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Lipschitz Stability in Variational Analysis

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Contents

The four theorems by Lawrence M. Graves

- The Hildebrand-Graves theorem (1927)
- The Lyusternik-Graves theorem (1932,1950)
- The Bartle-Graves theorem (1952)
- The Karush-Kuhn-Tucker theorem (1939)

Hildebrand–Graves inverse function theorem (TAMS 1927)

Lipschitz modulus

$$\mathrm{lip}(f;\bar{x}) := \limsup_{\substack{x',x\to\bar{x},\\ x\neq x'}} \frac{\|f(x')-f(x)\|}{\|x'-x\|}.$$

Theorem (Hildebrandt-Graves slightly extended)

Let X be a Banach space and consider a continuous function $f : X \to X$ with $f(\bar{x}) = 0$ and a linear bounded mapping $A : X \to X$ which is invertible. Suppose that

$$\mathrm{lip}(f(\cdot) - A(\cdot - \bar{x});\bar{x}) \cdot \|A^{-1}\| < 1.$$

Then the inverse f^{-1} has a **single-valued localization** around 0 for $\bar{x}$ which is Lipschitz continuous near 0.

Reformulation of H-G theorem

Theorem.

Let X be a Banach space and consider a continuous function $f : X \to X$ with $f(\bar{x}) = 0$ and a linear bounded mapping $A : X \to X$ such that

$$\operatorname{lip}(f(\cdot) - A(\cdot - \bar{x}); \bar{x}) \leq \mu.$$

Suppose that the mapping $(A(\cdot - \bar{x}))^{-1}$ has a Lipschitz continuous single-valued localization at 0 for $\bar{x}$ with Lipschitz modulus $\leq \kappa$. Let

$$\kappa\mu < 1.$$

Then the inverse f^{-1} has a **single-valued localization** around 0 for $\bar{x}$ which is Lipschitz continuous near 0.

From H-G theorem to Robinson's theorem

Theorem.

Let X be a Banach space and consider a continuous function $f : X \to X$ with $f(\bar{x}) = 0$ and a linear bounded mapping $A : X \to X$ such that

$$\mathrm{lip}(f(\cdot) - A(\cdot - \bar{x}); \bar{x}) \leq \mu.$$

Let G be any set-valued mapping from X to X with $0 \in G(\bar{x})$ for which the mapping $(A(\cdot - \bar{x}) + G(\cdot))^{-1}$ has a Lipschitz continuous single-valued localization around 0 for $\bar{x}$ with Lipschitz constant κ. Let

$$\kappa\mu < 1.$$

Then the inverse $(f + G)^{-1}$ has a **single-valued localization** around 0 for $\bar{x}$ which is Lipschitz continuous near 0.

The smooth Robinson's inverse function theorem

F is **strongly metrically regular** at $\bar{x}$ for $\bar{y}$ if $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and F^{-1} has a **Lipschitz continuous single-valued localization** around $\bar{y}$ for $\bar{x}$. The Lipschitz modulus of the localization is denoted $\operatorname{reg}(F; \bar{x} \,|\, \bar{y})$.

Theorem.

Let X be a Banach space, let $f : X \to X$ be strictly differentiable at $\bar{x}$ and let $f(\bar{x}) = 0$. Let $F : X \rightrightarrows X$ be a set-valued mapping with $\bar{y} \in F(\bar{x})$. Then FAE:

1) $Df(\bar{x})(\cdot - \bar{x}) + F$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$;

2) $f + F$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$.

Moreover,

$$\operatorname{reg}(Df(\bar{x})(\cdot - \bar{x}) + F; \bar{x} \,|\, \bar{y}) = reg(f + F; \bar{x} \,|\, \bar{y}).$$

A general Robinson's theorem in metric spaces

Theorem.

Let X be a complete metric space, Y be a linear normed space
1) κ and μ positive constants with $\kappa\mu < 1$.
2) $F : X \rightrightarrows Y$ is such that F is strongly regular at $\bar{x}$ for $\bar{y}$ with $\mathrm{reg}(F; \bar{x} \,|\, \bar{y}) \le \kappa$.
3) $f : X \to Y$ with $f(\bar{x}) = 0$ and $\mathrm{lip}(f; \bar{x}) \le \mu$.
Then $f + F$ is strongly regular at $\bar{x}$ for $\bar{y}$ with

$$\mathrm{reg}(f + F; \bar{x} \,|\, \bar{y}) \le (\kappa^{-1} - \mu)^{-1}.$$

Robinson's theorem for Newton's Method

Newton's method for a parameterized VI

$$x_0 = a, \quad f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x) \ni p$$

Consider the mapping

$$\boldsymbol{R}^n \times \boldsymbol{R}^n \ni (a, p) \mapsto \Xi(u, p) = \Bigg\{ \{x_k\} \in l_\infty \,\Bigg|\, x_0 = a,$$

$$f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \ni p, \ \forall k = 1, 2, \dots \Bigg\}$$

Theorem.

The mapping Ξ has a Lipschitz continuous single-valued localization around $(\bar{x}, 0)$ for $\{\bar{x}\}$ if and only if $f + F$ is strongly regular at $\bar{x}$ for 0. Moreover, each value of the localization of Ξ is a quadratically convergent sequence.

A nonsmooth Robinson's theorem (A. Izmailov)

Theorem (finite dimensions).

Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be Lipschitz continuous around $\bar{x}$, let $F : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$, and let $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Suppose for every $A \in \partial f(\bar{x})$ the mapping

$$G_A : x \mapsto f(\bar{x}) + A(x - \bar{x}) + F(x)$$

is strongly metrically regular at $\bar{x}$ for $\bar{y}$. Then the mapping $f + F$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$.

For $F \equiv 0$ reduces to Clarke's IFT. For f smooth reduces to Robinson's theorem.

A nonsmooth Robinson's theorem 1 (AD and R. Cibulka)

Theorem (Banach spaces).

Let X and Y be Banach spaces, let $(\bar{x}, \bar{y}) \in X \times Y$, let $f : X \to Y$ and $F : X \rightrightarrows Y$ be such that $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Suppose that there exist a convex subset $\mathcal{A}$ of $\mathcal{L}(X, Y)$ and a constant $c > 0$ such that

1) there exists $r > 0$ such that for each u and v in $B_r(\bar{x})$ one can find $A \in \mathcal{A}$ such that

$$\|f(v) - f(u) - A(v - u)\| \leq c\|v - u\|;$$

....

A nonsmooth Robinson's theorem 2 (AD and R. Cibulka)

....

2) for every $A \in \mathcal{A}$ the mapping

$$G_A : x \mapsto f(\bar{x}) + A(x - \bar{x}) + F(x)$$

is strongly metrically regular at $\bar{x}$ for $\bar{y}$; moreover, if s_A is a single-valued graphical localization of G_A^{-1} around $\bar{y}$ for $\bar{x}$, then

$$(c + \chi(\mathcal{A})) \sup_{A \in \mathcal{A}} \mathrm{reg}(G_A; \bar{x} \,|\, \bar{y}) < 1.$$

Then the mapping $f + F$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$.

For $F \equiv 0$ reduces to an IFT by Páles (1997).

Proof in finite dimensions

Lemma 1. there exist constants α, β and ℓ such that for every $A \in \bar{\partial} f(\bar{x})$, the mapping

$$\boldsymbol{B}_{2\beta}(0) \ni y \mapsto s_A := G_A^{-1}(y) \cap \boldsymbol{B}_{2\ell\beta}(\bar{x})$$

is a Lipschitz continuous function with Lipschitz constant ℓ.

Lemma 2. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, x' \in \boldsymbol{B}_\delta(\bar{x})$ there exists $A \in \partial f(\bar{x})$ with the property

$$|f(x) - f(x') - A(x - x')| \leq \varepsilon |x - x'|.$$

Lemma 3. The function

$$\partial f(\bar{x}) \times \boldsymbol{B}_\beta(0) \ni (A, z) \mapsto \varphi(A, z) := s_A(z) \cap \boldsymbol{B}_{\ell\beta}(\bar{x})$$

has the following properties:

(a) $\mathrm{dom}\,\varphi = \bar{\partial} f(\bar{x}) \times \boldsymbol{B}_\beta(0)$;

(b) For each $A \in \partial f(\bar{x})$ the function $\varphi(A, \cdot) = s_A$ is Lipschitz continuous on $\boldsymbol{B}_\beta(0)$ with Lipschitz constant ℓ;

(c) For each $A \in \partial f(\bar{x})$ one has $\varphi(A, 0) = \bar{x} = s_A(0)$;

(d) φ is continuous in its domain.

Lemma 4. Fix $u \in \boldsymbol{B}_\delta(\bar{x})$ and define the function

$$\partial f(\bar{x}) \ni A \mapsto \Phi_u(A) = \varphi\big(A, y - f(u) + f(\bar{x}) + A(u - \bar{x})\big) - u.$$

Suppose that there exist $v \in \boldsymbol{B}_\delta(\bar{x}) \setminus \{u\}$ along with $\tilde{A} \in \partial f(\bar{x})$ satisfying

$$|f(v) - f(u) - \tilde{A}(v - u)| \leq \varepsilon |v - u| \quad \text{and} \quad f(v) + \tilde{A}(u - v) + F(u) \ni y.$$

Then

$$0 < |\Phi_u(A)| \leq \ell\varepsilon |u - v| \;\text{ whenever }\; A \in \bar{\partial} f(\bar{x}).$$

Define the mapping

$$K \ni h \mapsto \Psi_u(h) = \{A \in \partial f(\bar{x}) : \ |f(u+h) - f(u) - Ah| \le \varepsilon |h|\} .$$

Lemma 5. Given $u \in B_\delta(\bar{x})$, suppose that Φ_u maps $\partial f(\bar{x})$ into K. Then there exists a continuous selection ψ_u of the mapping Ψ_u such that the function defined as the composition $\psi_u \circ \Phi_u$ has a fixed point in K.

Lemma 6. There exist sequences $\{x_n\}$ in $\boldsymbol{R}^n$ and $\{A_n\}$ in $\partial f(\bar{x})$ whose entries have the following properties for each n:

(i) $|x_n - \bar{x}| < \delta$;

(ii) $0 < |x_{n+1} - x_n| \leq (l\varepsilon)^n |x_1 - x_0|$;

(iii) $|f(x_{n+1}) - f(x_n) - A_n(x_{n+1} - x_n)| \leq \varepsilon |x_{n+1} - x_n|$;

(iv) $f(x_n) + A_n(x_{n+1} - x_n) + F(x_{n+1}) \ni y$.

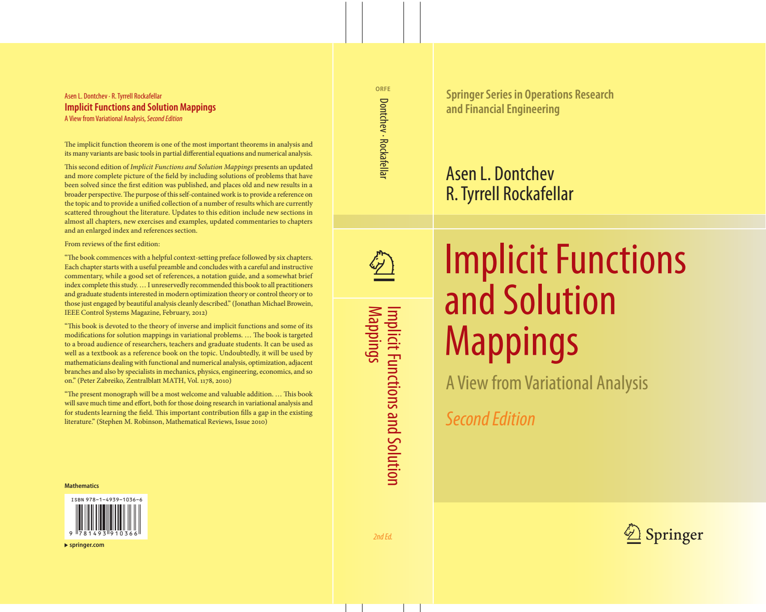
Lemma 7. The mapping

$$\boldsymbol{B}_{\varepsilon b}(0) \ni y \mapsto \sigma(y) := (f + F)^{-1}(y) \cap \boldsymbol{B}_{\delta}(\bar{x})$$

is a nonempty-valued localization of $(f + F)^{-1}$.

End of proof. The map σ is both single-valued and Lipschitz continuous.

Most of the results in this talk can be found in



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Implicit Functions and Solution Mappings

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Implicit Functions and Solution Mappings

A View from Variational Analysis

Second Edition

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Some Open Problems

1. Applications of the nonsmooth Robinson's theorem – PDE constrained optimization? (Ito and Kunish, Ulbrich)

2. Are there a nonsmooth Lyusternik-Graves theorem or a nonsmooth Bartle-Graves theorem?

3. Is there a Nash-Moser version of Robinson's or Lyusternik-Graves theorem?

4. Find the radius of strong regularity for mappings with specific structure, e.g. the KKT system.

THANK YOU!