Z. Zheng extended Knopp's identity to $s(c; a, b)$ by

$$\sum_{d|n} \sum_{r_1=1}^{d} \sum_{r_2=1}^{d} s\left(dc; \frac{n}{d}a + r_1 c, \frac{n}{d}b + r_2 c\right) = n\sigma(n)s(c; a, b).$$

Beck generalized this result for an arbitrary sum of "Dedekind type", which includes multiple Dedekind-Rademacher sum.

Purpose

• To introduce multiple Dedekind-Rademacher sums in function fields. These are related to A-lattices, which are associated to Drinfeld modules.
• To discuss the reciprocity law, the Petersson-Knopp identity, and the rationality.

Some results were obtained from the joint works with A. Bayad.

Notation

$$\begin{array}{lcl} A = \mathbb{F}_q[T] & \leftrightarrow & \mathbb{Z} \\ K = \mathbb{F}_q(T) & \leftrightarrow & \mathbb{Q} \\ K_\infty = \mathbb{F}_q((1/T)) & \leftrightarrow & \mathbb{R} \\ C_\infty = \overline{\widehat{\mathbb{F}_q((1/T))}} & \leftrightarrow & \mathbb{C} \end{array}$$

A-lattices

Λ is an A-lattice of rank r in C_∞ if it is a finitely generated A-submodule of rank r in C_∞ that is discrete in the topology of C_∞. Define

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right).$$

1. e_Λ is entire in the rigid analytic sense, and surjective.
2. e_Λ is $\mathbb{F}_q$-linear and Λ-periodic.
3. e_Λ has simple zeros at the points of Λ, and no other zeros.
4. $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$.

Drinfeld modules

$L\{\tau\}$: the non-commutative ring of polynomials in τ over the field L s.t. $\tau a = a^q \tau$ $(a \in L)$
An $\mathbb{F}_q$-algebra homomorphism

$$\phi : A \to L\{\tau\}, \qquad a \mapsto \phi_a$$

is said to be a Drinfeld module of rank r over L if ϕ satisfies

(i) $D \circ \phi = \iota$, where D is the derivation $D(f) = a_0$ for $f(\tau) = \sum_{i=0}^{l} a_i \tau^i \in L\{\tau\}$, and ι is the inclusion map $\iota : A \hookrightarrow C_\infty$.

(ii) For some $a \in A$, $\phi_a \neq \iota(a)\tau^0$.

(iii) For all $a \in A$, $\deg \phi_a = r \deg(a)$.

For any rank r A-lattice Λ, there exists a unique rank r Drinfeld module ϕ^{Λ} s.t.

$$e_{\Lambda}(az) = \phi_a^{\Lambda}(e_{\Lambda}(z)) \quad (\forall a \in A).$$

The association $\Lambda \mapsto \phi^{\Lambda}$ yields a bijection

the set of A-lattices of rank r in C_{∞}

$$\updownarrow$$

the set of Drinfeld modules of rank r over C_{∞}.

Multiple Dedekind-Rademacher sums

Λ: A-lattice
$a_1, a_2, \ldots, a_d \in A \setminus \{0\}$ is relatively prime to $a_0 \in A \setminus \{0\}$
$m_0, \ldots, m_d$: non-negative integers
We call

$$s_\Lambda\left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_d \\ m_0 & m_1, & \ldots, & m_d \end{array}\right) = \frac{1}{a_0^{m_0+1}} \sum_{0 \neq \lambda \in \Lambda/a_0\Lambda} e_\Lambda\left(\frac{\lambda a_1}{a_0}\right)^{-m_1-1} \cdots e_\Lambda\left(\frac{\lambda a_d}{a_0}\right)^{-m_d-1}$$

the multiple Dedekind-Rademacher sum.

In particular,

$$s_\Lambda(a_0; a_1, \ldots, a_d) = (-1)^d s_\Lambda\left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_d \\ 0 & 0, & \ldots, & 0 \end{array}\right)$$

$$= \frac{(-1)^d}{a_0} \sum_{0 \neq \lambda \in \Lambda/a_0\Lambda} e_\Lambda\left(\frac{\lambda a_1}{a_0}\right)^{-1} \cdots e_\Lambda\left(\frac{\lambda a_d}{a_0}\right)^{-1}$$

is the higher dimensional Dedekind sum of Zagier type.

$$s_\Lambda(c; a, b) = \frac{1}{c} \sum_{0 \neq \lambda \in \Lambda/c\Lambda} e_\Lambda\left(\frac{\lambda a}{c}\right)^{-1} e_\Lambda\left(\frac{\lambda b}{c}\right)^{-1}$$

is the homogeneous Dedekind sum, and

$$s_\Lambda(a,c) = \frac{1}{c} \sum_{0 \neq \lambda \in \Lambda/c\Lambda} e_\Lambda\left(\frac{\lambda}{c}\right)^{-1} e_\Lambda\left(\frac{\lambda a}{c}\right)^{-1}$$

is the inhomogeneous Dedekind sum.

The reciprocity law

Theorem 1

If $a_0, \ldots, a_d \in A \setminus \{0\}$ are pairwise coprime,

$$\sum_{i=0}^{d} \sum_{\substack{l_0,\ldots,\widehat{l_i},\ldots,l_d \geq 0 \\ l_0+\cdots+\widehat{l_i}+\cdots+l_d = m_i}} \left(\prod_{j \neq i} \binom{m_j + l_j}{m_j} (-a_j)^{l_j} \right)$$

$$\times s_\Lambda \left(\begin{array}{c|ccccc} a_i & a_0, & \ldots, & \widehat{a_i}, & \ldots, & a_d \\ m_i & m_0 + l_0, & \ldots, & \widehat{m_i + l_i}, & \ldots, & m_d + l_d \end{array} \right)$$

$$= \frac{(-1)^{m_0+\cdots+m_d+d}}{a_0^{m_0+1} \cdots a_d^{m_d+1}} \sum_{\substack{j_0,\ldots,j_d \geq 0 \\ j_0+\cdots+j_d \\ = m_0+\cdots+m_d+d}} A_{0,j_0} A_{1,j_1} \cdots A_{d,j_d}.$$

Here $\widehat{\bullet}$ is omitting of $\bullet$ and

$$A_{i,j_i} = \begin{cases} (-1)^{m_i+1} & (\text{if } j_i = 0) \\ \binom{j_i-1}{m_i} E_{j_i}(\phi[a_i]) & (\text{if } j_i \geq m_i) \\ 0 & (\text{otherwise}) \end{cases},$$

where $\phi[a] = \{x \in C_\infty \mid \phi_a(x) = 0\}$,

$$E_j(\phi[a]) = \sum_{0 \neq x \in \phi[a]} \frac{1}{x^j}.$$

Outline of Proof of Theorem 1

Consider

$$F(z) = \frac{1}{\phi_{a_0}(z)^{m_0+1} \cdots \phi_{a_d}(z)^{m_d+1}}.$$

Its poles are $R = \bigcup_{i=0}^{d} \phi[a_i]$.
Use

$$\sum_{x \in R} \mathrm{Res}(F(z)dz, z = x) = 0.$$

The Petersson-Knopp identity

$a_0, a_1, \ldots, a_d \in A \setminus \{0\}$, $0 \le m_1, \ldots, m_d \le d-1$.
Theorem 2 Let $n \in A \setminus \{0\}$. Then we have

$$\sum_{b|n} b^{m_0-m_1-\cdots-m_d-d+1}$$

$$\times \sum_{r_1,\ldots,r_d \in A/bA} s_A\left(\begin{array}{c|ccc} a_0 b & \frac{n}{b}a_1 + r_1 a_0, & \ldots, & \frac{n}{b}a_d + r_d a_0 \\ m_0 & m_1, & \cdots, & m_d \end{array}\right)$$

$$= |n| s_A\left(\begin{array}{c|ccc} a_0 & a_1, & \cdots, & a_d \\ m_0 & m_1, & \cdots, & m_d \end{array}\right) \sum_{c|n} |c|^{d-1} c^{-m_1-\cdots-m_d-d},$$

where $\sum_{b|n}$ means the sum over monic elements of A dividing n.

Outline of proof of Theorem 2

We have

Lemma 1(Distribution property) For $m = 1, \ldots, q$ and $a, c \in A \setminus \{0\}$,

$$\sum_{\lambda \in A/cA} e_A\left(z + \frac{a\lambda}{c}\right)^{-m} = \frac{|(a,c)|c^m}{(a,c)^m} e_A\left(\frac{cz}{(a,c)}\right)^{-m}.$$

Apply this to the left-hand side of the theorem.

The rationality

ϕ: rank r Drinfeld module associated to Λ

When is $s_\Lambda\left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_{q^i-1} \\ m_0 & m_1, & \ldots, & m_{q^i-1} \end{array}\right)$ rational, i.e,

$s_\Lambda\left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_{q^i-1} \\ m_0 & m_1, & \ldots, & m_{q^i-1} \end{array}\right) \in K$?

The Dedekind sum is not always rational.

Proposition 1 If ϕ is defined over K,

$s_\Lambda\left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_{q^i-1} \\ m_0 & m_1, & \ldots, & m_{q^i-1} \end{array}\right)$ is rational.

The converse is true?

Theorem 3 The following conditions are equivalent:

(i) For all d, $s_\Lambda \left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_d \\ 0 & 0, & \ldots, & 0 \end{array} \right)$ are rational.

(ii) ϕ is defined over K.

Outline of proofs of Prop. 1, Thm. 3

Prop. 1 and (i)⇒(ii) Apply Galois theory to

$$s_\Lambda \left(\begin{array}{c|ccc} a_0 & a_1, & \ldots, & a_{q^i-1} \\ m_0 & m_1, & \ldots, & m_{q^i-1} \end{array} \right) = \frac{1}{a_0^{m_0+1}} \sum_{x \in \phi[a_0] \setminus \{0\}} \frac{1}{\phi_{a_0}(x)^{m_0+1} \ldots \phi_{a_d}(x)^{m_d+1}}.$$

(ii)⇒(i) From the assumption, for all $a \in A \setminus \{0\}$ and $j < i$,

$$E_{q^i-q^j}(\phi[a]) = \sum_{x \in \phi[a] \setminus \{0\}} \frac{1}{x^{q^i-q^j}} \in K.$$

Then use the Newton formula for $\phi_a(z)$.

Thank you for your attention!