Graph algebras, groupoids and shifts of finite type

James Rout

University of Wollongong

26/07/2017

(Joint work with Toke Carlsen)

Introduction

E	Directed graph
$\mathcal{G}_E$	Graph groupoid
$C^*(E)$	Graph C^*-algebra
$L(E)$	Leavitt path algebra
$X_E, \overline{X}_E$	One- and two-sided edge shifts

Groupoids

A **groupoid** is a nonempty set $\mathcal{G}$ satisfying the following properties.

G1. There is a distinguished subset $G^{(0)} \subseteq G$, called the unit space. Elements of $\mathcal{G}^{(0)}$ are called **units**.

G2. There are maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ satisfying $r(u) = s(u) = u$ for all $u \in G^{(0)}$. These maps are called the **range** and **source** maps respectively.

G3. Setting $G^{(2)} := \{(\alpha, \beta) : \alpha, \beta \in \mathcal{G}, s(\alpha) = r(\beta)\} \subseteq \mathcal{G} \times \mathcal{G}$, there is a **'law of composition'**

$$G^{(2)} \to G : (\alpha, \beta) \mapsto \alpha\beta$$

that satisfies

(i) For every $(\alpha, \beta) \in \mathcal{G}^{(2)}$, $r(\alpha\beta) = r(\alpha)$ and $s(\alpha\beta) = s(\beta)$.

(ii) If (α, β) and (β, γ) belong to $\mathcal{G}^{(2)}$, then $(\alpha, \beta\gamma)$ and $(\alpha\beta, \gamma)$ also belong to $\mathcal{G}^{(2)}$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

(iii) For every $\alpha \in \mathcal{G}$, $r(\alpha)\alpha = \alpha = \alpha s(\alpha)$.

G4. For every $\alpha \in \mathcal{G}$ there is an **'inverse'** $\alpha^{-1} \in \mathcal{G}$ (necessarily unique) such that (α, α^{-1}) and (α^{-1}, α) belong to $\mathcal{G}^{(2)}$ and such that $\alpha\alpha^{-1} = r(\alpha)$ and $\alpha^{-1}\alpha = s(\alpha)$.

A **topological groupoid** is a groupoid $\mathcal{G}$ with a topology such that

i. The set of composable pairs $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ is **closed** under the relative topology (automatic when $\mathcal{G}$ is Hausdorff).

ii. Composition $\mathcal{G}^{(2)} \to \mathcal{G} : (\alpha, \beta) \mapsto \alpha\beta$ and inversion $\mathcal{G} \to \mathcal{G} : \alpha \mapsto \alpha^{-1}$ are **continuous**.

A topological groupoid G is called **étale** if the maps $r, s : \mathcal{G} \to \mathcal{G}$ are local homeomorphisms.

Homomorphisms of groupoids

Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A **groupoid homomorphism** is a map $\phi : \mathcal{G} \to \mathcal{H}$ such that $(\phi(\alpha), \phi(\beta)) \in \mathcal{H}^{(2)}$ whenever $(\alpha, \beta) \in \mathcal{G}^{(2)}$, and such that $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$. A **groupoid isomorphism** is a groupoid homomorphism that is bijective.

If $\mathcal{G}$ and $\mathcal{H}$ are topological groupoids, then a **topological groupoid isomorphism** is a groupoid isomorphism that is also a homeomorphism.

If $\mathcal{G}$ is a topological groupoid and $\mathcal{H} = \Gamma$ is a discrete group, then a continuous homomorphism $c : \mathcal{G} \to \Gamma$ is called a **continuous cocycle**. That is, $c : \mathcal{G} \to \Gamma$ carries composition in $\mathcal{G}$ to the group operation in Γ.

The reduced C^*-algebra of a groupoid

Let $\mathcal{G}$ be a locally compact, Hausdorff, étale groupoid. The space $C_c(\mathcal{G})$ is given the structure of a complex $*$-algebra with **convolution**

$$f \cdot g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta),$$

for $\gamma \in \mathcal{G}$, and **involution**

$$f^*(\alpha) = \overline{f(\alpha^{-1})}$$

for $\alpha \in \mathcal{G}$.

The reduced C^*-algebra of a groupoid

For $u \in \mathcal{G}^{(0)}$, there is a representation $\pi_u : C_c(\mathcal{G}) \to B(\ell^2(s^{-1}(u)))$ such that

$$\pi_u(f)\delta_\gamma = \sum_{s(\alpha)=r(\gamma)} f(\alpha)\delta_{\alpha\gamma},$$

for $\gamma \in s^{-1}(u)$.

There is a C^*-norm on $C_c(\mathcal{G})$ defined by

$$\|f\|_r := \sup\{\|\pi_u(f)\|_{\mathrm{op}} : u \in \mathcal{G}^{(0)}\}.$$

The **reduced C^*-algebra** $C_r^*(\mathcal{G})$ of $\mathcal{G}$ is the completion of $C_c(\mathcal{G})$ under this norm.

The boundary path space of a directed graph

Let $E = (E^0, E^1, r, s)$ be a directed graph. Denote by E^* the set of **finite** sequences μ in E and by E^∞ the set of **infinite** sequences x of edges in E such that $r(x_i) = s(x_{i+1})$ for all i.

The **boundary path space** ∂E of E is the space

$$\partial E := E^\infty \cup \{\mu \in E^* : s^{-1}(r(\mu)) \text{ is empty or infinite}\}.$$

For $\mu \in E^*$, define a **cylinder set** $Z(\mu)$ by

$$Z(\mu) := \{\mu x \in \partial E : x \in \partial E, s(x) = r(\mu)\} \subseteq \partial E.$$

The boundary path space of a directed graph

For $\mu \in E^*$ and a finite subset $F \subseteq E^1$ such that $s(f) = r(\mu)$ for all $f \in F$, we define

$$Z(\mu \backslash F) := Z(\mu) \backslash \Big(\bigcup_{f \in F} Z(\mu f) \Big).$$

Sam Webster showed that the collection of all such sets forms a basis for a **locally compact, Hausdorff** topology on ∂E, and each such set is compact and open.

For $n \in \mathbb{N}$, let $\partial E^{\geq n} := \{x \in \partial E : |x| \geq n\} \subseteq \partial E$. Define the **edge shift map** $\sigma_E : \partial E^{\geq 1} \to \partial E$ by

$$\sigma_E(x_1 x_2 x_3 \dots) = x_2 x_3 \dots$$

for $x_1 x_2 x_3 \dots \in \partial E^{\geq 2}$ and $\sigma_E(e) = r(e)$ for $e \in \partial E \cap E^1$.

The groupoid of a directed graph

The **graph groupoid** $\mathcal{G}_E$ of a directed graph E is given by

$$\mathcal{G}_E := \{(x, m-n, y) \in \partial E \times \mathbb{Z} \times \partial E : m, n \in \mathbb{N} \text{ and } \sigma_E^m(x) = \sigma_E^n(y)\},$$

with unit space

$$\mathcal{G}_E^{(0)} = \{(x, 0, x) : x \in \partial E\} \cong \partial E,$$

range and source maps

$$r(x, m-n, y) := x \quad \text{and} \quad s(x, m-n, y) := y,$$

composition

$$(x, m-n, y)(w, m'-n', z) := (x, m+m'-(n+n'), z)$$

whenever $y = w$ and undefined otherwise, and inverse

$$(x, m-n, y)^{-1} := (y, n-m, x).$$

The groupoid of a directed graph

Let $m, n \in \mathbb{N}$, let U be an open subset of $\partial E^{\geq m}$ such that $\sigma_E^m|_U$ is injective, let V be an open subset of $\partial E^{\geq n}$ such that $\sigma_E^n|_V$ is injective, and such that $\sigma_E^m(U) = \sigma_E^n(V)$. Define

$$Z(U, m, n, V) := \{(x, m-n, y) \in \mathcal{G}_E : x \in U, y \in V, \sigma_E^m(x) = \sigma_E^n(y)\}.$$

The graph groupoid $\mathcal{G}_E$ is a **locally compact, Hausdorff, étale** groupoid when equipped with the topology generated by subsets of the form $Z(U, m, n, V)$.

Graph C^*-algebras and general gauge actions

Kumjian, Pask, Raeburn and Renault originally used graph groupoids to construct C^*-algebras for a large class of directed graphs. For a directed graph E, the **graph C^*-algebra** $C^*(E)$ is the groupoid C^*-algebra of $\mathcal{G}_E$.

For $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, let

$$Z(\mu, \nu) := Z(Z(\mu), |\mu|, |\nu|, Z(\nu)).$$

The **graph C^*-algebra** $C^*(E)$ is

$$C^*(\mathcal{G}_E) = \overline{\text{span}}\{1_{Z(\mu,\nu)} : \mu, \nu \in E^*, r(\mu) = r(\nu)\}.$$

The **diagonal subalgebra** of $C^*(E)$, denoted $\mathcal{D}(E)$, is

$$C_0(\partial E) = \overline{\text{span}}\{1_{Z(\mu)} : \mu \in E^*\}.$$

Cocycles and generalised gauge-actions

Let E be a directed graph, and let $k : E^1 \to \mathbb{R}$ be a function. Then k extends to a function $k : E^* \to \mathbb{R}$ given by $k(v) = 0$ for $v \in E^0$ and $k(e_1 \dots e_n) = k(e_1) + \dots + k(e_n)$ for $e_1 \dots e_n \in E^n$, $n \geq 1$.

We then get a **continuous cocycle** $c_k : \mathcal{G}_E \to \mathbb{R}$ given by

$$c_k((\mu x, |\mu| - |\nu|, \nu x)) = k(\mu) - k(\nu)$$

and a **generalised gauge action** $\gamma^{E,k} : \mathbb{R} \to \mathrm{Aut}(C^*(E))$ satisfying

$$\gamma_t^{E,k}(1_{Z(\mu,\nu)}) = e^{it(k(\mu)-k(\nu))} 1_{Z(\mu,\nu)}$$

for $\mu, \nu \in E^*$ and $t \in \mathbb{R}$.

Define $k_E : E^1 \to \mathbb{R}$ by $k_E(e) = 1$ for all $e \in E^1$. Then c_{k_E} is standard continuous cocycle and γ^{E,k_E} is the standard gauge action.

Diagonal-preserving gauge-invariant isomorphism

Theorem (Brownlowe–Carlsen–Whittaker, Carlsen–R)

Let E and F be directed graphs and let $k : E^1 \to \mathbb{R}$ and $l : F^1 \to \mathbb{R}$ be functions. TFAE.

1. *There is an isomorphism $\Phi : \mathcal{G}_E \to \mathcal{G}_F$ satisfying*

$$c_l(\Phi(\eta)) = c_k(\eta)$$

for $\eta \in \mathcal{G}_E$.

2. *There is a $*$-isomorphism $\Psi : C^*(E) \to C^*(F)$ satisfying*

$$\Psi(\mathcal{D}(E)) = \mathcal{D}(F) \text{ and } \gamma_t^{F,l} \circ \Psi = \Psi \circ \gamma_t^{E,k}$$

for $t \in \mathbb{R}$.

Stabilised graphs

Let E be a graph. **"Adding a head"** to a vertex $v \in E^0$ means attaching the following graph to v.

$$\cdots \xrightarrow{e_{4,v}} w_{3,v} \xrightarrow{e_{3,v}} w_{2,v} \xrightarrow{e_{2,v}} w_{1,v} \xrightarrow{e_{1,v}} v$$

Form a **"stabilised graph"** SE by "adding a head" to each vertex and defining

$$(SE)^0 := E^0 \cup \bigcup_{v \in E^0} \{w_{1,v}, w_{2,v}, \dots\}$$

and

$$(SE)^1 := E^1 \cup \bigcup_{v \in E^0} \{e_{1,v}, e_{2,v}, \dots\}$$

and setting $r(e_{i,v}) := w_{i-1,v}$ and $s(e_{i,v}) := w_{i,v}$ for each $v \in E^0$ and $i = 1, 2, \dots$.

Stabilised graph C^*-algebras

Denote by $\mathcal{K}$ the **compact operators** on $\ell^2(\mathbb{N})$ which are generated by the rank-one operators $\{\theta_{i,j} : i, j \in \mathbb{N}\}$. Denote by $\mathcal{C}$ the maximal abelian subalgebra of $\mathcal{K}$ consisting of **diagonal operators** which are generated by the rank-one operators $\{\theta_{i,i} : i \in \mathbb{N}\}$.

Mark Tomforde showed that $C^*(SE)$ is isomorphic to $C^*(E) \otimes \mathcal{K}$. In fact, $\mathcal{D}(SE)$ is isomorphic to $\mathcal{D}(E) \otimes \mathcal{C}$.

For a function $k : E^1 \to \mathbb{R}$, we define a function $\bar{k} : (SE)^1 \to \mathbb{R}$ by $\bar{k}(e) = k(e)$ for $e \in E^1$, and $\bar{k}(e_{i,v}) = 0$ for $v \in E^0$ and $i = 1, 2, \ldots$.

The isomorphism between $C^*(SE)$ and $C^*(E) \otimes \mathcal{K}$ intertwines the gauge actions $\gamma^{SE,\bar{k}}$ and $\gamma^{E,K} \otimes \mathrm{Id}_{\mathcal{K}}$.

Corollary

Let E and F be directed graphs and $k : E^1 \to \mathbb{R}$ and $l : F^1 \to \mathbb{R}$ functions. TFAE.

1. *There is an isomorphism $\Phi : \mathcal{G}_{SE} \to \mathcal{G}_{SF}$ satisfying*

$$c_{\bar{l}}(\Phi(\eta)) = c_{\bar{k}}(\eta)$$

for $\eta \in \mathcal{G}_{SE}$.

2. *There is a $*$-isomorphism $\Psi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ satisfying $\Psi(\mathcal{D}(E) \otimes \mathcal{C}) = \mathcal{D}(F) \otimes \mathcal{C}$ and*

$$(\gamma_t^{F,l} \otimes \mathrm{Id}_{\mathcal{K}}) \circ \Psi = \Psi \circ (\gamma_t^{E,k} \otimes \mathrm{Id}_{\mathcal{K}})$$

for $t \in \mathbb{R}$.

Two-sided edge shifts

Let E be a finite directed graph with no sinks or sources. The **two-sided edge shift** $\overline{X}_E$ is the space

$$\overline{X}_E := \{(x_n)_{n\in\mathbb{Z}} : x_n \in E^1 \text{ and } r(x_n) = s(x_{n+1}) \text{ for all } n \in \mathbb{Z}\}.$$

The **shift map** is the homeomorphism $\overline{\sigma}_E : \overline{X}_E \to \overline{X}_E$ given by

$$\overline{\sigma}_E(\dots x_{-1}x_0x_1\dots) = \dots x_0x_1x_2\dots.$$

If E and F are finite directed graphs with no sinks or sources, then $\overline{X}_E$ and $\overline{X}_F$ are **conjugate** if there is a homeomorphism $\phi : \overline{X}_E \to \overline{X}_F$ such that

$$\overline{\sigma}_F \circ \phi = \phi \circ \overline{\sigma}_E.$$

Two-sided edge shifts and groupoids

Theorem

Let E and F are finite graphs with no sinks or sources. TFAE.

1. *The two-sided edge shifts $\overline{X}_E$ and $\overline{X}_F$ are conjugate.*
2. *There is an isomorphism $\Phi : \mathcal{G}_{SE} \to \mathcal{G}_{SF}$ satisfying*

$$c_{\overline{k}_F}(\Phi(\gamma)) = c_{\overline{k}_E}(\gamma)$$

for $\gamma \in \mathcal{G}_{SE}$.

Corollary (Cuntz–Krieger, Carlsen–R)

Let E and F are finite graphs with no sinks or sources. TFAE.

1. *The two-sided edge shifts $\overline{X}_E$ and $\overline{X}_F$ are conjugate.*
2. *There is a $*$-isomorphism $\Psi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ satisfying $\Psi(\mathcal{D}(E) \otimes \mathcal{C}) = \mathcal{D}(F) \otimes \mathcal{C}$ and*

$$(\gamma_t^{F,k_F} \otimes \mathrm{Id}_{\mathcal{K}}) \circ \Psi = \Psi \circ (\gamma_t^{E,k_E} \otimes \mathrm{Id}_{\mathcal{K}})$$

for $t \in \mathbb{R}$

Let E be a graph and R a commutative integral domain with identity. The **Leavitt path algebra** $L_R(E)$ is the R-algebra

$$\mathrm{span}_R\{1_{Z(\mu,\nu)} : \mu, \nu \in E^*, r(\mu) = r(\nu)\},$$

and the **diagonal subalgebra** $D_R(E)$ is the R-algebra

$$\mathrm{span}_R\{1_{Z(\mu)} : \mu \in E^*\}.$$

Let $k : E^1 \to \mathbb{R}$ be a function and write Γ_k for the additive subgroup $\{k(\mu) - k(\nu) : \mu, \nu \in E^*, r(\mu) = r(\nu)\}$ of $\mathbb{R}$. For $g \in \Gamma_k$, setting

$$L_R(E)_g := \mathrm{span}_R\{1_{Z(\mu,\nu)} : \mu, \nu \in E^*, r(\mu) = r(\nu), k(\mu) - k(\nu) = g\},$$

gives a **Γ_k-grading** $L_R(E) = \bigoplus_{g \in \Gamma_k} L_R(E)_g$. The standard $\mathbb{Z}$-grading is obtained when $k = k_E$.

Theorem (Ara–Bosa–Hazrat–Sims, Carlsen–R)

Let E and F be directed graphs and let $k : E^1 \to \mathbb{R}$ and $l : F^1 \to \mathbb{R}$ be functions. TFAE.

1. *There is an isomorphism $\Phi : \mathcal{G}_E \to \mathcal{G}_F$ satisfying*

$$c_l(\Phi(\eta)) = c_k(\eta)$$

for $\eta \in \mathcal{G}_E$.

2. *There is a ring-isomorphism $\Psi : L_R(E) \to L_R(F)$ satisfying*

$$\Psi(D(E)) = D(F) \text{ and } \Psi(L_R(E)_g) = L_R(F)_g$$

for $g \in c_k(\mathcal{G}_E)$.

Let $M_\infty(R)$ denote the ring of finitely supported, countable **infinite square matrices** over R, and $D_\infty(R)$ the abelian subring of $M_\infty(R)$ consisting of **diagonal matrices**.

Corollary

If E and F are finite graphs with no sinks or sources and R is a commutative integral domain with identity, then TFAE.

1. *The two-sided edge shifts $\overline{X}_E$ and $\overline{X}_F$ are conjugate.*
2. *There is a ring-isomorphism*

$\Psi : L_R(E) \otimes M_\infty(R) \to L_R(F) \otimes M_\infty(R)$ *such that*

$$\Psi(D_R(E) \otimes D_\infty(R)) = D_R(F) \otimes D_\infty(R))$$

and

$$\Psi(L_R(E)_n \otimes M_\infty(R)) = L_R(F)_n \otimes M_\infty(R)$$

References

- P. Ara, J. Bosa, R. Hazrat and A. Sims, *Reconstruction of graded groupoids from graded Steinberg algebras*, to appear in Forum Math. (2016).
- N. Brownlowe, T.M. Carlsen, and M.F. Whittaker, *Graph algebras and orbit equivalence*, Ergodic Theory Dynam. Systems (10) **37** (2017) 389–417.
- T.M. Carlsen and J. Rout, *Diagonal-preserving gauge-invariant isomorphisms of graph C^*-algebras*, to appear in J. Funct. Anal. (2017).
- T.M. Carlsen and J. Rout, *Diagonal–preserving graded isomorphisms of Steinberg algebras*, to appear in Commun. Contemp. Math. (2017).

References

- J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980) 251–268.
- A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids, and Cuntz–Krieger algebras*, J. Funct. Anal. **144** (1997) 505–541.
- M. Tomforde, *Stability of C^*-algebras associated to graphs*, Proc. Amer. Math. Soc. **132** (2004), 1787–1795.
- S. Webster, *The path space of a directed graph*, Proc. Amer. Math. Soc. **142** (2014), 213–225.

Thanks!