# Group actions on graphs and their $C^*$–algebras

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# Crossed products

An *action* of $G$ on a $C^*$–algebra $A$ is a homomorphism $\alpha : G \to \mathrm{Aut}(A)$ and gives rise to a *$C^*$–dynamical system* $(A, \alpha, G)$.

Let $B$ be a $C^*$–algebra, a *covariant representation* of $(A, \alpha, G)$ in $B$ is a pair $(\psi, \pi)$ of maps $\psi : A \to M(B)$, $\pi : G \to UM(B)$ such that

$$\psi(\alpha_g(a)) = \pi(g)\psi(a)\pi(g)^*$$

The *crossed product* $A \times_\alpha G$ is generated by a universal covariant representation of $(A, \alpha, G)$.

# Coactions

When $G$ is abelian, the crossed product $A \times_\alpha G$, carries a natural action $\widehat{\alpha}$ of $\widehat{G}$. When $G$ is nonabelian, there is a dual coaction $\widehat{\alpha}$ of $G$ on $A \times_\alpha G$.

A *coaction* of $G$ on a $C^*$-algebra $A$ is an injective nondegenerate homomorphism $\delta : A \to A \otimes C^*(G)$ such that $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$. Where $\delta_G$ is the canonical coaction of $G$ on $C^*(G)$.

There is a notion of a covariant representation of $(A, \delta, G)$ however it is a bit technical.

The *crossed product* $A \times_\delta G$ is generated by a universal covariant representation of $(A, \delta, G)$ which carries a natural dual action $\widehat{\delta}$ of $G$.

# Nonabelian Duality

**Theorem 1 (Takesaki, Takai)**

*Let $A$ be a $C^*$–algebra and $G$ a group.*

(1) *Let $\alpha$ be an action of $G$ on $A$, then the dual coaction $\widehat{\alpha}$ of $G$ on $A \times_\alpha G$ such that $(A \times_\alpha G) \times_{\widehat{\alpha}} G \cong A \otimes \mathcal{K}(L^2(G))$.*

(2) *Let $\delta$ be a coaction of $G$ on $A$, then the dual action $\widehat{\delta}$ of $G$ on $A \times_\delta G$ such that $(A \times_\delta G) \times_{\widehat{\delta}} G \cong A \otimes \mathcal{K}(L^2(G))$.*

# Directed graphs

## Definition 2

A *directed graph* $E$ consists of a set $E^0$ of vertices, a set $E^1$ of edges and maps $r, s : E^1 \to E^0$ giving the direction of each edge.

Let $E^n$ denote the directed paths $\lambda$ with length $|\lambda| = n$, then $E^* = \cup_{n \geq 0} E^i$ denotes the collection of all finite paths in $E$.

## Definition 3

A *graph morphism* $\phi : E \to F$ is a pair $\phi = (\phi^0, \phi^1)$ of maps $\phi^i : E^i \to F^i$ for $i = 1, 2$ such that for all $e \in E^1$

$$s(\phi^1(e)) = \phi^0(s(e)) \quad r(\phi^1(e)) = \phi^0(r(e))$$

## Group actions

**Definition 4**

An *action* of a group $G$ on a directed graph $E$ is a group homomorphism $\alpha : G \to \mathrm{Aut}(E)$.

For $v \in E^0$ and $e \in E^1$ let

$$[u] = \{v \in E^0 : v = \alpha_g^0 u \text{ for some } g \in G\}$$
$$[e] = \{f \in E^1 : f = \alpha_g^1 e \text{ for some } g \in G\}.$$

If we put $E^0/G = \{[u] : u \in E^0\}$, $E^1/G = \{[e] : e \in E^1\}$ and set

$$r'([e]) = [r(e)] \quad s'([e]) = [s(e)] \text{ for } [e] \in E^1/G$$

then $E/G = (E^0/G, E^1/G, r', s')$ is a directed graph, called the *quotient graph*.

# Cuntz–Krieger families

A directed graph $E$ is *essential* if every vertex receives and emits a finite, nonzero number of edges.

## Definition 5

Let $E$ be an essential directed graph and $B$ a $C^*$–algebra. A *Cuntz–Krieger $E$–family* is a function $t : \lambda \mapsto t_\lambda$ from $E^*$ to $B$ such that

(1) $\{t_v : v \in E^0\}$ are mutually orthogonal projections

(2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $r(\mu) = s(\lambda)$

(3) $t_\lambda^* t_\lambda = t_{s(\mu)}$

(4) for all $v \in E^0$ and $n \geq 1$ we have $t_v = \sum_{|\lambda|=n} t_\lambda t_\lambda^*$.

$C^*(E)$ is then defined to be the universal $C^*$–algebra generated by a Cuntz-Krieger $E$–family.

## Group actions

The universal property of $C^*(E)$ is such that if $s : \lambda \mapsto s_\lambda$ is any Cuntz–Krieger $E$–family in a $C^*$–algebra $D$, then there is a homomorphisn $\pi_s : C^*(E) \to D$ satisfying $\pi_s(t_\lambda) = s_\lambda$ for all $\lambda \in E^*$. One may show (non–trivially!) that such a $C^*$–algebra exists.

An action $\alpha$ of $G$ on $E$ induces an action of $G$ on $E^*$ which transforms a Cuntz–Krieger $E$–family $t$ in a $C^*$–algebra $B$ into a Cuntz–Krieger $E$–family $t \circ \alpha$ in $B$. By the universal property of $C^*(E)$, this induces an action $\alpha_*$ of $G$ on $C^*(E)$.

Hence we may form the crossed product $C^*$–algebra $C^*(E) \times_{\alpha_*} G$.

# Free actions, skew product graphs

The action $\alpha$ of $G$ on $E$ is *free* if $\alpha_g^0 v = v$ for all $v \in E^0$ then $g = 1_G$.

Let $E$ be a directed graph, $G$ a group and $c : E^1 \to G$ a function. The *skew–product graph* $E \times_c G$ has vertices $E^0 \times G$, edges $E^1 \times G$ and range and source maps

$$r(e,g) = (r(e), g) \quad s(e,g) = (s(e), gc(e)).$$

There is a natural free action $\lambda$ of $G$ on $E \times_c G$ given by

$$\lambda_h^i(x,g) = (x, hg) \text{ for } i = 0, 1 \text{ and } h \in G.$$

The quotient $(E \times_c G)/G$ is isomorphic to $E$.

## Gross–Tucker Theorem

The Gross–Tucker Theorem says that the situation on the previous slide is generic: if a group acts freely on a graph it is acting on a skew–product graph.

**Theorem 6 (Gross–Tucker)**

*Let $E$ be a directed graph and $\alpha$ a free action of a group $G$. Let $\eta : (E/G)^0 \to E^0$ be a section for the quotient map $q^0 : E^0 \to (E/G)^0$, then there is a function $c_\eta : (E/G)^1 \to G$ such that $(E/G) \times_{c_\eta} G$ is equivariantly isomorphic to $E$.*

# Back to graph algebras

**Theorem 7 (Kumjian-P,P–Raeburn)**

*Let $E$ be an essential directed graph and $G$ a countable group.*

(1) *Let $\alpha$ be a free action of $G$ on $E$, then* $C^*(E) \times_{\alpha_*} G \cong C^*(E/G) \otimes \mathcal{K}(\ell^2(G))$. *Indeed* $C^*(E) \times_{\alpha_*} G \sim_{sme} C^*(E/G)$.

(2) *Let $c : E^1 \to G$ be a function, then* $C^*(E \times_c G) \times_{\lambda_*} G \cong C^*(E) \otimes \mathcal{K}(\ell^2(G))$.

The connection between Theorem 7 and Theorem 1 is explained by:

**Theorem 8 (Kaliszewski-Quigg-Raeburn)**

*Let $E$ be an essential directed graph, $G$ a group and $c : E^1 \to G$ a function. Then there is a coaction $\delta_c$ of $G$ on $C^*(E)$ such that $C^*(E) \times_{\delta_c} G$ is equivariantly isomorphic to $C^*(E \times_c G)$.*

# Relative skew–product graphs

Let $E$ be a directed graph, $G$ a group and $c : E^1 \to G$ a function. Then for any subgroup $H$ of $G$, the *relative skew product* $E \times_c (G/H)$ is the graph with $(E \times_c (G/H))^i = E^i \times (G/H)$ for $i = 0, 1$,

$$r(e, Hg) = (r(e), Hg) \text{ and } s(e, Hg) = (s(e), Hgc(e)).$$

When $H$ is the trivial subgroup, this is the usual skew product $E \times_c G$

## Coactions of symmetric spaces

It turns out that one can define the coaction of a symmetric space on a $C^*$–algebra, and define the associated crossed–product. The following is then a generalisation of Theorem 8.

**Theorem 9 (Deicke–P–Raeburn)**

*Let $E$ be an essential directed graph, $G$ a group and $c : E^1 \to G$ a function and suppose $H$ is a subgroup of $G$. Then*

$$C^*(E \times_c (G/H)) \cong C^*(E) \times_{\delta_c} (G/H).$$

# More coactions

The subgroup $H$ acts on the right of $E \times_c G$, and one can show that $E \times_c (G/H)$ is isomorphic to the quotient $(E \times_c G)/H$. Hence from Theorem 8 we have

**Theorem 10 (Deicke–P–Raeburn)**

*Let $\delta_c$ be the coaction of $G$ on $C^*(E)$ induced by a function $c : E^1 \to G$, and let $H$ be a subgroup of $G$.*

- *There is an action $\widehat{\delta_c}$ of $H$ on $C^*(E) \times_{\delta_c} G$ such that*

$$C^*(E) \times_{\delta_c} (G/H) \sim_{sme} (C^*(E) \times_{\delta_c} G) \times_{\widehat{\delta_c}} H.$$

- *There is a coaction $\delta_d$ of $H$ on $C^*(E) \times_{\delta_c} (G/H)$ such that*

$$C^*(E) \times_{\delta_c} G \cong (C^*(E) \times_{\delta_c} (G/H)) \times_{\delta_d} H.$$

## Generalisations

Similar results have been given for

- Group actions on $k$–graphs: P–Quigg–Raeburn.
- Ore semigroup actions on $k$-graphs: Maloney–P–Raeburn.
- Group actions on topological graphs: Kaliszewski–Kumjian–Quigg, also Kaliszewski–Robertson–Quigg.

Future directions: Non–free actions on graphs and $k$–graphs: Brownlowe–Kumjian–P–Thomas.

THANK YOU!