Water wave scattering by asymmetric trench beneath ice cover

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1 Introduction

In this study we have considered the propagation of normally incident water wave over a rectangular submarine trench in an ice covered ocean. The primary objective of this work is to study the effects of thin ice sheet of various configuration on scattering phenomena. Such study provides insights to help understand how waves are transformed in a marine environment with floating ice sheet in the presence of a rectangular submarine trench. The mathematical model that we analyze treats the ice as a thin elastic plate resting on an inviscid irrotational fluid. Solution of various problems on wave ice interaction considered by a number of researchers in last few decades. One of such study can be found in the work of Meylan and Squire(1993) which describe the reflection and transmission of water waves by a floe of finite length by applying time reversal to the semi-infinite ice sheet problem.

Studies on diffraction of surface waves by an asymmetric submarine trench can be found in the work of Lassister(1972), Kirby and Dalrymple(1983). General investigation of the problem is divided into two parts. First, the fluid domain is divided into three regions with uniform depth and the solution in each region are found. Using conditions along the boundary of the trench a set of integral equations are formed. Finally, the solutions of the integral equations are obtained by using multi-term Galerkin approximations involving ultraspherical Gegenbauer polynomials. The expansions of the velocity potentials are expressed in terms of eigenfunctions. For this we have used the mode coupling relations given in Manam et al.(2006). A set of basis function for the problem of water wave scattering by a thick barrier was considered by Kanoria et al.(1999). Recently, Roy et al.(2017) studied the problem of obliquely incident wave propagation over an asymmetric trench and the reflection as well as transmission coefficients are calculated. In this work we have reconstructed their problem by considering an infinitely large ice sheet at the surface of the fluid. Once the integral equations are solved, the reflection as well as transmission coefficients are obtained easily. An analysis on the numerical results are given for the reflection coefficient for various parameters.

2 Formulation of the problem

We consider an asymmetric trench in an ocean which is covered by an infinite ice sheet of thickness d floats at the surface of water. We take two dimensional coordinate system with y-axis is taken vertically downwards and x-axis is taken along the ice covered surface. The trench being situated between $x = 0$ to $x = l$. The fluid domain is divided into three regions: $R_1(x < 0, 0 < y < h_1)$, $R_2(x > 0, 0 < y < h_2)$ and $R_3(0 < x < l, 0 < y < h_3)$. The fluid is assumed to be inviscid and incompressible. The motion in the fluid is assumed to be irrotational. The velocity potential is $\Phi(\mathrm{x}, \mathrm{y}; \mathrm{t}) = \mathrm{Re}\left[\phi\left(\mathrm{x}, \mathrm{y}\right) \mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}\right]$, where ω being the angular frequency of the wave. Then $\phi(x, y)$ satisfies the Laplace's equation within the fluid region, i.e.,

$$\nabla^2\phi = 0, \quad \text{in the fluid region} \tag{1}$$

along with the ice covered ocean surface condition

$$\left(D\frac{\partial^4}{\partial x^4} + 1 - \epsilon K\right)\phi_y + K\phi = 0, \text{ on } y = 0 \tag{2}$$

where $K = \frac{\omega^2}{g}$, g is the acceleration due to gravity. $\epsilon = \frac{\rho_i}{\rho_w}d$, ρ_w is the density of water and ρ_i is the density of ice sheet. $D = Ed^3/\left\{12\left(1-\nu^2\right)\rho g\right\}$ is the flexural rigidity of ice sheet, E is the Young's modulus, ν is the Poisson's ratio. The boundary conditions at the bottom and on the trench walls are

$$\phi_y = 0, \text{ on } x < 0, y = h_1; 0 < x < l, y = h_3; x > l, y = h_2; \qquad \phi_x = 0, \text{ on } x = 0, h_1 < y < h_3; x = l, h_2 < x < h_3 \tag{3}$$

The edge conditions are given by

$$r^{1/3}\nabla\phi \text{ is bounded as } r \to 0, \text{where } r \text{ being the distance from submerged edge of the trench.} \tag{4}$$

Let a simple harmonic time dependent progressive wave from negative infinity be incident on the trench in R_1. For an incident wave with wavelength λ we denote the wavenumber by $k_0^{(1)}\left(=\frac{2\pi}{\lambda}\right)$ satisfying the dispersion equation

$$u\left(Du^4+1-\epsilon K\right)\tanh uh_1-K=0 \tag{5}$$

The eigenfunction expansion of the velocity potential $\phi(x,y)$ in each of the region can be given by:

$$\phi(x,y)=\begin{cases} Re^{-\mathrm{i}k_0^{(1)}(x-l)}Y_0^{(1)}(y)+\sum\limits_{n=-2}^{\infty}A_n e^{-\mathrm{i}k_n^{(1)}x}Y_n^{(1)}(y), \text{ in } R_1 \\ \sum\limits_{n=-2}^{\infty}B_n e^{\mathrm{i}k_n^{(2)}(x-l)}Y_n^{(2)}(y), \text{ in } R_2 \\ \sum\limits_{n=-2}^{\infty}\left\{C_n\cosh k_n^{(3)}x+D_n\sinh k_n^{(3)}\left(x-l\right)\right\}Y_n^{(3)}(y), \text{ in } R_3 \end{cases} \tag{6}$$

where $B_0=Te^{\mathrm{i}k_0^{(2)}l}$. The parameters R and T described above are the reflection and transmission coefficients to be determined, respectively.

In the expansions we have defined $Y_n^{(q)}=\frac{\cosh k_n^{(q)}(h_q-y)}{\cosh k_n^{(q)}h_q}$, for $q=1,2,3$ and $n=-2,-1,0,1,...\infty$. The eigenfunctions are not orthogonal in general but satisfies the mode-coupling relation given by Manam et al.(2006). The wavenumber $k_n^{(q)}$ satisfies the dispersion equation given in Eqn.(5). In region q we denote the real roots by $k_0^{(q)}$, complex conjugate roots by $k_n^{(q)}, n=-2,-1$ and purely imaginary roots by $k_n^{(q)}, n=1,2,3,....$ The boundary conditions as $x\longrightarrow\pm\infty$ for incident wave of unit amplitude from negative infinity are given by

$$\phi(x,y)\rightarrow\begin{cases}\left(e^{ik_0^{(1)}x}+Re^{-ik_0^{(1)}x}\right)Y_0^{(1)}, \text{ as } x\longrightarrow-\infty \\ Te^{ik_0^{(2)}x}Y_0^{(2)}, \quad \text{as} \quad x\longrightarrow\infty\end{cases} \tag{7}$$

The solutions must satisfy the following matching conditions:
(i) ϕ, ϕ_x are continuous at $x=0, 0<y<h_1$ and (ii) ϕ, ϕ_x are continuous at $x=l, 0<y<h_2$

We consider

$$f_1(y)=\left.\phi_x\right|_{0-}=\left.\phi_x\right|_{0+}, y\in(0,h_1), \qquad f_2(y)=\left.\phi_x\right|_{l-}=\left.\phi_x\right|_{l+}, y\in(0,h_2), \tag{8}$$

The reflection and transmission coefficients described in Eqn.(7) must satisfy the following energy identity

$$|R|^2+J\,|T|^2=1, \text{ where } J=\left(k_0^{(2)}Y_0^{(2)}\right)/\left(k_0^{(1)}Y_0^{(1)}\right) \tag{9}$$

3 Method of solution

The unknown coefficients involved in Eqn.(6) can be obtained by using Eqn.(8) and the mode-coupling relations. Moreover, continuity of $\phi(x,y)$ at $x=0,l$ leads to a set of integral equations which are given by

$$\sum_{q=1}^{2}\int_0^{h_q}M_{pq}(y,u)f_q(u)du=\chi_p(y), p=1,2 \tag{10}$$

where

$$\chi_1(y)=(1+Re^{ik_0^1l})k_0^1\left\langle Y_0^1,Y_0^1\right\rangle Y_0^1(y), \qquad \chi_2(y)=-Te^{ik_0^2l}k_0^2\left\langle Y_0^2,Y_0^2\right\rangle Y_0^2(y) \tag{11}$$

$$\begin{aligned}M_{pq}(y,u)=\delta_0^q&\left[\frac{\cot k_0^3l}{\delta_0^3}Y_0^3(y)Y_0^3(u)+\sum_{n=-2}^{-1}\left\{\frac{\cot k_n^3l}{\delta_n^3}Y_n^3(y)Y_n^3(u)-i\frac{1}{\delta_n^q}Y_n^q(y)Y_n^q(u)\right\}\right.\\ &\left.-\sum_{n=1}^{\infty}\left\{\frac{\coth k_n^3l}{\delta_n^3}Y_n^3(y)Y_n^3(u)-i\frac{1}{\delta_n^q}Y_n^q(y)Y_n^q(u)\right\}\right], \text{ for } p=q=1,2\end{aligned} \tag{12}$$

$$M_{pq}(y,u)=\delta_0^q\left[-\sum_{n=-2}^{0}\frac{\csc k_n^3l}{\delta_n^3}Y_n^3(y)Y_n^3(u)+\sum_{n=-2}^{0}\frac{\csc k_n^3l}{\delta_n^3}Y_n^3(y)Y_n^3(u)\right], \text{ for } p\neq q=1,2 \tag{13}$$

in which $\delta_n^q=k_n^q\left\langle Y_n^q,Y_n^q\right\rangle=\frac{\left(Dk_n^{q^4}+1-\epsilon K\right)2k_n^qh_q+\left(5Dk_n^{q^4}+1-\epsilon K\right)\sinh 2k_n^qh_q}{4k_n^q\left(Dk_n^{q^4}+1-\epsilon K\right)\cosh^2 k_n^qh_q}$ for $q=1,2,3$. The unknown coefficients R and T are given by

$$R=e^{-ik_0^1l}\left(1+i\frac{\left\langle f_1(u),Y_0^1(u)\right\rangle}{\delta_0^1}\right), T=-e^{-ik_0^2l}\frac{\left\langle f_2(u),Y_0^2(u)\right\rangle}{\delta_0^2} \tag{14}$$

We consider $f_p(u) = \left(1 + Re^{ik_0^1 l}\right)\delta_0^1 g_{p1}(u) - Te^{ik_0^2 l}\delta_0^2 g_{p2}(u)$ for $p = 1, 2$, so that are obtain the following set of integral equations:

$$\sum_{q=1}^{2} \int_0^{h_q} M_{pq}(y,u) g_{qr}(u) du = \delta_{pr} Y_0^p(y), 0 < y < h_p \tag{15}$$

and the system of equations:

$$(1 - ib_{11})\, e^{ik_0^1 l}\delta_0^1 R + ie^{ik_0^2 l} b_{12}\delta_0^2 T = (1 + ib_{11})\,\delta_0^1, \quad ie^{ik_0^1 l} b_{21}\delta_0^1 R + (1 - ib_{22})\, e^{ik_0^3 l}\delta_0^2 = -b_{21}\delta_0^1 \tag{16}$$

where δ_{pr} denotes the Kronecker delta symbol and $b_{pq} = \int_0^{h_p} Y_0^p(u) g_{pq}(u) du, p, q = 1, 2$.

4 The multi-term Galerkin approximation technique

At $x = 0, l$ we approximate the functions $g_{pq}(y), p, q = 1, 2$ by another function $\zeta_{pq}(y), p, q = 1, 2$ which are expressed in the form of the truncated series as given by

$$\zeta_{pq}(y) = \sum_{j=0}^{N} c_n^{(pq)} \bar{\zeta}_n^{(p)}(y), \text{ for } p, q = 1, 2 \tag{17}$$

where $\bar{\zeta}_n^{(p)}(y), p = 1, 2$ are suitably chosen basis functions. In this work we have considered a similar set of basis functions as in Kanoria et al.(1999) and we obtain the following set of linear equations

$$\sum_{j=0}^{N} \sum_{q=1}^{2} K_{pq}^{(mn)} c_n^{qr}(u) du = \delta_{pr} X_m^{(p)}, p, r = 1, 2 \tag{18}$$

where

$$X_m^{(p)} = 2I_{2m+1/6}\left(k_0^p h_p\right) / \left(k_0^p h_p\right)^{1/6} \tag{19}$$

$$\begin{aligned} K_{pq}^{(mn)} = 4\delta_0^q \Bigg[& \frac{\cot k_0^3 l}{\delta_0^3} \frac{I_{2m+1/6}\left(k_0^3 h_p\right) I_{2n+1/6}\left(k_0^3 h_p\right)}{\left(k_0^3 h_p\right)^{1/3}} + \sum_{r=-2}^{-1} \Bigg\{ \frac{\cot k_r^3 l}{\delta_r^3} \frac{I_{2m+1/6}\left(k_r^3 h_p\right) I_{2n+1/6}\left(k_r^3 h_p\right)}{\left(k_r^3 h_p\right)^{1/3}} \\ & -i \frac{1}{\delta_r^q} \frac{I_{2m+1/6}\left(k_r^q h_p\right) I_{2n+1/6}\left(k_r^q h_p\right)}{\left(k_r^q h_p\right)^{1/3}} \Bigg\} - \sum_{r=1}^{\infty} \Bigg\{ \frac{\coth k_r^3 l}{\delta_r^3} \frac{J_{2m+1/6}\left(k_r^3 h_p\right) J_{2n+1/6}\left(k_r^3 h_p\right)}{\left(k_r^3 h_p\right)^{1/3}} \\ & -i \frac{1}{\delta_r^q} \frac{J_{2m+1/6}\left(k_r^q h_p\right) J_{2n+1/6}\left(k_r^q h_p\right)}{\left(k_r^q h_p\right)^{1/3}} \Bigg\} \Bigg], \text{ for } p = q = 1, 2 \end{aligned} \tag{20}$$

$$K_{pq}^{(mn)} = 4\delta_0^q \left[-\sum_{r=-2}^{0} \frac{\csc k_r^3 l}{\delta_r^3} \frac{I_{2m+1/6}\left(k_r^3 h_p\right) I_{2n+1/6}\left(k_r^3 h_q\right)}{\left(k_r^3 h_p\right)^{1/6} \left(k_r^3 h_q\right)^{1/6}} + \sum_{r=-2}^{0} \frac{\mathtt{csch} k_r^3 l}{\delta_r^3} \frac{J_{2m+1/6}\left(k_r^3 h_p\right) J_{2n+1/6}\left(k_r^3 h_q\right)}{\left(k_r^3 h_p\right)^{1/6} \left(k_r^3 h_q\right)^{1/6}} \right] \text{ for } p \neq q = 1, 2 \tag{21}$$

$$b_{pq} = \sum_{j=0}^{N} c_j^{pq} X_j^{(p)}, p, q = 1, 2. \tag{22}$$

5 Analysis

To solve the linear system given in Eqn.(18) numerically we choose $N = 40$. Throughout the study we consider $\frac{\epsilon}{h_1} = 0.001$. In Figure 1 the reflection coefficients are drawn for different values of the flexural rigidity. In these figures we observe that the amplitude of reflection coefficients gradually increases with increasing values of $\frac{D}{h_1^4}$.

The graphs of $|R|$ versus $k_0^{(1)} h_1$ for different depth of the trench are given in Figure 2(a). It is observed that the amplitude of $|R|$ increases with increasing values of depth of the rectangular trench.

A comparison of the results with Roy et al.(2017) is obtained in Figure 2(b). It is clear that, in absence of ice cover the present method agree closely with those given in Roy et al.(2017).

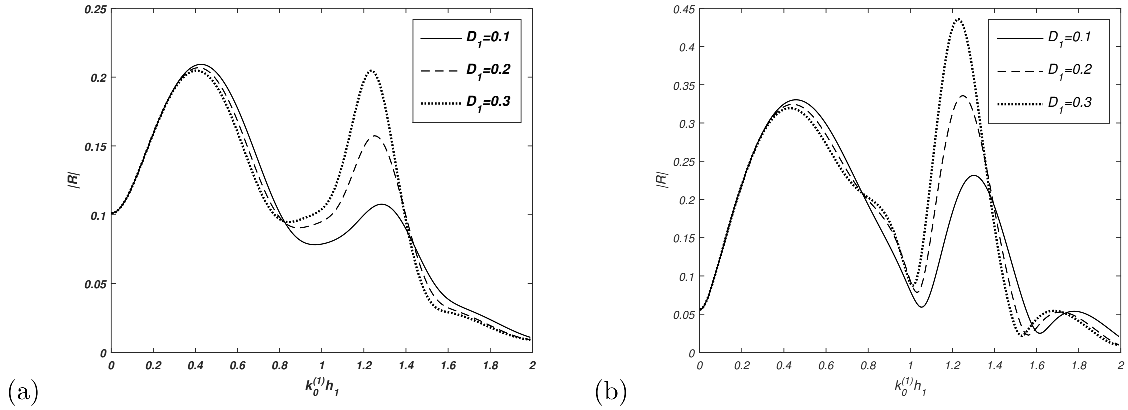

Figure 1: $|R|$ for different values of $D_1(\equiv D/h_1^4)$ for $l/h_1 = 5$, $h_3/h_1 = 2.0$,(a) $h_2/h_1 = 0.5$, (b) $h_2/h_1 = 1.5$.

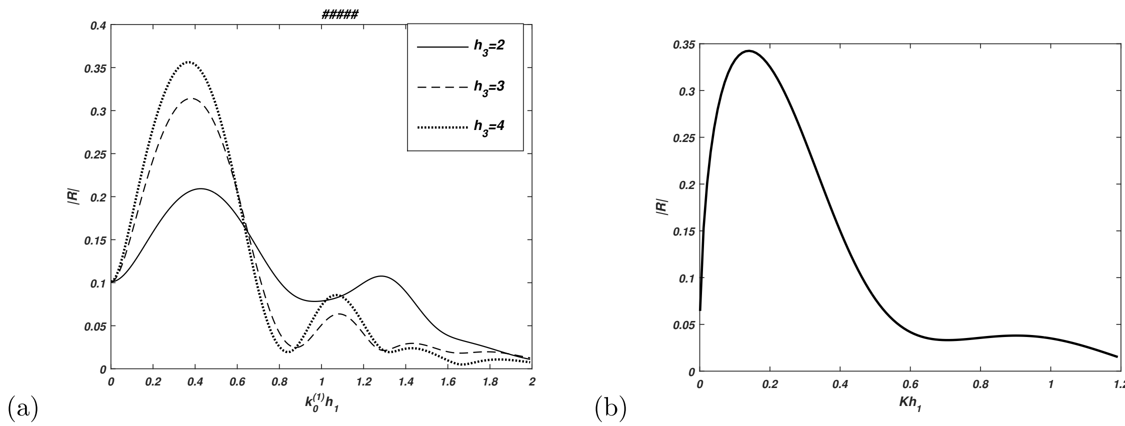

Figure 2: Reflection coefficients for $l/h_1 = 4$, $h_2/h_1 = 2$, (a) $D/h_1^4 = 0.1$,$h_3/h_1 = 2, 3, 4$, (b) $\frac{D}{h_1^4} = 0$, $h_3/h_1 = 5$.

6 Conclusions

The two dimensional velocity potential is formulated in terms of eigenfunctions. Reflection and transmission coefficients are obtained analytically in terms of inner products involving unknown functions. The coefficients are calculated numerically using multi-term Galerkin approximation. From the results it is observed that in presence of ice cover zero or complete reflection is not possible. Moreover, the ice sheet may increase the amplitude of reflection coefficients relative to a specific incident wave mode.

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