A Proof that Euler Missed ...

Apéry's Proof of the Irrationality of $\zeta(3)$

An Informal Report

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1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 "Sur l'irrationalité de $\zeta(3)$." Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

Exercise
Prove the following amazing claims:

① For all $a_1, a_2, \ldots$

$$\sum_{k=1}^{\infty} \frac{a_1 a_2 \ldots a_{k-1}}{(x+a_1)\ldots(x+a_k)} = \frac{1}{x}.$$

② $$\zeta(3) =: \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}. \qquad (1)$$

③ Consider the recursion:

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1}, \quad n \geqslant 2. \qquad (2)$$

Let $\{b_n\}$ be the sequence defined by $b_0 = 1$, $b_1 = 5$, and $b_n = u_n$ for all n; then the b_n all are integers! Let $\{a_n\}$ be the sequence defined by $a_0 = 0$, $a_1 = 6$, and $a_n = u_n$ for all n; then the a_n are rational numbers with denominator dividing $2[1, 2, \ldots, n]^3$ (here $[1, 2, \ldots, n]$ is the lcm (lowest common multiple) of $1, \ldots, n$).

④ $a_n/b_n \to \zeta(3)$; indeed the convergence is so fast as to prove that $\zeta(3)$ cannot be rational. To be precise, for all integers p, q with q sufficiently large relative to $\epsilon > 0$,

$$\left|\zeta(3) - \frac{p}{q}\right| > \frac{1}{q^{\theta+\epsilon}}, \quad \theta = 13.41782\ldots.$$

Exercise *(continued)*

②' $$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} \qquad (3)$$

③' Consider the recursion:

$$n^2 u_n - (n-1)^2 u_{n-2} = (11n^2 - 11n + 3)u_{n-1}, \quad n \geqslant 2. \qquad (4)$$

Let $\{b'_n\}$ be the sequence defined by $b'_0 = 1$, $b'_1 = 3$ and the recursion; then the b'_n all are integers! Let $\{a'_n\}$ be the sequence defined by $a'_0 = 0$, $a'_1 = 5$ and the recursion; then the a'_n are rational numbers with denominator dividing $[1, 2, \ldots, n]^2$.

④' $a'_n/b'_n \to \zeta(2) = \pi^2/6$; indeed the convergence is so fast as to imply that for all integers p, q with q sufficiently large relative to $\epsilon > 0$

$$\left|\pi^2 - \frac{p}{q}\right| > \frac{1}{q^{\theta'+\epsilon}}, \quad \theta' = 11.85078\ldots.$$

I heard with some incredulity that, for one, Henri Cohen (Bordeaux, now Grenoble) believed that these claims might well be valid. Very much intrigued, I joined Hendrik Lenstra (Amsterdam) and Cohen in an evening's discussion in which Cohen explained and demonstrated most of the details of the proof. We came away convinced that Professeur Apéry had indeed found a quite miraculous and magnificent demonstration of the irrationality of $\zeta(3)$. But we remained unable to prove a critical step.

2. For the Nonexpert Reader

A number β is *irrational* if it is not of the form p_0/q_0; p_0, q_0 integers ($\in \mathbb{Z}$). A rational number b is characterised by the property that for $p, q \in \mathbb{Z}$ ($q > 0$) and $b \neq p/q$ there exists an integer q_0 (> 0, of course) such that

Alf van der Poorten

AWESOME ALF

Alf receiving the award of a Doctorat
Honoris Causa bv the Universit´e Bordeaux I in 1998

Prof Von der Poorten

MATHS IS 'BEAUTIFUL AND FUN' SAYS PROF

Maths can be fu
claims a new professo
at Macquarie Univer
sity.

Prof Alfred van de
Poorten said today
"Maths is beautifu
fun.

"If you can learn
read and write you ca
do maths."

Youngest

At 36, he is one
the youngest men
the university to hold
professorial position.

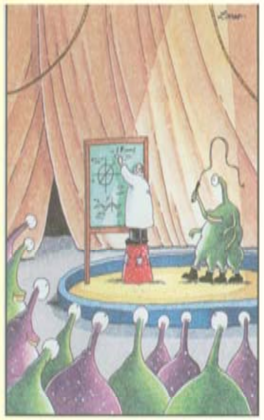
"I believe maths
badly taught at scho
level," he said.

"People can't belie
the subject can
attractive — they'
frightened of it."

What the Mirro

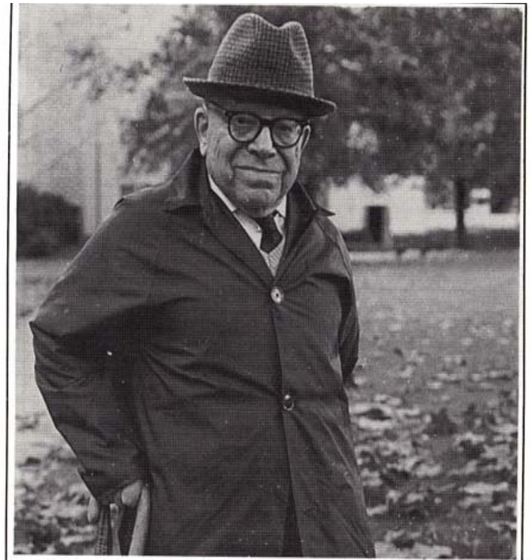
Go the Dragon's, Alf's favourite team

Alf at work on the IMU ECCEIC Meeting



Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.

Alf was a student of Kurt Mahler and helped set up the room in his honour



A room of his own

Later this month, a section of the Mathematical Sciences Department in the Research School of Physical Sciences is to be dedicated to Professor Kurt Mahler, who celebrates his eighty-fourth birthday on 26 July. A wall between two offices has been pulled down and the remaining space converted into a seminar room. On Friday 24 July, to mark his birthday, friends and colleagues will gather for a small ceremony to name the area the Kurt Mahler Meeting Room.

FOLDS!

III. More Morphisms[0]

Michel Dekking, Michel Mendès France, and A

In which we become quite convoluted

Our story thus far In Folds! I. Folds! (Vol. 4, No. 3) we met certain paperfolding sequences, finite automata and dragon curves. In FOLDS! II. Symmetry Disturbed we passed from chaos to determinism and read of continued fractions, automata, and Mahler functions. Here we meet more folds . . .

3.1. Generalised paperfolding
3.2. Latticefilling curves
3.3. Acknowledgements

3.1 . . . and rarely simple[1]

One *can* fold paper other than 'in half'. Since it is our object to justify displaying some attractive pictures we will emphasise the curves that ensue rather than the generalised folding sequences themselves. So we return to the count sequences (c_h) that determine dragon curves. It is enough to have c_h (mod 4) because the direction of each segment of the curve is given by

$$i^{c(h)}.$$

Thus we may take the c_h in $\{0, 1, 2, 3\}$.

The positive paperfolding sequence

1 1 0 1 1 0 0 1 1 1 0 0 1 0 0 1 1 1 0 1 1 0 0 0 1 1 0 0 1 0 0 1 . . .

and its count (of the excess of ones over zeros, mod 4)

0 1 2 1 2 3 2 1 2 3 0 3 2 3 2 1 2 3 0 3 0 1 0 3 2 3 0 3 2 3 2 1 . . .

already signals the procedure we should follow. We notice

# folds	*count sequence (dragon curve)*
0	0
1	0 1
2	0 1 2 1
3	0 1 2 1 2 3 2 1
4	0 1 2 1 2 3 2 1 2 3 0 3 2 3 2 1
⋮	

To add a fold to the count sequence of the positive paperfolding sequence one adds on the reverse of the previous sequence, having added 1 (mod 4) *to each symbol.*

Geometrically speaking, this is no more than advice to add the reverse of the previous dragon curve after rotating it through $\pi/2$:

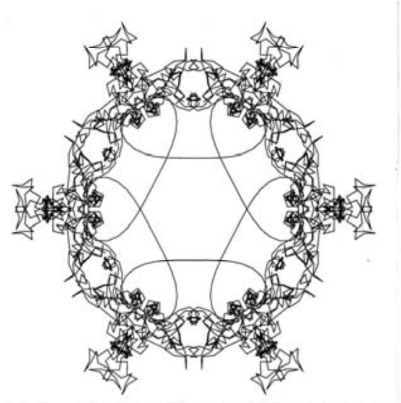
the curve 01212321 — duplicated . . . — and the duplicate swung through $\pi/2$ yielding: 0121232123032321

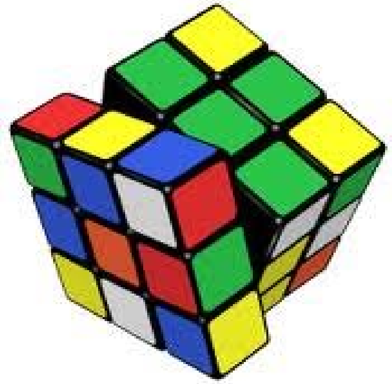
Nevertheless, the clumsiness of the statement of Observation 3.1 suggests that some notation is needed. We recall the operator $R : \vec{w} \mapsto \overleftarrow{w}$ which reverses words, and we introduce the operator $\sigma : abc \ldots \mapsto a+1, b+1, c+1, \ldots$ which pushes each symbol up by 1 (mod 4), The appropriately conditioned reader will already have seen:

Observation 3.2. The positive paperfolding dragon curve is generated by the 2-substitution
$\theta : 0 \mapsto 01, 1 \mapsto 21, 2 \mapsto 23, 3 \mapsto 03.$

0 This is a vulgarisation of a manuscript 'Iterated paperfolding and plane filling curves' creased by F. M. Dekking.
1 Truth is never pure and rarely simple: Oscar Wilde *The Importance of being Earnest*.

The final part of Alf's folding trilogy – written in unmistakable & beautiful Alfism





Alf had one of the first ever rubik's cubes in Sydney, he was a fanatic about it and delighted in showing off his expertise

Alf 's great friends George and Esther Szekeres