Expectations on Fractal IFS Attractors

Michael Rose

Principal Supervisor: Laureate Professor Jon Borwein
Co-supervisor: Associate Professor Brailey Sims

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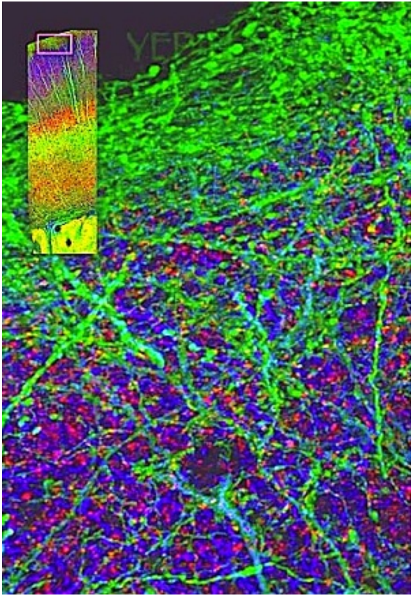
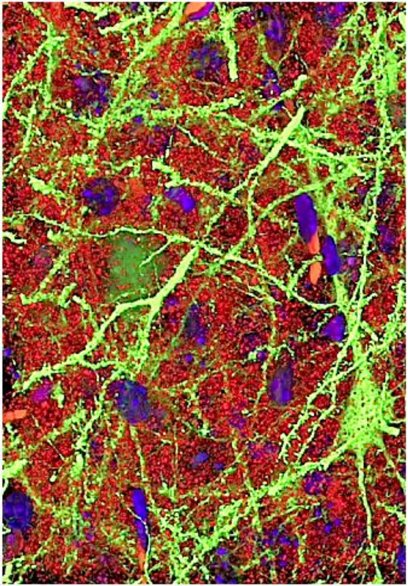
CARMA

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

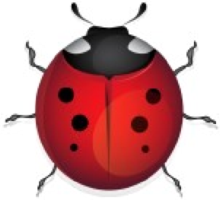
Synapse spatial distributions





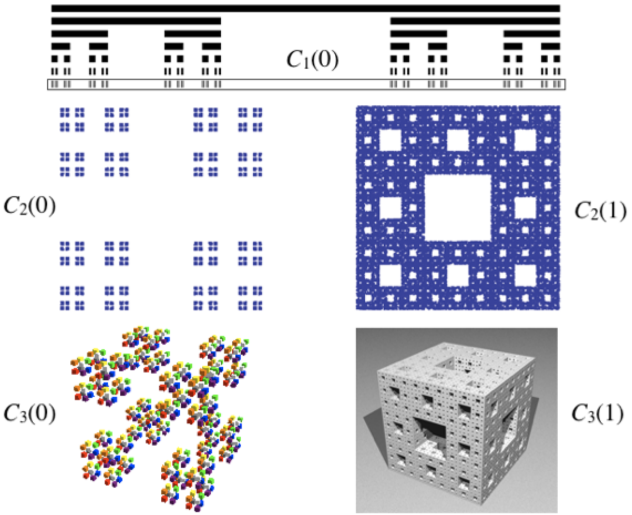
Outline:

1. Summary of SCS results
2. Extension to IFS attractors
3. Examples



Summary of SCS results

String-generated Cantor Sets



D.H. Bailey, J.M. Borwein, R.E. Crandall and M.G. Rose,
Expectations on fractal sets, J. Appl. Math. Comput. 220 (2013).

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A helpful **counting function**:

$$\begin{aligned} U(c) &:= \#\{1\text{'s in ternary vector } c\} \\ Z(b) &:= \#\{0\text{'s in ternary vector } b\} \end{aligned}$$

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with notational periodicity assumed: $P_{p+k} := P_k$ *for all* $k \geq 1$*.*

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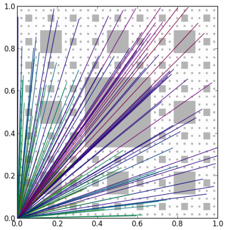
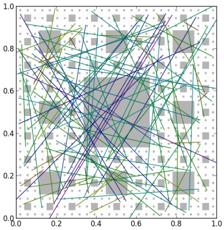
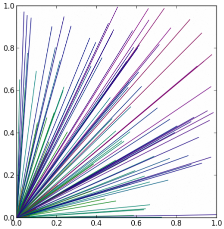
Fractal Box Integrals



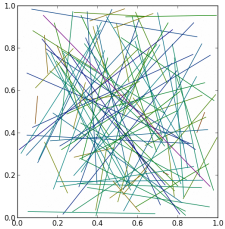
$B(2, C_2(1)) = \frac{11}{16} = 0.6875$



$\Delta(2, C_2(1)) = \frac{3}{8} = 0.375$



$B(2, C_2(2)) = \frac{2}{3} = 0.66\ldots$



$\Delta(2, C_2(2)) = \frac{1}{3} = 0.33\ldots$

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$$\langle\, F(r)\,\rangle_{r\in C_n(P)} \;:=\; \lim_{j\to\infty} \frac{1}{N_1\cdots N_j} \sum_{U(c_i)\le P_i} F(c_1/3 + c_2/3^2 + \cdots + c_j/3^j)$$

$$\langle\, F(r-q)\,\rangle_{r,q\in C_n(P)} \;:=$$

$$\lim_{j\to\infty} \frac{1}{N_1^2\cdots N_j^2} \sum_{\substack{U(c_i)\le P_i \\ U(d_i)\le P_i}} F((c_1-d_1)/3 + \cdots + (c_j-d_j)/3^j)$$

when the respective limits exist.

Useful formulation of expectation

Next, determine a probability measure such that

$$\langle\, F(r)\,\rangle_{r\in C_n(P)} \;=\; \int_{r\in[0,1]^n} F(r)\,\phi(r)\,\mathcal{D}r$$

where ϕ is a **probability density** that vanishes on inadmissible $r \in [0,1]^n \setminus C_n(P)$.

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$$\phi(r) \;=\; \frac{3^{pn}}{\prod_{k=1}^{p} N_k} \sum_{U(c_k)\leq P_k} \phi(3^p(r - \sum_{j=1}^{p} \frac{c_j}{3^j}))$$

$$\Phi(d := r - q) \;=\; \frac{3^{pn}}{\prod_{k=1}^{p} N_k^2} \sum_{\substack{Z(b_k)\leq P_k \\ Z(a_k)\leq P_k}} \Phi\left(3^p(d - \sum_{j=1}^{p} \frac{(b_j - a_j)}{3^j})\right)$$

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Special case - second moments

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Theorem (Closed forms for $B(2, C_n(P))$ and $\Delta(2, C_n(P))$)

For any embedding dimension n and SCS $C_n(P)$ the box integral $B(2, C_n(P))$ is **rational**, *given by the closed form:*

$$B(2, C_n(P)) \;=\; \frac{n}{4} + \frac{1}{1-9^{-p}} \sum_{k=1}^{p} \frac{1}{9^k} \frac{\sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}(n-j)}{\sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}}$$

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and the corresponding box integral $\Delta(2, C_n(P))$ is also **rational**, *given by:*

$$\Delta(2, C_n(P)) \;=\; 2B(2, C_n(P)) - \frac{n}{2}$$

Special case - second moments

The first few cases for period-1 strings P are:

$$B(2, C_n(0)) = \frac{3}{8}n$$

$$B(2, C_n(1)) = \frac{n(3n+5)}{8n+16}$$

$$B(2, C_n(2)) = \frac{n(3n^2+7n+22)}{8n^2+24n+64}$$

$$B(2, C_n(n-1)) = \frac{n}{4}\left(1 + \frac{3^{n-1}}{3^n - 1}\right)$$

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The classical box integrals over the unit n-cube are:

$$B_n(2) \ = \ \frac{n}{3} \quad \text{and} \quad \Delta_n(2) \ = \ \frac{n}{6}$$

which matches the output of our closed forms when $P = n$.

Special case - complex poles

Powerful **self-similarity relations** for $C_1(0)$ follow from the functional expectation relations:

$$B(s, C_1(0)) := \langle |r^s| \rangle_{r \in C_1(0)} = \frac{1}{2} \left\langle \left(\frac{r}{3}\right)^s \right\rangle + \frac{1}{2} \left\langle \left(\frac{r+2}{3}\right)^s \right\rangle$$

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Theorem (Poles of $B(s, C_n(P))$)

For any embedding dimension n and any SCS $C_n(P)$, the (analytically continued) box integral $B(s, C_n(P))$ has a **pole** *at*

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Note that for the full unit n-cube $[0, 1]^n$, the pole is at $s = -n$. This is consistent with the classical theory.

Extension to IFS attractors

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Definition

For each $i \in \{1, 2, \ldots, m\}$ (where $m \geq 2$), let $f_i : X \to X$ be a contraction mapping with contractivity factor $0 < c_i < 1$ and associated probability $0 < p_i < 1$ (where $\sum_{i=1}^{m} p_i = 1$). A (hyperbolic) iterated function system (IFS) with probabilities is the collection

$$\{X; w_1, \ldots, w_m; c_1, \ldots, c_m; p_1, \ldots, p_m\}$$

.

IFS Attractors

Theorem

Let $\{X; w_1, \ldots, w_m\}$ *be an IFS with contractivity factor* c*. Then the transformation* $f : H(X) \to H(X)$ *defined by* $f(S) = \bigcup_{n=1}^{m} f_n(S)$ *for all* $S \in H(X)$ *is a contraction mapping on* $H(X)$ *with contractivity factor* s*.*

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Theorem (The Contraction Mapping Theorem)

The mapping f possesses a unique fixed point $A \in H(X)$, which satisfies:

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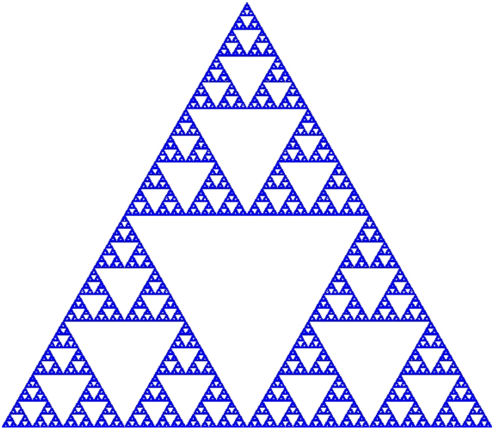
We will take as our 'fractal sets' those sets that can be expressed as the attractor of a (non-overlapping) IFS.

IFS Attractors



$$f_1(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$$

$$f_2(x, y) = \left(\frac{x+1}{2}, \frac{y+\sqrt{3}}{2}\right)$$

$$f_3(x, y) = \left(\frac{x+2}{2}, \frac{y}{2}\right)$$

SCS in IFS framework

Any given SCS can be expressed as the attractor of an IFS in the following manner:

Proposition

The IFS corresponding to the SCS $C_n(P)$ is:

$$\{[0,1]^n \subset \mathbb{R}^n; f_1, f_2, \ldots, f_i, \ldots, f_m\} \tag{1}$$

where $f_i(x) = \left(\frac{1}{3}\right)^p x + \left(\frac{1}{3}\right) c_{1_i} + \left(\frac{1}{3}\right)^2 c_{2_i} + \ldots + \left(\frac{1}{3}\right)^p c_{p_i}$ *for* $i \in \{1, 2, \ldots, m\}$ *ranging over all admissible columns* c_k*, where* $m = \prod_{k=1}^{p} N_k$ *and* $N_k = \sum_{j=0}^{P_k} \binom{n}{j} 2^{n-j}$*.*

Expectations on IFS Attractors

Expectations over IFSs

Definition (Fundamental definition of expectation)

Let $\{X; f_1, \ldots, f_m\}$ be an IFS with attractor $A \in H(X)$. Let $F : X \to \mathbb{C}$ be a complex-valued function over X. The expectation of F over A, $\langle F(x) \rangle_{x \in A}$, is defined as:

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for any $x_0 \in A$, when the limit exists.

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Intuitively we evaluate the expectation over ever-finer pre-fractal sets and examine the limit as the resolution grows ever-finer. This definition is more elegantly stated using code-space ideas.

Code Space

Definition

Given an IFS $\{X; f_1, \ldots, f_m\}$, the associated code space Σ_m is defined as:

$$\Sigma_m := \{\sigma = \sigma_1\sigma_2 \ldots \ : \ \sigma_i \in \{0, 1, \ldots, m-1\} \quad \forall i \in \mathbb{N}\}$$

Code Space

Definition

Given an IFS $\{X; f_1, \ldots, f_m\}$, the associated code space Σ_m is defined as:

$$\Sigma_m := \{\sigma = \sigma_1\sigma_2 \ldots \ : \ \sigma_i \in \{0, 1, \ldots, m-1\} \quad \forall i \in \mathbb{N}\}$$

The code space metric is defined by:

$$d_\Sigma(\sigma, \omega) = d_\Sigma(\sigma_1\sigma_2 \ldots, \omega_1\omega_2 \ldots) := \sum_{k=1}^{\infty} \frac{|\sigma_k - \omega_k|}{(m+1)^k}$$

Expectations on IFS Attractors

Definition (Fundamental definition of expectation (using code-space))

Let $\{X; f_1, \ldots, f_m\}$ be an IFS with attractor $A \in H(X)$. Let $F : X \to \mathbb{C}$ be a complex-valued function over X. The expectation of f over A, $\langle f(x) \rangle_{x \in A}$, is defined as:

$$\langle F(x) \rangle_{x \in A} \; := \; \lim_{j \to \infty} \frac{1}{m^j} \sum_{\sigma_j \in \Sigma_m(j)} F\left(\phi(\sigma_j)\right)$$

when the limit exists.

Expectations on IFS Attractors

Corollary

(Fundamental definition of separation (using code-space)) Let $\{X; f_1, \ldots, f_m\}$ *be an IFS with attractor* $A \in H(X)$*. Let* $F : X \to \mathbb{C}$ *be a complex-valued function over* X*. The separation expectation of* F *over* A*,* $\langle F(x-y)\rangle_{x,y\in A}$*, is defined as:*

$$\langle F(x,y)\rangle_{x,y\in A} \; := \; \lim_{j\to\infty} \frac{1}{m^{2j}} \sum_{\sigma_j \in \Sigma_m(j)} \sum_{\tau_j \in \Sigma_m(j)} F\left(\phi(\sigma_j - \tau_j)\right)$$

when the limit exists.

The invariant IFS measure

Definition

(Falconer) A measure μ on X is invariant for a mapping $f : X \to X$ if for every subset $B \subset X$ we have

$$\mu\left(f^{-1}(A)\right) = \mu\left(A\right)$$

A measure μ on X is normalised if $\mu(X) = 1$.

Definition

Let B be a Borel subset of a metric space (X, d). The residence measure is defined as:

$$\mu(B) := \lim_{n\to\infty} \frac{1}{n} \# \left\{ k \ : \ f^k(x) \in B, \ 1 \leq k \leq n \right\} \tag{3}$$

Ergodic theory shows that this limit exists and is identical for μ-almost all points in the basin of attraction.

The invariant IFS measure

Corollary

The residence measure is an invariant measure over the attractor of any IFS.

The invariant IFS measure

Theorem (Elton's Theorem - special case)

Let (X, d) *be a compact metric space and let* $\{X; w_1, \ldots, w_m; c_1, \ldots, c_m; p_1, \ldots, p_m\}$ *be a hyperbolic IFS. Let* $\{x_n\}_{n=0}^{\infty}$ *denote a chaos game orbit of the IFS starting at* $x_0 \in X$*, that is,* $x_m = w_{\sigma_n} \circ \ldots \circ w_{\sigma_1}(x_0)$ *where the maps are chosen independently according to the probabilities* $p_1, \ldots, p_m$ *for* $n \in \mathbb{N}$*. Let* μ *be the unique invariant measure for the IFS. Then, with probability* 1 *(i.e. for all code sequences excepting a set having probability* 0*),*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x_k) = \int_X f(x)\mathrm{d}\mu(x) \tag{4}$$

Elton's Theorem

Corollary

(Barnsley) Let B be a Borel subset of X and let $\mu(B') = 0$ (where B' is the boundary of B. Then, with probability 1,

$$\mu(B) = lim_{n\to\infty} \frac{\#\{x_0, x_1, \ldots, x_n\} \cap B}{n+1} \tag{5}$$

for all $x_0 \in X$.

Elton's Theorem

Corollary

(Barnsley) Let B be a Borel subset of X and let $\mu(B') = 0$ (where B' is the boundary of B. Then, with probability 1*,*

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for all $x_0 \in X$.

It follows that our definition of the expectation leads naturally to the equivalence of integration with respect to the residence measure. Elton's Theorem and Barnsley's corollary implies a similarly fast Chaos-Game algorithms for numerical estimation of the residence measure of Borel sets, as well as our expectations.

Expectations over IFS attractors

Corollary

Let $\{X; f_1, f_2, \ldots, f_N\}$ *be a contractive IFS with attractor* $A \in \mathbb{H}(X)$*. Given a complex-valued function* $F : X \to \mathbb{C}$*, the expectation of* F *over* A *is given by the integral:*

$$\langle f(x) \rangle_{x \in A} = \int_X f(x) \mathrm{d}\mu(x) \tag{6}$$

Functional equations

If the IFS is non-overlapping, the measure separates as follows:

Proposition (**Measure scaling relation**)

The invariant measure μ on a subset S of the attractor A of a totally-disconnected IFS satisfies the scaling relation:

$$\mu(S) \; = \; \sum_{k=1}^{m} \mu\left(f_k(S)\right) \tag{7}$$

Functional equations

The functional equations for expectations are:

Proposition (**Function equations for expectations**)

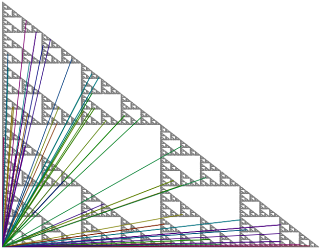
For points x, y in the attractor A of a non-overlapping IFS, the expectations for a complex-valued function F satisfy the functional equations (respectively pertaining to the box-integrals B and Δ):

$$\langle F(x)\rangle = \frac{1}{m}\sum_{j=1}^{m}\langle F\left(f_j(x)\right)\rangle \tag{8}$$

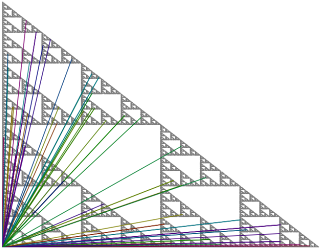
$$\langle F(x-y)\rangle = \frac{1}{m^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\langle F\left(f_j(x)-f_k(y)\right)\rangle \tag{9}$$

Examples

Orthogonal Sierpinski Triangle

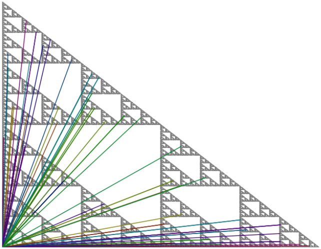


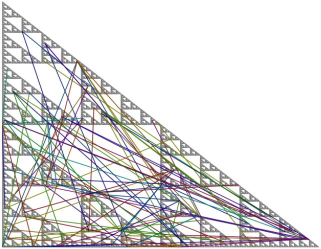
Orthogonal Sierpinski Triangle



$$B_2 = \frac{10}{27}$$

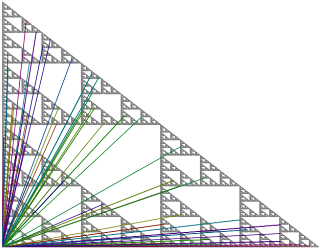
Orthogonal Sierpinski Triangle



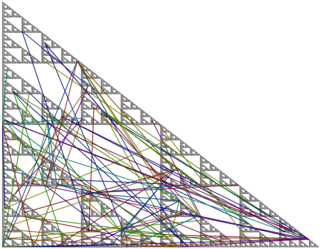


$$B_2 = \frac{10}{27}$$

Orthogonal Sierpinski Triangle

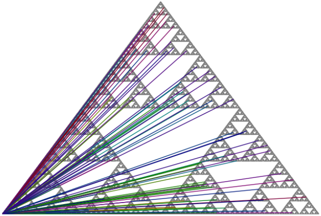


$$B_2 = \frac{10}{27}$$

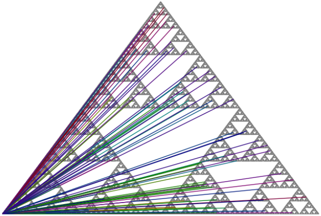


$$\Delta_2 = \frac{8}{27}$$

Equilateral Sierpinski Triangle

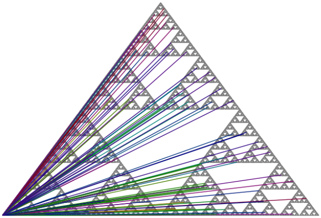


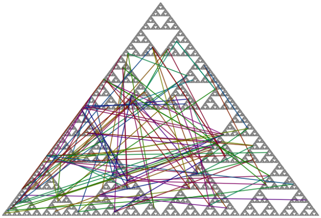
Equilateral Sierpinski Triangle



$$B_2 = \frac{4}{9}$$

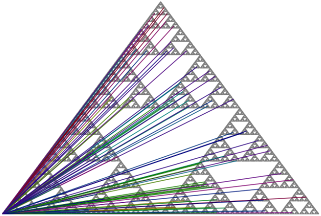
Equilateral Sierpinski Triangle

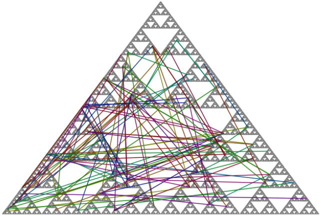




$$B_2 = \frac{4}{9}$$

Equilateral Sierpinski Triangle

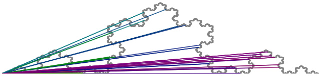




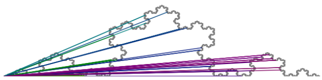
$$B_2 = \frac{4}{9}$$

$$\Delta_2 = \frac{2}{9}$$

Koch Curve

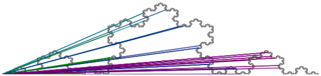


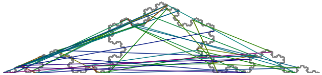
Koch Curve



$$B_2 = \frac{1}{3}$$

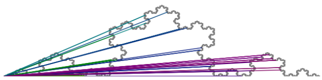
Koch Curve



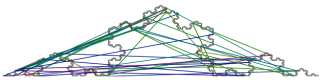


$$B_2 = \frac{1}{3}$$

Koch Curve



$$B_2 = \frac{1}{3}$$

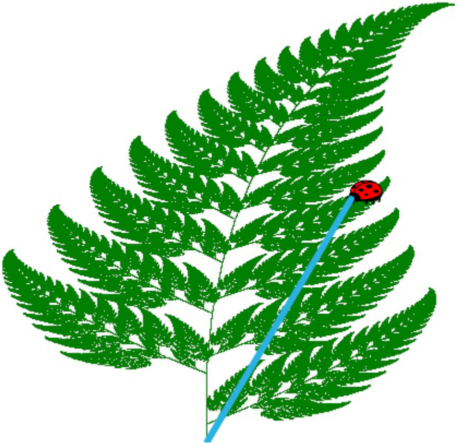


$$\Delta_2 = \frac{4}{27}$$

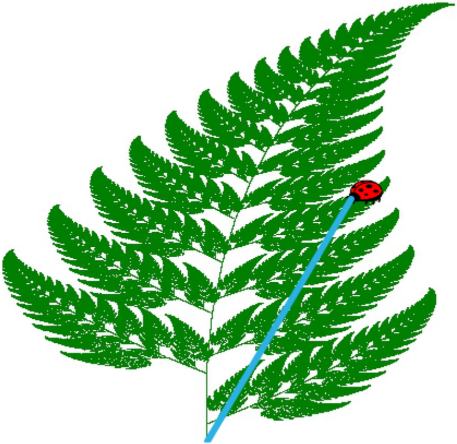
Barnsley Fern

Barnsley Fern

Barnsley Fern



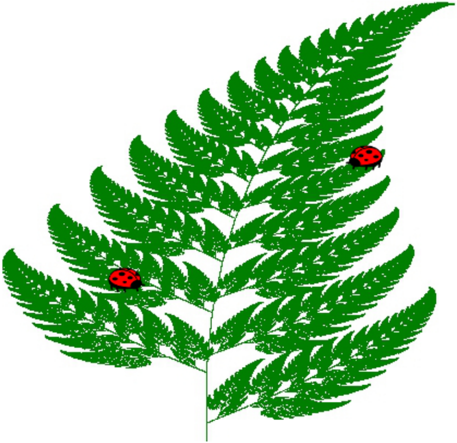
Barnsley Fern



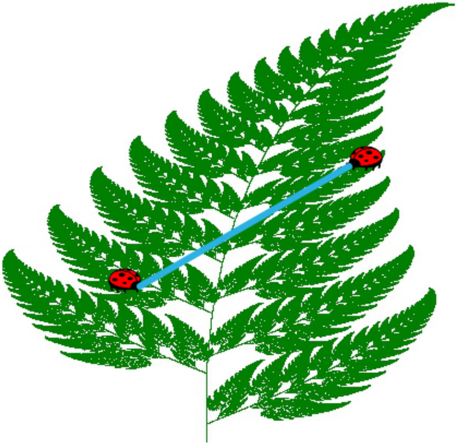
$$B_2 = \frac{2049440803137681904}{580160660775546421} \approx 3.5$$

Barnsley Fern

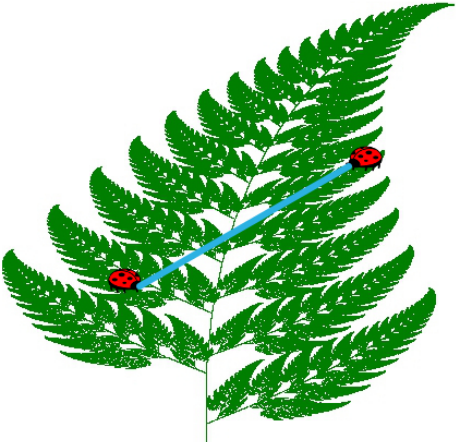
Barnsley Fern



Barnsley Fern



Barnsley Fern



$$\Delta_2 = \frac{1561818604387599983932186}{541130352321871535527225} \approx 2.9$$

Future directions

- Applications to Daubechies wavelets
- Evaluation of odd moments
- NMR diffusion studies

Thanks!