# Moving Pressure Distribution in an Ice Channel

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## 1 INTRODUCTION

In this paper, we present the solution of a steady 3-D problem for flexural-gravity waves (FGW) and surface ones generated by a local pressure distribution moving with uniform speed on open water lead between two semi-infinite floating ice sheets (ice channel). The problem is formulated within linear hydroelastic theory. The fluid is assumed to be inviscid and incompressible and its motion is potential. The ice sheets are treated as viscoelastic thin plates. The external load simulates the motion of an air-cushion vehicle (ACV). The solution of this problem is constructed using the Fourier transform and the Wiener-Hopf technique. The displacements of free surface and ice sheets are determined as well as strains along the ice edges and wave resistance acting on ACV at various speeds of its motion: subcritical and supercritical ones relative to the minimum phase velocity of FGW in the ice cover. Special attention is paid to the characteristics of edge waves, the existence of which in this problem was shown earlier by Marchenko (1997) and Porter (2018). The uniform motion of the external pressure on semi-infinite free surface along the rectilinear edge of the ice sheet was considered earlier by Sturova and Tkacheva (2018) and Tkacheva (2018).

## 2 MATHEMATICAL FORMULATION

The water is taken to be of constant density $\rho_0$ and uniform depth $H$. The ice sheets are modeled by two thin viscoelastic semi-infinite plates of identical thickness $h$ floating on the water surface and separated by an open water lead of width $L$. The edges of plates are free. The plate drafts are ignored. The pressure distribution $q(x, y)$ moves with constant speed $U$ along the rectilinear edges of plates. The moving Cartesian coordinate system $x, y, z$ is considered with the $x$-axis passing through the center of the pressure region perpendicular to the plate edges, the $y$-axis is directed along the edge of the left plate and the $z$-axis is directed vertically upwards.

The boundary-value problem for the velocity potential $\varphi(x, y, z)$ and the free surface elevation or plate deflection $w(x, y)$ can be written as

$$\Delta_3\varphi = 0 \quad (|x|, |y| < \infty,\ -H \le z \le 0), \quad \Delta_3 \equiv \Delta_2 + \partial^2/\partial z^2, \quad \Delta_2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 \ , \tag{1}$$

$$D\left(1 - \tau U\frac{\partial}{\partial y}\right)\Delta_2^2 w + \rho h U^2\frac{\partial^2 w}{\partial y^2} + g\rho_0 w - \rho_0 U\frac{\partial\varphi}{\partial y}\bigg|_{z=0} = 0 \quad (x < 0,\ x > L,\ |y| < \infty), \tag{2}$$

$$gw - U\frac{\partial\varphi}{\partial y}\bigg|_{z=0} = -\frac{q(x,y)}{\rho_0} \quad (0 < x < L,\ |y| < \infty), \quad \frac{\partial\varphi}{\partial z}\bigg|_{z=0} = -U\frac{\partial w}{\partial y}, \quad \frac{\partial\varphi}{\partial z}\bigg|_{z=-H} = 0, \tag{3}$$

$$\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)w = 0, \quad \frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial x^2} + (2-\nu)\frac{\partial^2}{\partial y^2}\right]w = 0 \quad (x = 0-, L+,\ |y| < \infty). \tag{4}$$

Here $D = Eh^3/[12(1-\nu^2)]$; $E,\ \nu,\ \rho$ are Young's modulus, Poisson's ratio, the density of ice sheets, respectively; $\tau = \eta/E$ is the retardation time, $\eta$ is the viscosity of ice; $g$ is the acceleration due to gravity. For wave motion the decaying conditions should be satisfied far from the pressure region.

We restrict our consideration to the pressure distribution in the form (Doctors and Sharma, 1972)

$$q(x, y) = q_0\{\tanh[\varkappa(y + b)] - \tanh[\varkappa(y - b)]\}[\mathcal{H}(x - x_0 + a) - \mathcal{H}(x - x_0 - a)]/2, \tag{5}$$

where $q_0$ is a nominal pressure, $a$ and $b$ are respectively the half-beam and half-length of the pressure region whose center is located at the point $(x = x_0 > a,\ y = 0)$, $\kappa$ is the smoothing parameter, $\mathcal{H}(\cdot)$ is the Heaviside function.

## 3 METHOD OF SOLUTION

We describe briefly the solution of problem (1)–(4) by the Wiener-Hopf technique. The formulated problem is considered in non-dimensional variables where the fluid depth $H$ is taken as the length scale. The non-dimensional parameters and functions are introduced

$$\beta = \frac{D}{\rho_0 g H^4},\ F = \frac{U}{\sqrt{gH}},\ \sigma = \frac{\rho h}{\rho_0 H},\ Q_0 = \frac{q_0}{\rho_0 g H},\ \epsilon = \frac{\tau U}{H},\ \phi(x,y,z) = \frac{\varphi}{UH},\ W(x,y) = \frac{w}{H}.$$

We use the Fourier transform to the variables $x$ and $y$ in the form

$$\Phi(\alpha,s,z) = \int_{-\infty}^{\infty} \mathrm{e}^{-isy} dy \int_{-\infty}^{\infty} \phi(x,y,z) \mathrm{e}^{i\alpha x} dx.$$

From the Laplace equation (1) and no-flux bottom condition in (3), we have

$$\Phi(\alpha,s,z) = C(\alpha,s) Z(\alpha,s,z), \quad Z = \cosh[(z+1)\sqrt{\alpha^2+s^2}]/\cosh\sqrt{\alpha^2+s^2}, \tag{6}$$

where $C(\alpha,s)$ is unknown function. We introduce the functions $G_\pm(\alpha,s),\ G_1(\alpha,s)$ in the following manner:

$$G_- = \int_{-\infty}^{\infty} \mathrm{e}^{-isy} dy \int_{-\infty}^{0} \Big[\Big(\beta\Delta_2^2 + 1 + \sigma F^2 \frac{\partial^2}{\partial y^2}\Big)\frac{\partial \phi}{\partial z} + F^2 \frac{\partial^2 \phi}{\partial y^2}\Big]_{z=0} \mathrm{e}^{i\alpha x} dx, \tag{7}$$

$$G_1 = \int_{-\infty}^{\infty} \mathrm{e}^{-isy} dy \int_{0}^{L} \Big[\Big(\beta\Delta_2^2 + 1 + \sigma F^2 \frac{\partial^2}{\partial y^2}\Big)\frac{\partial \phi}{\partial z} + F^2 \frac{\partial^2 \phi}{\partial y^2}\Big]_{z=0} \mathrm{e}^{i\alpha x} dx, \tag{8}$$

$$G_+ = \int_{-\infty}^{\infty} \mathrm{e}^{-isy} dy \int_{L}^{\infty} \Big[\Big(\beta\Delta_2^2 + 1 + \sigma F^2 \frac{\partial^2}{\partial y^2}\Big)\frac{\partial \phi}{\partial z} + F^2 \frac{\partial^2 \phi}{\partial y^2}\Big]_{z=0} \mathrm{e}^{i\alpha (x-L)} dx. \tag{9}$$

Similar to these functions, we also introduce functions $D_\pm(\alpha,s),\ D_1(\alpha,s)$ assuming $\beta = \sigma = 0$ in (7)-(9). Using (6), we can write

$$D_-(\alpha,s) + D_1(\alpha,s) + \mathrm{e}^{i\alpha L} D_+(\alpha,s) = C(\alpha,s) K_1(\alpha,s), \tag{10}$$

$$G_-(\alpha,s) + G_1(\alpha,s) + \mathrm{e}^{i\alpha L} G_+(\alpha,s) = C(\alpha,s) K_2(\alpha,s), \tag{11}$$

where $K_1(\alpha,s)$ and $K_2(\alpha,s)$ are the dispersion functions for the surface waves and FGW in a moving coordinate system:

$$K_1(\alpha,s) = \sqrt{\alpha^2+s^2}\tanh\sqrt{\alpha^2+s^2} - F^2 s^2,$$

$$K_2(\alpha,s) = [\beta(1-i\epsilon s)(\alpha^2+s^2)^2 + 1 - \sigma F^2 s^2]\sqrt{\alpha^2+s^2}\tanh\sqrt{\alpha^2+s^2} - F^2 s^2.$$

From boundary conditions (2), (3), we have

$$G_-(\alpha,s) = 0,\ G_+(\alpha,s) = 0,\ D_1(\alpha,s) = isQ(\alpha,s), \tag{12}$$

where $Q(\alpha,s)$ is the Fourier transform of the function $q(x,y)$ in (5)

$$Q(\alpha,s) = \int_{x_0-a}^{x_0+a} dx \int_{-\infty}^{\infty} q(x,y) \mathrm{e}^{i(\alpha x - sy)} dy = \frac{2Q_0 \mathrm{e}^{i\alpha x_0}\sin(\alpha a)\sin(sb)}{\alpha\varkappa\sinh[\pi s/(2\varkappa)]}.$$

Using (10)-(12), we have

$$G_1(\alpha,s) = C(\alpha,s)K_2(\alpha,s),\ D_-(\alpha,s) + isQ(\alpha,s) + \mathrm{e}^{i\alpha L} D_+(\alpha,s) = C(\alpha,s)K_1(\alpha,s).$$

Excluding the function $C(\alpha,s)$ from these equations, we obtain

$$D_-(\alpha,s) + isQ(\alpha,s) + \mathrm{e}^{i\alpha L} D_+(\alpha,s) = G_1(\alpha,s)K(\alpha,s),\ K(\alpha,s) = K_1(\alpha,s)/K_2(\alpha,s).$$

It is known that the dispersion relation for the free surface waves $\mathcal{K}_1(\gamma) \equiv \gamma\tanh\gamma - F^2 s^2 = 0$ has two real roots $\pm\gamma_0$ and the countable set of imaginary roots $\pm\gamma_j,\ j = 1,2,\ldots$. At $\epsilon = 0$, the dispersion relation for FGW $\mathcal{K}_2(\gamma) \equiv (\beta\mu^4 + 1 - \sigma F^2 s^2)\mu\tanh\mu - F^2 s^2 = 0$ has two real roots $\pm\mu_0$, four complex roots $\pm\mu_{-1}$, $\pm\mu_{-2}$, $\mu_{-2} = -\bar{\mu}_{-1}$ (the bar denotes complex conjugation), and the countable set of imaginary roots $\pm\mu_j$, $j = 1, 2, ....$ At $\epsilon \neq 0$, the values of roots are shifted from the real and imaginary axes. It is difficult to find them numerically since they can be close with complex roots $\pm\mu_{-1}$, $\pm\mu_{-2}$. Therefore, we will take structural damping into account approximately, only for the root $\mu_0$, in order to shift the real root into the complex domain. Then the roots of the dispersion relations $K_n(\alpha, s) = 0$ are $\chi_j = \sqrt{\gamma_j^2 - s^2}$ $(n = 1)$ and $\alpha_j = \sqrt{\mu_j^2 - s^2}$ $(n = 2)$. We will take these values in the upper half-plane.

In accordance with the Wiener-Hopf technique, we factorize the function $K(\alpha, s)$:

$$K(\alpha, s) = K_-(\alpha, s)K_+(\alpha, s),\ K_\pm(\alpha, s) = \frac{\mu_{-1}\mu_{-2}}{(\alpha \pm \alpha_{-1})(\alpha \pm \alpha_{-2})} N_\pm(\alpha, s),\ N_\pm = \prod_{j=0}^{\infty} \frac{\mu_j(\alpha \pm \chi_j)}{\gamma_j(\alpha \pm \alpha_j)},$$

where the functions $K_\pm$ are analytical in the upper/lower parts of the complex plane $\alpha$, respectively.

After some algebra we obtain the equation

$$P(\alpha)\left[\frac{D_+(\alpha, s)}{N_+(\alpha, s)} + V_+(\alpha, s) + q_1(s)\Omega_+(\alpha, s)\right] = G_1(\alpha, s)K_-(\alpha, s)\mathrm{e}^{-i\alpha L} - P(\alpha)[V_-(\alpha, s) + q_1(s)\Omega_-(\alpha, s)], \tag{13}$$

where

$$q_1(s) = \frac{\pi s Q_0 \sin(sb)}{\varkappa \sinh[\pi s/(2\varkappa)]},\ \psi(\alpha) = \frac{\mathrm{e}^{i\alpha(x_0+a)} - \mathrm{e}^{i\alpha(x_0-a)}}{\alpha},\ P(\alpha) = \frac{(\alpha + \alpha_{-1})(\alpha + \alpha_{-2})}{\mu_{-1}\mu_{-2}},$$

$$V_\pm(\alpha, s) = \pm\frac{1}{2i\pi}\int_{-\infty\mp i\lambda}^{\infty\mp i\lambda} \frac{D_-(\zeta, s)\mathrm{e}^{-i\zeta L}d\zeta}{(\zeta - \alpha)N_+(\zeta, s)},\quad \Omega_\pm(\alpha, s) = \pm\frac{1}{2i\pi}\int_{-\infty\mp i\lambda}^{\infty\mp i\lambda} \frac{\psi(\zeta)\mathrm{e}^{-i\zeta L}d\zeta}{(\zeta - \alpha)N_+(\zeta, s)}.$$

The functions on the left-hand and right-hand sides of Eq. (13) are analytical in the lower and upper parts of the complex plane $\alpha$, respectively. Then we have analytical function over the entire complex plane $\alpha$. By Liouville's theorem, this function is a polynomial. The degree of the polynomial is determined by the behavior of this function as $|\alpha| \to \infty$ and is equal to one. Consequently, we can write

$$\frac{D_+(\alpha, s)}{N_+(\alpha, s)} + V_+(\alpha, s) + q_1(s)\Omega_+(\alpha, s) = q_1(s)\frac{[c_1(s) + c_2(s)\alpha]}{P(\alpha)}. \tag{14}$$

Similar to Eq. (13), we can get the equation

$$P(-\alpha)\left[\frac{D_-(\alpha, s)}{N_-(\alpha, s)} + R_-(\alpha, s) + q_1(s)\Psi_-(\alpha, s)\right] = G_1(\alpha, s)K_+(\alpha, s) - P(-\alpha)[R_+(\alpha, s) + q_1(s)\Psi_+(\alpha, s)], \tag{15}$$

where

$$R_\pm(\alpha, s) = \pm\frac{1}{2i\pi}\int_{-\infty\mp i\lambda}^{\infty\mp i\lambda} \frac{D_+(\zeta, s)\mathrm{e}^{i\zeta L}d\zeta}{(\zeta - \alpha)N_-(\zeta, s)},\quad \Psi_\pm(\alpha, s) = \pm\frac{1}{2i\pi}\int_{-\infty\mp i\lambda}^{\infty\mp i\lambda} \frac{\psi(\zeta)d\zeta}{(\zeta - \alpha)N_-(\zeta, s)}.$$

Reasoning as above, we can write

$$\frac{D_-(\alpha, s)}{N_-(\alpha, s)} + R_-(\alpha, s) + q_1(s)\Psi_-(\alpha, s) = q_1(s)\frac{[d_1(s) + d_2(s)\alpha]}{P(-\alpha)}. \tag{16}$$

Equations (14), (16) compose the system of two integral equations. Integrals are evaluated by the calculus of residues. Unknown functions $c_1(s)$, $c_2(s)$ in (14) and $d_1(s)$, $d_2(s)$ in (16) are defined from the edge conditions (4). We introduce new variables

$$\xi_j(s) = D_+(\chi_j, s)/[q_1(s)N_+(\chi_j, s)],\quad \zeta_j(s) = D_-(-\chi_j, s)/[q_1(s)N_-(-\chi_j, s)],$$

and as the result we obtain the infinite system of linear algebraic equations for determination of coefficients $\xi_j$, $\zeta_j$ $(j = 0, 1, ...)$; $c_k$, $d_k$ $(k = 1, 2)$, which is solved by the reduction method. If the vehicle moves along the central line of the channel, then $\xi_j = \zeta_j$ and the system of equations becomes simpler.

For some values of the parameter $s$, the corresponding homogeneous system of these equations has nontrivial solutions, which is explained by the existence of edge modes in this problem. The vertical displacements of ice sheets and free surface $W(x, y)$ are determined by performing the inverse Fourier transform.

## 4 NUMERICAL RESULTS

The following input data are used: $E = 5\ GPa$, $\rho = 900\ kg/m^3$, $\nu = 1/3$, $\rho_0 = 10^3\ kg/m^3$, $\tau = 0.7\ s$, $a = 10\ m$, $b = 20\ m$, $\varkappa b = 5$, $x_0 = 25\ m$, $L = 50\ m$, $P_0 = 10^3\ Pa$, $H = 100\ m$. The minimum phase velocities of FGW for these parameters are equal to $12.06,\ 15.59,\ 20.09\ m/s$ for the ice sheet thicknesses $h = 0.5,\ 1,\ 2\ m$, respectively.

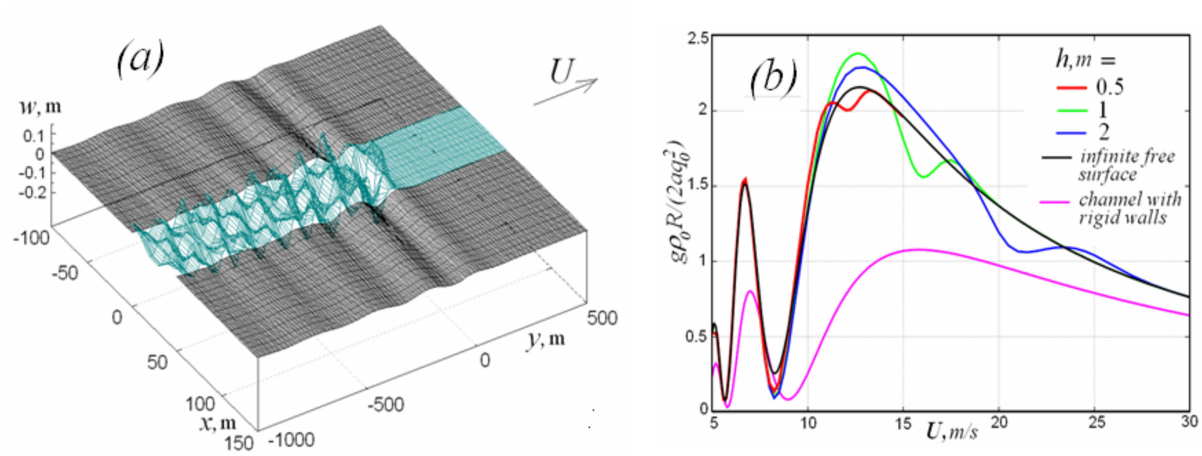


**Figure 1.**

Fig. 1(a) shows the 3-D plot for the function $w(x, y)$ at $h = 2\ m$ for supercritical load speed $U = 21\ m/s$. The dependence of non-dimensional wave resistance of ACV on load speed is presented for different thicknesses of ice in Fig. 1(b), where

$$R = \int_{-b}^{b} \int_{x_0-a}^{x_0+a} q(x, y) \frac{\partial w}{\partial y} dx dy.$$

For comparison, the values of the wave resistance for infinite free surface (Doctors and Sharma, 1972) and for the channel with rigid vertical walls at $x = 0$ and $x = L$ (Newman and Poole, 1962) are shown.

More detailed numerical results will be presented at the Workshop.

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