Monotone operators and "bigger conjugate" functions

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Abstract

We study a question posed by Stephen Simons in his 2008 monograph involving "bigger conjugate" (BC) functions and the partial infimal convolution. As Simons demonstrated in his monograph, these function have been crucial to the understanding and advancement of the state-of-the-art of harder problems in monotone operator theory, especially the sum problem.

In this paper, we provide some tools for further analysis of BC–functions which allow us to answer Simons' problem in the negative. We are also able to refute a similar but much harder conjecture which would have generalized a classical result of Brézis, Crandall and Pazy. Our work also reinforces the importance of understanding unbounded skew linear relations to construct monotone operators with unexpected properties.

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1 Introduction

Throughout this paper, we assume that X is a real reflexive Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$.

Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as a multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of A. The *domain* of A is $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$, and $\operatorname{ran} A := A(X)$ for the *range* of A. Recall that A is *monotone* if

$$(1) \qquad \langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (x,x^*) \in \operatorname{gra} A\ \forall (y,y^*) \in \operatorname{gra} A,$$

and *maximally monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). Let $S \subseteq X \times X^*$. We say S is *a monotone set* if there exists a monotone operator $A : X \rightrightarrows X^*$ such that $\operatorname{gra} A = S$, and S is *a maximally monotone set* if there exists a maximally monotone operator A such that $\operatorname{gra} A = S$. Let $A : X \rightrightarrows X^*$ be monotone and $(x,x^*) \in X \times X^*$. We say (x,x^*) is *monotonically related to* $\operatorname{gra} A$ if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (y,y^*) \in \operatorname{gra} A.$$

Maximally monotone operators have proven to be a potent class of objects in modern Optimization and Analysis; see, e.g., [6, 7, 8], the books [2, 9, 10, 13, 16, 17, 15, 19] and the references therein.

We adopt standard notation used in these books especially [9, Chapter 2] and [6, 16, 17]: Given a subset C of X, $\operatorname{int} C$ is the *interior* of C, $\overline{C}$ is the *norm closure* of C. The *support function* of C, written as σ_C, is defined by $\sigma_C(x^*) := \sup_{c\in C}\langle c, x^*\rangle$. The *indicator function* of C, written as ι_C, is defined at $x \in X$ by

$$(2) \qquad \iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

For every $x \in X$, the *normal cone operator* of C at x is defined by $N_C(x) = \big\{x^* \in X^* \mid \sup_{c\in C}\langle c-x, x^*\rangle \leq 0\big\}$, if $x \in C$; and $N_C(x) = \varnothing$, if $x \notin C$. For $x, y \in X$, we set $[x,y] = \{tx + (1-t)y \mid 0 \leq t \leq 1\}$. The *closed unit ball* is $B_X := \big\{x \in X \mid \|x\| \leq 1\big\}$, and $\mathbb{N} := \{1, 2, 3, \ldots\}$.

If Z is a real Banach space with dual Z^* and a set $S \subseteq Z$, we denote $S^\perp$ by $S^\perp := \{z^* \in Z^* \mid \langle z^*, s\rangle = 0, \quad \forall s \in S\}$. The *adjoint* of an operator A, written A^*, is defined by

$$\operatorname{gra} A^* := \big\{(x,x^*) \in X \times X^* \mid (x^*, -x) \in (\operatorname{gra} A)^\perp\big\}.$$

We say A is a *linear relation* if $\operatorname{gra} A$ is a linear subspace. We say that A is *skew* if $\operatorname{gra} A \subseteq \operatorname{gra}(-A^*)$; equivalently, if $\langle x, x^* \rangle = 0$, $\forall (x, x^*) \in \operatorname{gra} A$. Furthermore, A is *symmetric* if $\operatorname{gra} A \subseteq \operatorname{gra} A^*$; equivalently, if $\langle x, y^* \rangle = \langle y, x^* \rangle$, $\forall (x, x^*), (y, y^*) \in \operatorname{gra} A$.

Let $f\colon X \to \left]-\infty, +\infty\right]$. Then $\operatorname{dom} f := f^{-1}(\mathbb{R})$ is the *domain* of f, and $f^*\colon X^* \to [-\infty, +\infty] : x^* \mapsto \sup_{x\in X}(\langle x, x^* \rangle - f(x))$ is the *Fenchel conjugate* of f. We say f is proper if $\operatorname{dom} f \neq \varnothing$. Let f be proper. The *subdifferential* of f is defined by

$$\partial f\colon X \rightrightarrows X^*\colon x \mapsto \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^* \rangle + f(x) \leq f(y)\}.$$

2 BC-functions

We now turn to the objects of the present paper: *representative* and *BC-functions.* Let $F : X \times X^* \to \left]-\infty, +\infty\right]$, and define $\operatorname{pos} F$ [17] by

$$\operatorname{pos} F := \big\{(x, x^*) \in X \times X^* \mid F(x, x^*) = \langle x, x^* \rangle\big\}.$$

We say F is a *BC–function* (BC stands for "bigger conjugate") [17] if F is proper and convex with

$$F^*(x^*, x) \geq F(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*. \tag{3}$$

The prototype for a BC function is the Fitzpatrick function [11, 17, 9].

Let now Y be another real Banach space. We set $P_X : X \times Y \to X\colon (x, y) \mapsto x$. Let $F_1, F_2\colon X\times Y \to \left]-\infty, +\infty\right]$. Then the *partial inf-convolution* $F_1 \Box_2 F_2$ is the function defined on $X \times Y$ by

$$F_1 \Box_2 F_2\colon (x, y) \mapsto \inf_{v\in Y} F_1(x, y - v) + F_2(x, v).$$

The importance of BC-functions associated with monotone operators is that along with appropriate partial convolutions, they provide the most powerful current method to establish the maximality of the sum of two maximally monotone operators [17, 9]. The two problems considered below are closely related to constructions of maximally monotone operators as sums (see also Remark 5.4).

The following question was posed by S. Simons [17, Problem 34.7]:

Problem 2.1 (Simons) Let $F_1, F_2 : X \times X^* \to \left]-\infty, +\infty\right]$ be proper lower semicontinuous and convex functions with $P_X \operatorname{dom} F_1 \cap P_X \operatorname{dom} F_2 \neq \varnothing$. Assume that F_1, F_2 are BC–functions and that there exists an increasing function $j : \left[0, +\infty\right[\to \left[0, +\infty\right[$ such that the implication

$$(x, x^*) \in \operatorname{pos} F_1, (y, y^*) \in \operatorname{pos} F_2, x \neq y \text{ and } \langle x - y, y^* \rangle = \|x - y\| \cdot \|y^*\|$$
$$\Rightarrow \|y^*\| \leq j\big(\|x\| + \|x^* + y^*\| + \|y\| + \|x - y\| \cdot \|y^*\|\big)$$

holds. Then, is it true that, for all $(z, z^*) \in X \times X^*$, there exists $x^* \in X^*$ such that

$$F_1^*(x^*, z) + F_2^*(z^* - x^*, z) \leq (F_1 \Box_2 F_2)^*(x^*, z)?$$

In Example 4.4 of this paper, we construct a comprehensive negative answer to Problem 2.1. This in turn prompts another question:

Problem 2.2 Let $F_1, F_2 : X \times X^* \to \ ]-\infty, +\infty]$ be proper lower semicontinuous and convex functions with $P_X \operatorname{dom} F_1 \cap P_X \operatorname{dom} F_2 \neq \varnothing$. Assume that F_1, F_2 are BC–functions and that there exists an increasing function $j : [0, +\infty[\ \to [0, +\infty[$ such that the implication

$$(x, x^*) \in \operatorname{pos} F_1, (y, y^*) \in \operatorname{pos} F_2, x \neq y \text{ and } \langle x - y, y^* \rangle = \|x - y\| \cdot \|y^*\|$$
$$\Rightarrow \|y^*\| \leq j\big(\|x\| + \|x^* + y^*\| + \|y\| + \|x - y\| \cdot \|y^*\|\big)$$

holds. Then, is it true that, for all $(z, z^*) \in X \times X^*$, there exists $v^* \in X^*$ such that

$$F_1^*(v^*, z) + F_2^*(z^* - v^*, z) \leq (F_1 \Box_2 F_2)^*(z^*, z)? \tag{4}$$

This is a quite reasonable question and somewhat harder to answer. An affirmative response to Problem 2.2 would rederive Simons' theorem (Fact 3.4). Precisely, when the latter conjecture holds, we can deduce that $F := F_1 \Box_2 F_2$ is a BC-function. It follows that $\operatorname{pos} F$ (i.e., M in Fact 3.4) is a maximally monotone set; by Simons' result [17, Theorem 21.4]. However, Example 5.2 shows that the conjecture fails in general.

We are now ready to set to work. The remainder of the paper is organized as follows. In Section 3, we collect auxiliary results for future reference and for the reader's convenience. Our main result (Theorem 4.3) is established in Section 4. In Example 4.4, we provide the promised negative answer to Problem 2.1. In Section 5, we provide a negative answer to Problem 2.2.

3 Auxiliary results

Fact 3.1 (Rockafellar) (See [14, Theorem A], [19, Theorem 3.2.8], [17, Theorem 18.7] or [12, Theorem 2.1]) *Let $f : X \to \ ]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then ∂f is maximally monotone.*

We now turn to prerequisite results on Fitzpatrick functions, monotone operators, and linear relations.

Fact 3.2 (Fitzpatrick) (See [11, Corollary 3.9 and Proposition 4.2] and [6, 9].) *Let $A\colon X \rightrightarrows X^*$ be maximally monotone, and set*

$$F_A\colon X\times X^* \to \left]-\infty,+\infty\right] : (x,x^*) \mapsto \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x,a^*\rangle + \langle a,x^*\rangle - \langle a,a^*\rangle\big), \tag{5}$$

which is the Fitzpatrick function *associated with A. Then F_A is a BC–function and* $\operatorname{pos} F_A = \operatorname{gra} A$.

Fact 3.3 (Simons and Zălinescu) (See [18, Theorem 4.2] or [17, Theorem 16.4(a)].) *Let Y be a real Banach space and $F_1, F_2\colon X\times Y \to \left]-\infty,+\infty\right]$ be proper, lower semicontinuous, and convex. Assume that for every $(x,y)\in X\times Y$,*

$$(F_1\Box_2 F_2)(x,y) > -\infty$$

and that $\bigcup_{\lambda>0}\lambda\left[P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2\right]$ is a closed subspace of X. Then for every $(x^*,y^*)\in X^*\times Y^*$,

$$(F_1\Box_2 F_2)^*(x^*,y^*) = \min_{u^*\in X^*}\left[F_1^*(x^*-u^*,y^*) + F_2^*(u^*,y^*)\right].$$

The following Simons' result generalizes the result of Brézis, Crandall and Pazy [5].

Fact 3.4 (Simons) (See [17, Theorem 34.3].) *Let $F_1, F_2 : X\times X^* \to \left]-\infty,+\infty\right]$ be proper lower semicontinuous and convex functions with $P_X \operatorname{dom} F_1 \cap P_X \operatorname{dom} F_2 \neq \varnothing$. Assume that F_1, F_2 are BC–functions and that there exists an increasing function $j : \left[0,+\infty\right[\to \left[0,+\infty\right[$ such that the implication*

$$\begin{aligned}&(x,x^*)\in \operatorname{pos} F_1, (y,y^*)\in \operatorname{pos} F_2, x\neq y \text{ and } \langle x-y,y^*\rangle = \|x-y\|\cdot\|y^*\|\\ &\quad\Rightarrow \|y^*\| \leq j\big(\|x\| + \|x^*+y^*\| + \|y\| + \|x-y\|\cdot\|y^*\|\big)\end{aligned}$$

holds. Then $M := \big\{(x,x^*+y^*) \mid (x,x^*)\in \operatorname{pos} F_1, (x,y^*)\in \operatorname{pos} F_2\big\}$ is a maximally monotone set.

4 Our main result

We start with two technical tools which relate Fitzpatrick functions and skew operators. We first give a direct proof of the following result.

Fact 4.1 (See [1, Corollary 5.9].) *Let C be a nonempty closed convex subset of X. Then $F_{N_C} = \iota_C \oplus \iota_C^*$.*

Proof. Let $(x, x^*) \in X \times X^*$. Then we have

$$
\begin{aligned}
F_{N_C}(x, x^*) &= \sup_{(c,c^*)\in \operatorname{gra} N_C} [\langle x, c^*\rangle + \langle c, x^*\rangle - \langle c, c^*\rangle] \\
&= \sup_{(c,c^*)\in \operatorname{gra} N_C, k\geq 0} [\langle x, kc^*\rangle + \langle c, x^*\rangle - \langle c, kc^*\rangle] \\
&= \sup_{(c,c^*)\in \operatorname{gra} N_C, k\geq 0} [k(\langle x, c^*\rangle - \langle c, c^*\rangle) + \langle c, x^*\rangle] \qquad (6)
\end{aligned}
$$

By (6),

$$
\begin{aligned}
&(x, x^*) \in \operatorname{dom} F_{N_C} \Rightarrow \sup_{(c,c^*)\in \operatorname{gra} N_C} [\langle x, c^*\rangle - \langle c, c^*\rangle] \leq 0 \\
&\Leftrightarrow \inf_{(c,c^*)\in \operatorname{gra} N_C} [-\langle x, c^*\rangle + \langle c, c^*\rangle] \geq 0 \\
&\Leftrightarrow \inf_{(c,c^*)\in \operatorname{gra} N_C} [\langle c - x, c^* - 0\rangle] \geq 0 \\
&\Leftrightarrow (x, 0) \in \operatorname{gra} N_C \quad \text{(by Fact 3.1)} \\
&\Leftrightarrow x \in C. \qquad (7)
\end{aligned}
$$

Now assume $x \in C$. By (6),

$$F_{N_C}(x, x^*) = \iota_C^*(x^*). \qquad (8)$$

Combine (7) and (8), $F_{N_C} = \iota_C \oplus \iota_C^*$. ■

Fact 4.2 (See [3, Proposition 5.5].) *Let $A \colon X \rightrightarrows X^*$ be a monotone linear relation such that $\operatorname{gra} A \neq \varnothing$ and $\operatorname{gra} A$ is closed. Then*

$$F_A^*(x^*, x) = \iota_{\operatorname{gra} A}(x, x^*) + \langle x, x^*\rangle, \ \forall (x, x^*) \in X \times X^*. \qquad (9)$$

We are now ready to establish our main result.

Theorem 4.3 *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation that is at most single-valued, and let $C \neq \{0\}$ be a bounded closed and convex subset of X such that $\bigcup_{\lambda>0} \lambda [\operatorname{dom} A - C]$ is a closed subspace of X. Let $j : [0, +\infty[\ \to [0, +\infty[$ be an increasing function such that $j(\gamma) \geq \gamma$ for every $\gamma \in [0, +\infty[$. Then the following hold.*

(i) *F_A and $F_{N_C} = \iota_C \oplus \sigma_C$ are BC-functions.*

(ii) $F_A^*(x^*,x)+F_{N_C}^*(y^*-x^*,x)=\iota_{\operatorname{gra} A\cap C\times X^*}(x,x^*)+\langle x,x^*\rangle+\sigma_C(y^*-x^*), \quad \forall(x,x^*,y^*)\in X\times X^*\times X^*$.

(iii) *For every* $(x,x^*)\in X\times X^*$,

$$(F_A\Box_2 F_{N_C})^*(x^*,x)=\begin{cases}\langle x,Ax\rangle+\sigma_C(x^*-Ax), & \text{if } x\in C\cap\operatorname{dom} A;\\ +\infty, & \text{otherwise.}\end{cases} \tag{10}$$

(iv) *There exists* $(z,z^*)\in X\times X^*$ *such that* $z\in\operatorname{dom} A\cap C$ *and* $\sigma_C(z^*-Az)>0$.

(v) *Assume that* $(z,z^*)\in X\times X^*$ *satisfies* $z\in\operatorname{dom} A\cap C$ *and* $\sigma_C(z^*-Az)>0$. *Then*

$$F_A^*(x^*,z)+F_{N_C}^*(z^*-x^*,z)>(F_A\Box_2 F_{N_C})^*(x^*,z), \quad \forall x^*\in X^*. \tag{11}$$

(vi) *Moreover, assume that* X *is a Hilbert space and* $C=B_X$. *Then the implication*

$$\begin{aligned}&(x,x^*)\in\operatorname{pos} F_A,(y,y^*)\in\operatorname{pos} F_{N_C}, x\neq y \text{ and } \langle x-y,y^*\rangle=\|x-y\|\cdot\|y^*\|\\ &\quad\Rightarrow \|y^*\|\leq\|x^*+y^*\|\leq j\big(\|x\|+\|x^*+y^*\|+\|y\|+\|x-y\|\cdot\|y^*\|\big)\end{aligned} \tag{12}$$

holds.

Proof. (i): Combine Fact 4.1 and Fact 3.2.

(ii): Let $(x,x^*,y^*)\in X\times X^*\times X^*$. Then by Fact 4.2 and (i), we have

$$\begin{aligned}F_A^*(x^*,x)+F_{N_C}^*(y^*-x^*,x)&=\iota_{\operatorname{gra} A}(x,x^*)+\langle x,x^*\rangle+(\iota_C^*\oplus\sigma_C^*)(y^*-x^*,x)\\ &=\iota_{\operatorname{gra} A}(x,x^*)+\langle x,x^*\rangle+\iota_C(x)+\sigma_C(y^*-x^*)\\ &=\iota_{\operatorname{gra} A\cap C\times X^*}(x,x^*)+\langle x,x^*\rangle+\sigma_C(y^*-x^*).\end{aligned}$$

(iii): By [3, Lemma 5.8], we have

$$\bigcup_{\lambda>0}\lambda\big(P_X(\operatorname{dom} F_A)-P_X(\operatorname{dom} F_{N_C})\big) \text{ is a closed subspace of } X. \tag{13}$$

Then for every $(x,x^*)\in X\times X^*$ and $u^*\in X^*$, by (i),

$$F_A(x,u^*)+F_{N_C}(x,x^*-u^*)\geq\langle x,u^*\rangle+\langle x,x^*-u^*\rangle=\langle x,x^*\rangle.$$

Hence

$$(F_A\Box_2 F_{N_C})(x,x^*)\geq\langle x,x^*\rangle>-\infty. \tag{14}$$

By (13), (14), Fact 3.3, and (ii), for every $(x, x^*) \in X \times X^*$, there exists $z^* \in X^*$ such that

$$(F_A \Box_2 F_{N_C})^*(x^*, x) = \min_{y^* \in X^*} F_A^*(y^*, x) + F_{N_C}^*(x^* - y^*, x)$$

$$= \iota_{\operatorname{gra} A \cap C \times X^*}(x, z^*) + \langle x, z^* \rangle + \sigma_C(x^* - z^*). \tag{15}$$

This implies (10).

(iv): By the assumption, there exists $z \in \operatorname{dom} A \cap C$. Since $C \neq \{0\}$, there exists $z^* \in X^*$ such that $\sigma_C(z^* - Az) > 0$.

(v): Let $x^* \in X^*$. By the assumptions, (iii) and the boundedness of C, we have

$$(F_A \Box_2 F_{N_C})^*(x^*, z) = \langle z, Az \rangle + \sigma_C(x^* - Az) < +\infty. \tag{16}$$

We consider two cases.

Case 1: $x^* \neq Az$.

Then $(z, x^*) \notin \operatorname{gra} A$ and so $\iota_{\operatorname{gra} A \cap C \times X^*}(z, x^*) = +\infty$. In view of (ii) and (16), (11) holds.

Case 2: $x^* = Az$.

By (ii) and (16), we have

$$F_A^*(x^*, z) + F_{N_C}^*(z^* - x^*, z) = \langle z, Az \rangle + \sigma_C(z^* - Az) > \langle z, Az \rangle + 0 = \langle z, Az \rangle + \sigma_C(0)$$
$$= (F_A \Box_2 F_{N_C})^*(x^*, z).$$

Hence (11) holds as well.

(vi): We start with a well known formula whose short proof we include for completeness. Let $x \in X$. Then

$$N_{B_X}(x) = \begin{cases} 0, & \text{if } \|x\| < 1; \\ [0, \infty[\cdot x, & \text{if } \|x\| = 1; \\ \varnothing, & \text{otherwise.} \end{cases} \tag{17}$$

Clearly, $N_{B_X}(x) = 0$ if $\|x\| < 1$, and $N_{B_X}(x) = \varnothing$ if $x \notin B_X$. Assume $\|x\| = 1$. Then

$$x^* \in N_{B_X}(x) \Leftrightarrow \|x^*\| = \|x^*\| \cdot \|x\| \geq \langle x^*, x \rangle \geq \sup\langle x^*, B_X \rangle = \|x^*\|$$
$$\Leftrightarrow \langle x^*, x \rangle = \|x^*\| \cdot \|x\|$$
$$\Leftrightarrow x^* = \gamma x, \quad \gamma \geq 0.$$

Hence (17) holds.

Now let $(x, x^*) \in \operatorname{pos} F_A, (y, y^*) \in \operatorname{pos} F_{N_C}$ and $x \neq y$ be such that $\langle x - y, y^* \rangle = \|x - y\| \cdot \|y^*\|$. By Fact 3.2,

$$x^* = Ax \text{ and } y^* \in N_{B_X}(y). \tag{18}$$

Now we show that

$$\|x^* + y^*\| \geq \|y^*\|. \tag{19}$$

Clearly, (19) holds if $y^* = 0$. Thus, we assume that $y^* \neq 0$. By (18) and (17), there exists $\gamma_0 > 0$ such that

$$y^* = \gamma_0 y, \tag{20}$$

where

$$\|y\| = 1. \tag{21}$$

Since $\langle x - y, y^* \rangle = \|x - y\| \cdot \|y^*\|$, we have

$$y^* = \frac{\|y^*\|}{\|x - y\|}(x - y). \tag{22}$$

We claim that

$$x \neq 0. \tag{23}$$

Suppose to the contrary that $x = 0$. Then by (22) and (21), we have $y^* = -\frac{\|y^*\|}{\|y\|} y = -\|y^*\| y$, which contradicts (20). Hence (23) holds.

By (20), (22) and (23), we have

$$\frac{x}{\|x\|} = \frac{y^*}{\|y^*\|}. \tag{24}$$

Then (18) and the monotonicity of A imply

$$\|x^* + y^*\| \geq \langle x^* + y^*, \frac{x}{\|x\|} \rangle \geq \langle y^*, \frac{y^*}{\|y^*\|} \rangle = \|y^*\|.$$

Therefore, (19) holds.

Then by the assumption, we have

$$\begin{aligned} j\big(\|x\| + \|x^* + y^*\| + \|y\| + \|x - y\| \cdot \|y^*\|\big) &\geq j\big(\|x^* + y^*\|\big) \\ &\geq \|x^* + y^*\| \\ &\geq \|y^*\|. \end{aligned}$$

Hence (12) holds, ■

We are now ready to exploit Theorem 4.3 to resolve Problem 2.1.

Example 4.4 Suppose that X is a Hilbert space, and let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation that is at most single-valued, and set $C = B_X$. Let $j : [0, +\infty[\to [0, +\infty[$ be an increasing function such that $j(\gamma) \geq \gamma$ for every $\gamma \in [0, +\infty[$. Then the following hold.

(i) Let $z^* \neq 0$. Then

$$F_A^*(x^*, 0) + F_{N_C}^*(z^* - x^*, 0) > (F_A \Box_2 F_{N_C})^*(x^*, 0), \quad \forall x^* \in X.$$

(ii) The implication

$$\begin{aligned}(x, x^*) \in \operatorname{pos} F_A, (y, y^*) \in \operatorname{pos} F_{N_C}, x \neq y \text{ and } \langle x - y, y^* \rangle = \|x - y\| \cdot \|y^*\| \\ \Rightarrow \|y^*\| \leq \|x^* + y^*\| \leq j\big(\|x\| + \|x^* + y^*\| + \|y\| + \|x - y\| \cdot \|y^*\|\big)\end{aligned}$$

holds.

Proof. Set $z = 0$. Then $Az = 0 \Rightarrow z^* - Az = z^* \neq 0 \Rightarrow \sigma_C(z^* - Az) = \sigma_C(z^*) = \|z^*\| > 0$. Now apply Theorem 4.3(v)&(vi). ■

Remark 4.5 Example 4.4 yields a negative answer to Simons' Problem 2.1 ([17, Problem 34.7]) for many linear relations — including the rotation by 90 degrees in the plane.

5 Resolution of Problem 2.2

We now move to the second problem. Its resolution depends on the following fact concerning a maximally monotone operator on ℓ^2, the real Hilbert space of square-summable sequences.

Fact 5.1 (See [4, Propositions 3.5, 3.6 and 3.7 and Lemma 3.18].) *Suppose that $X = \ell^2$, and that $A : \ell^2 \rightrightarrows \ell^2$ is given by*

$$Ax := \frac{\left(\sum_{i<n} x_i - \sum_{i>n} x_i\right)_{n\in\mathbb{N}}}{2} = \left(\sum_{i<n} x_i + \tfrac{1}{2}x_n\right)_{n\in\mathbb{N}}, \quad \forall x = (x_n)_{n\in\mathbb{N}} \in \operatorname{dom} A, \tag{25}$$

where $\operatorname{dom} A := \left\{x := (x_n)_{n\in\mathbb{N}} \in \ell^2 \mid \sum_{i\geq 1} x_i = 0, \left(\sum_{i\leq n} x_i\right)_{n\in\mathbb{N}} \in \ell^2\right\}$ *and* $\sum_{i<1} x_i := 0$. *Then*

$$A^*x = \left(\tfrac{1}{2}x_n + \sum_{i>n} x_i\right)_{n\in\mathbb{N}}, \tag{26}$$

where
$$x=(x_n)_{n\in\mathbb{N}}\in\operatorname{dom}A^*=\left\{x=(x_n)_{n\in\mathbb{N}}\in\ell^2\;\middle|\;\left(\sum_{i>n}x_i\right)_{n\in\mathbb{N}}\in\ell^2\right\}.$$
Then A provides an at most single-valued linear relation such that the following hold.

(i) *A is maximally monotone and skew.*

(ii) *A^* is maximally monotone but not skew.*

(iii) $F^*_{A^*}(x^*,x)=F_{A^*}(x,x^*)=\iota_{\operatorname{gra}A^*}(x,x^*)+\langle x,x^*\rangle,\quad\forall(x,x^*)\in X\times X.$

(iv) $\langle A^*x,x\rangle=\frac{1}{2}s^2,\quad\forall x=(x_n)_{n\in\mathbb{N}}\in\operatorname{dom}A^*$ *with* $\quad s:=\sum_{i\geq1}x_i.$

We are now ready for the main construction of this section.

Example 5.2 Suppose that X and A are as in Fact 5.1. Set $e_1:=(1,0,\ldots,0,\ldots)$, i.e., there is a 1 in the first place and all others entries are 0, and $C:=[0,e_1]$. Let $j:[0,+\infty[\,\to[0,+\infty[$ be an increasing function such that $j(\gamma)\geq\frac{\gamma}{2}$ for every $\gamma\in[0,+\infty[$. Then the following hold.

(i) F_{A^*} and $F_{N_C}=\iota_C\oplus\sigma_C$ are BC–functions.

(ii) $(F_{A^*}\Box_2F_{N_C})(x,x^*)=\begin{cases}\langle x,A^*x\rangle+\sigma_C(x^*-A^*x), & \text{if } x\in C;\\ +\infty, & \text{otherwise,}\end{cases}\quad\forall(x,x^*)\in X\times X^*.$

(iii) Then
$$F^*_{A^*}(x^*,0)+F^*_{N_C}(A^*e_1-x^*,0)>(F_{A^*}\Box_2F_{N_C})^*(A^*e_1,0),\quad\forall x^*\in X.$$

(iv) The implication
$$\begin{aligned}&(x,x^*)\in\operatorname{pos}F_{N_C},(y,y^*)\in\operatorname{pos}F_{A^*},x\neq y\text{ and }\langle x-y,y^*\rangle=\|x-y\|\cdot\|y^*\|\\&\quad\Rightarrow\|y^*\|\leq\tfrac{1}{2}\|y\|\leq j\big(\|x\|+\|x^*+y^*\|+\|y\|+\|x-y\|\cdot\|y^*\|\big)\end{aligned}$$
holds.

(v) A^*+N_C is maximally monotone.

Proof. (i): Combine Fact 5.1(ii), Fact 3.2 and Fact 4.1.

(ii): Using Fact 5.1(iii), we see that for every $(x,x^*) \in X \times X^*$,

$$(F_{A^*} \Box_2 F_{N_C})(x,x^*) = \inf_{y^* \in X^*} \iota_{\operatorname{gra} A^*}(x,y^*) + \langle x, y^* \rangle + \iota_C(x) + \sigma_C(x^* - y^*)$$
$$= \begin{cases} \langle x, A^*x \rangle + \sigma_C(x^* - A^*x), & \text{if } x \in \operatorname{dom} A^* \cap C; \\ +\infty, & \text{otherwise,} \end{cases}.$$

The identity now follows since $C \subseteq \operatorname{dom} A^*$.

(iii): Let $x^* \in X$. Then by Fact 5.1(iii) we have

$$\begin{aligned} F^*_{A^*}(x^*,0) + F^*_{N_C}(A^*e_1 - x^*, 0) &= \iota_{\{0\}}(x^*) + \sigma_C(A^*e_1 - x^*) \\ &= \sigma_C(A^*e_1) + \iota_{\{0\}}(x^*) \\ &= \sup_{t \in [0,1]} \left\{ t \langle e_1, A^*e_1 \rangle \right\} + \iota_{\{0\}}(x^*) \\ &= \langle e_1, A^*e_1 \rangle + \iota_{\{0\}}(x^*) \\ &= \tfrac{1}{2} + \iota_{\{0\}}(x^*) \quad \text{(by Fact 5.1(iv))}. \end{aligned} \tag{27}$$

On the other hand, by (ii) and $C \subseteq \operatorname{dom} A^*$ by Fact 5.1, we have

$$\begin{aligned} (F_{A^*} \Box_2 F_{N_C})^*(A^*e_1, 0) &= \sup_{x \in C, x^* \in X} \left\{ \langle A^*e_1, x \rangle - \langle x, A^*x \rangle - \sigma_C(x^* - A^*x) \right\} \\ &\leq \sup_{x \in C, x^* \in X} \left\{ \langle A^*e_1, x \rangle - \langle x, A^*x \rangle \right\} \quad (\text{by } 0 \in C) \\ &= \sup_{t \in [0,1]} \left\{ t \langle A^*e_1, e_1 \rangle - t^2 \langle e_1, A^*e_1 \rangle \right\} \\ &= \tfrac{1}{4} \langle A^*e_1, e_1 \rangle \\ &= \tfrac{1}{8} \quad \text{(by Fact 5.1(iv))} \\ &< F^*_{A^*}(x^*,0) + F^*_{N_C}(A^*e_1 - x^*, 0) \quad \text{(by (27))}. \end{aligned}$$

Hence (iii) holds.

(iv): Let $(x,x^*) \in \operatorname{pos} F_{N_C}, (y,y^*) \in \operatorname{pos} F_{A^*}$, and $x \neq y$ be such that $\langle x - y, y^* \rangle = \|x - y\| \cdot \|y^*\|$. By Fact 3.2,

$$x^* \in N_C(x) \text{ and } y^* = A^*y. \tag{28}$$

Now we show

$$\tfrac{1}{2}\|y\| \geq \|y^*\|. \tag{29}$$

Clearly, (29) holds if $y^* = 0$. Now assume that $y^* \neq 0$. Then by $\langle x - y, y^* \rangle = \|x - y\| \cdot \|y^*\|$ and $x \in C$, there exist $t_0 \geq 0$ and $\gamma_0 > 0$ such that

$$x = t_0 e_1 \text{ and } y^* = \gamma_0 (t_0 e_1 - y). \tag{30}$$

Write $y = (y_n)_{n\in\mathbb{N}}$. By (26) and (30), we have

$$\sum_{i>n} y_i = -\gamma_0 y_n - \tfrac{1}{2} y_n, \quad \forall n \geq 2. \tag{31}$$

Thus

$$\sum_{i>n+1} y_i = -\gamma_0 y_{n+1} - \tfrac{1}{2} y_{n+1}, \quad \forall n \geq 1. \tag{32}$$

Subtracting (32) from (31), we obtain

$$y_{n+1} = (-\gamma_0 - \tfrac{1}{2})(y_n - y_{n+1}), \quad \forall n \geq 2. \tag{33}$$

Since $\gamma_0 > 0$, by (33), we have

$$y_{n+1} \frac{\gamma_0 - \frac{1}{2}}{\gamma_0 + \frac{1}{2}} = y_n, \quad \forall n \geq 2. \tag{34}$$

Now we claim that

$$y_n = 0, \quad \forall n \geq 2. \tag{35}$$

Suppose to the contrary that there exists $i_0 \geq 2$ such that

$$y_{i_0} \neq 0. \tag{36}$$

Then by (34), we have $y_{i_0} = y_{i_0+1} \frac{\gamma_0 - \frac{1}{2}}{\gamma_0 + \frac{1}{2}}$. Thus,

$$\gamma_0 \neq \frac{1}{2}. \tag{37}$$

Then by (34), we have

$$y_{n+1} = \frac{\gamma_0 + \frac{1}{2}}{\gamma_0 - \frac{1}{2}} y_n, \quad \forall n \geq 2. \tag{38}$$

Set $\alpha := \frac{\gamma_0 + \frac{1}{2}}{\gamma_0 - \frac{1}{2}}$. Then by $\gamma_0 > 0$ again,

$$|\alpha| > 1. \tag{39}$$

By (38) and Fact 5.1, we have $\sum_{i>2} y_i = y_2 \sum_{i\geq 1} \alpha^i$ and the former series is convergent. Thus (39) implies that $y_2 = 0$ and then $y_n = 0, \forall n > 2$ by (38), which contradicts (36). Hence (35) holds. Then by Fact 5.1,

$$y^* = (\tfrac{1}{2} y_1, 0, 0, \ldots, 0, \ldots). \tag{40}$$

Hence $\|y^*\| \leq \frac{1}{2}\|y\|$ and thus (29) holds. Then by the assumption, we have

$$\begin{aligned}\|y^*\| &\leq \tfrac{1}{2}\|y\| \leq \tfrac{1}{2}\big(\|x\| + \|y\| + \|x^* + y^*\| + \|x - y\| \cdot \|y^*\|\big)\\ &\leq j(\|x\| + \|y\| + \|x^* + y^*\| + \|x - y\| \cdot \|y^*\|).\end{aligned}$$

Hence the implication (iv) holds.

(v): By Fact 3.2 and Fact 5.1(ii), $\operatorname{pos} F_{A^*} = \operatorname{gra} A^*$ and $\operatorname{pos} F_{N_C} = \operatorname{gra} N_C$. Then directly apply (i)&(iv) and Fact 3.4. ■

Remark 5.3 Example 5.2 provides a negative answer to Problem 2.2 as asserted.

Remark 5.4 It is not as easy to find a counterexample to Problem 2.2 as it is for Problem 2.1. Indeed, Fact 3.4 and Fact 3.3 imply that, to find a counterexample, we need to start with two maximally monotone operators $A, B : X \rightrightarrows X^*$ such that $A + B$ is maximally monotone but it does not satisfy the well known sufficient transversality condition for the maximal monotonicity of the sum operator in a reflexive space [18, Lemma 5.1] and [4, Lemma 5.8], that is:

$$\bigcup_{\lambda>0} \lambda\,[\operatorname{dom} A - \operatorname{dom} B] \quad \text{is a closed subspace of } X. \tag{41}$$

Otherwise, (41) ensures that (4) in Problem 2.2 holds by Fact 3.3 and [4, Lemma 5.8].

Finally, as we mentioned in Section 2 an affirmative answer to Problem 2.2 would rederive Simons' theorem (Fact 3.4). Indeed, Simons [17, Corollary 34.5] shows in detail how to deduce the classic result of Brézis, Crandall and Pazy [5] from his result.

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