Convex Functions: Constructions, Characterizations and Counterexamples

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Convex Functions: Constructions, Characterizations and Counterexamples

Like differentiability, convexity is a natural and powerful property of functions that plays a significant role in many areas of mathematics, both pure and applied. It ties together notions from typology, algebra, geometry and analysis, and is an important tool in optimization, mathematical programming and game theory. This book, which is the product of a collaboration of over 15 years, is unique in that it focuses on convex functions themselves, rather than on convex analysis. The authors explore the various classes and their characteristics, treating convex functions in both Euclidean and Banach spaces.

They begin by demonstrating, largely by way of examples, the ubiquity of convexity. Chapter 2 then provides an extensive foundation for the study of convex functions in Euclidean (finite-dimensional) space, and Chapter 3 reprises important special structures such as polyhedrality, selection theorems, eigenvalue optimization and semidefinite programming. Chapters 4 and 5 play the same role in (infinite-dimensional) Banach space. Chapter 6 discusses a number of other basic topics, such as selection theorems, set convergence, integral and trace class funcitonals, and convex functions on Banach lattices.

Chapters 7 and 8 examine Legendre functions and their relation to the geometry of Banach spaces. The final chapter investigates the application of convex functions to (maximal) monotone operators through the use of a recently discovered class of convex representive functions of which the Fitzpatrick function is the progenitor.

The book can either be read sequentially as a graduate text, or dipped into by researchers and practitioners. Each chapter contains a variety of concrete examples and over 600 exercises are included, ranging in difficulty from early graduate to research level.

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To our wives
Judith and Judith

Contents

Preface

This book on *convex functions* emerges out of 15 years of collaboration between the authors. It is far from being the first on the subject nor will it be the last. It is neither a book on *convex analysis* such as Rockafellar's foundational 1970 book [369] nor a book on *convex programming* such as Boyd and Vandenberghe's excellent recent text [128]. There are a number of fine books – both recent and less so – on both those subjects or on *convexity* and relatedly on *variational analysis*. Books such as [371, 255, 378, 256, 121, 96, 323, 332] complement or overlap in various ways with our own focus which is to explore the interplay between the structure of a normed space and the properties of convex functions which can exist thereon. In some ways, among the most similar books to ours are those of Phelps [349] and of Giles [229] in that both also straddle the fields of geometric functional analysis and convex analysis – but without the convex function itself being the central character.

We have structured this book so as to accommodate a variety of readers. This leads to some intentional repetition. Chapter 1 makes the case for the ubiquity of convexity, largely by way of examples, many but not all of which are followed up in later chapters. Chapter 2 then provides a foundation for the study of convex functions in Euclidean (finite-dimensional) space, and Chapter 3 reprises important special structures such as polyhedrality, eigenvalue optimization and semidefinite programming.

Chapters 4 and 5 play the same role in (infinite-dimensional) Banach space. Chapter 6 comprises a number of other basic topics such as Banach space selection theorems, set convergence, integral functionals, trace-class spectral functions and functions on normed lattices.

The remaining three chapters can be read independently of each other. Chapter 7 examines the structure of *Legendre functions* which comprises those barrier functions which are essentially smooth and essentially strictly convex and considers how the existence of such barrier functions is related to the geometry of the underlying Banach space; as always the nicer the space (e.g. is it reflexive, Hilbert or Euclidean?) the more that can be achieved. This coupling between the space and the convex functions which may survive on it is attacked more methodically in Chapter 8.

Chapter 9 investigates (maximal) monotone operators through the use of a specialized class of convex *representative functions* of which the *Fitzpatrick function* is the progenitor. We have written this chapter so as to make it more useable as a stand-alone source on convexity and its applications to monotone operators.

In each chapter we have included a variety of concrete examples and exercises – often guided, some with further notes given in Chapter 10. We both believe strongly that general understanding and intuition rely on having fully digested a good cross-section of particular cases. Exercises that build required theory are often marked with *, those that include broader applications are marked with † and those that take excursions into topics related – but not central to – this book are marked with **.

We think this book can be used as a text, either primary or secondary, for a variety of introductory graduate courses. One possible half-course would comprise Chapters 1, 2, 3 and the finite-dimensional parts of Chapters 4 through 10. These parts are listed at the end of Chapter 3. Another course could encompass Chapters 1 through 6 along with Chapter 8, and so on. We hope also that this book will prove valuable to a larger group of practitioners in mathematical science; and in that spirit we have tried to keep notation so that the infinite-dimensional and finite-dimensional discussion are well comported and so that the book can be dipped into as well as read sequentially. This also requires occasional intentional redundancy. In addition, we finish with a 'bonus chapter' revisiting the boundary between Euclidean and Banach space and making comments on the earlier chapters.

We should like to thank various of our colleagues and students who have provided valuable input and advice. We should also like to thank Cambridge University Press and especially David Tranah who has played an active and much appreciated role in helping shape this work. Finally, we have a companion web-site at `http://projects.cs.dal.ca/ddrive/ConvexFunctions/` on which various related links and addenda (including any subsequent errata) may be found.

1

Why convex?

The first modern formalization of the concept of convex function appears in J. L. W. V. Jensen, "Om konvexe funktioner og uligheder mellem midelvaerdier." Nyt Tidsskr. Math. B ***16** (1905), pp. 49–69. Since then, at first referring to "Jensen's convex functions," then more openly, without needing any explicit reference, the definition of convex function becomes a standard element in calculus handbooks.* (A. Guerraggio and E. Molho)[1]

Convexity theory . . . reaches out in all directions with useful vigor. Why is this so? Surely any answer must take account of the tremendous impetus the subject has received from outside of mathematics, from such diverse fields as economics, agriculture, military planning, and flows in networks. With the invention of high-speed computers, large-scale problems from these fields became at least potentially solvable. Whole new areas of mathematics (game theory, linear and nonlinear programming, control theory) aimed at solving these problems appeared almost overnight. And in each of them, convexity theory turned out to be at the core. The result has been a tremendous spurt in interest in convexity theory and a host of new results. (A. Wayne Roberts and Dale E. Varberg)[2]

1.1 Why 'convex'?

This introductory polemic makes the case for a study focusing on convex functions and their structural properties. We highlight the centrality of convexity and give a selection of salient examples and applications; many will be revisited in more detail later in the text – and many other examples are salted among later chapters. Two excellent companion pieces are respectively by Asplund [15] and by Fenchel [212]. A more recent survey article by Berger has considerable discussion of convex geometry [53].

It has been said that most of number theory devolves to the Cauchy–Schwarz inequality and the only problem is deciding 'what to Cauchy with'. In like fashion, much mathematics is tamed once one has found the right convex 'Green's function'. Why convex? Well, because . . .

- For convex sets topological, algebraic, and geometric notions often coincide; one sees this in the study of the simplex method and of continuity of convex functions. This allows one to draw upon and exploit many different sources of insight.

[1] A. Guerraggio and E. Molho, "The origins of quasi-concavity: a development between mathematics and economics," *Historia Mathematica*, **31**, 62–75, (2004).
[2] Quoted by Victor Klee in his review of [366], *SIAM Review*, **18**, 133–134, (1976).

- In a computational setting, since the *interior-point revolution* [331] in linear optimization it is now more or less agreed that 'convex' = 'easy' and 'nonconvex' = 'hard' – both theoretically and computationally. A striking illustration in combinatorial optimization is discussed in Exercise 3.3.9. In part this easiness is for the prosaic reason that local and global minima coincide.
- 'Differentiability' is understood and has been exploited throughout the sciences for centuries; 'convexity' less so, as the opening quotations attest. It is not emphasized in many instances in undergraduate courses – convex principles appear in topics such as the second derivative test for a local extremum, in linear programming (extreme points, duality, and so on) or via Jensen's inequality, etc. but often they are not presented as part of any general corpus.
- Three-dimensional convex pictures are surprisingly often realistic, while two-dimensional ones are frequently not as their geometry is too special. (Actually in a convex setting even two-dimensional pictures are much more helpful compared to those for nonconvex functions, still three-dimensional pictures are better. A good illustration is Figure 2.16. For example, working two-dimensionally, one may check convexity along lines, while seeing equal right-hand and left-hand derivatives in all directions implies differentiability.)

1.2 Basic principles

First we define some of the fundamental concepts. This is done more methodically in Chapter 2. Throughout this book, we will typically use E to denote the finite-dimensional real vector space $\mathbb{R}^n$ for some $n \in \mathbb{N}$ endowed with its usual norm, and typically X will denote a real infinite-dimensional Banach space – and sometimes merely a normed space. In this introduction we will tend to state results and introduce terminology in the setting of the Euclidean space E because this more familiar and concrete setting already illustrates their power and utility.

A set $C \subset E$ is said to be *convex* if it contains all line segments between its members: $\lambda x + (1 - \lambda)y \in C$ whenever $x, y \in C$ and $0 \leq \lambda \leq 1$. Even in two dimensions this deserves thought: every set S with $\{(x, y) : x^2 + y^2 < 1\} \subset S \subset \{(x, y) : x^2 + y^2 \leq 1\}$ is convex.

The *lower level sets* of a function $f : E \to [-\infty, +\infty]$ are the sets $\{x \in E : f(x) \leq \alpha\}$ where $\alpha \in \mathbb{R}$. The *epigraph* of a function $f : E \to [-\infty, +\infty]$ is defined by

$$\operatorname{epi} f := \{(x, t) \in E \times \mathbb{R} : f(x) \leq t\}.$$

We should note that we will use ∞ and $+\infty$ interchangeably, but we prefer to use $+\infty$ when $-\infty$ is nearby.

Consider a function $f : E \to [-\infty, +\infty]$; we will say f is *closed* if its epigraph is closed; whereas f is *lower-semicontinuous* (lsc) if $\liminf_{x \to x_0} f(x) \geq f(x_0)$ for all $x_0 \in E$. These two concepts are intimately related for convex functions. Our primary focus will be on *proper functions*, those functions $f : E \to [-\infty, +\infty]$ that do not take the value $-\infty$ and whose *domain* of f, denoted by $\operatorname{dom} f$, is defined by $\operatorname{dom} f := \{x \in E : f(x) < \infty\}$. The *indicator function* of a nonempty set D

is the function δ_D defined by $\delta_D(x) := 0$ if $x \in D$ and $\delta_D(x) := +\infty$ otherwise. These notions allow one to study convex functions and convex sets interchangeably, however, our primary focus will center on convex functions.

A sketch of a real-valued differentiable convex function very strongly suggests that the derivative of such a function is monotone increasing, in fact this is true more generally – but in a nonobvious way. If we denote the derivative (or *gradient*) of a real function g by ∇g, then using the inner product the monotone increasing property of ∇g can be written as

$$\langle \nabla g(y) - \nabla g(x), y - x \rangle \geq 0 \text{ for all } x \text{ and } y.$$

The preceding inequality leads to the definition of the *monotonicity* of the gradient mapping on general spaces. Before stating our first basic result, let us recall that a set $K \subset E$ is a *cone* if $tK \subset K$ for every $t \geq 0$; and an *affine mapping* is a translate of a linear mapping.

We begin with a recapitulation of the useful preservation and characterization properties convex functions possess:

Lemma 1.2.1 (Basic properties). *The convex functions form a convex cone closed under pointwise suprema: if f_γ is convex for each $\gamma \in \Gamma$ then so is $x \mapsto \sup_{\gamma \in \Gamma} f_\gamma(x)$.*

(a) A function g is convex if and only if $\operatorname{epi} g$ is convex if and only if $\delta_{\operatorname{epi} g}$ is convex.
(b) A differentiable function g is convex on an open convex set D if and only if ∇g is a monotone operator on D, while a twice differentiable function g is convex if and only if the Hessian $\nabla^2 g$ is a positive semidefinite matrix for each value in D.
(c) $g \circ A$ and $m \circ g$ are convex when g is convex, α is affine and m is monotone increasing and convex.
(d) For $t > 0$, $(x,t) \mapsto tg(x/t)$ and $(x,t) \mapsto g(xt)/t$ are convex if and only if g is and if in the latter case $g(0) \geq 0$.

Proof. See Lemma 2.1.8 for (a), (c) and (d). Part (b) is developed in Theorem 2.2.6 and Theorem 2.2.8, where we are more precise about the form of differentiability used. In (d) one may be precise also about the lsc hulls, see [95] and Exercise 2.3.9. □

Before introducing the next result which summarizes many of the important continuity and differentiability properties of convex functions, we first introduce some crucial definitions. For a proper function $f : E \to (-\infty, +\infty]$, the *subdifferential* of f at $\bar{x} \in E$ where $f(\bar{x})$ is finite is defined by

$$\partial f(\bar{x}) := \{\phi \in E : \langle \phi, y - \bar{x} \rangle \leq f(y) - f(\bar{x}), \text{ for all } y \in E\}.$$

If $f(\bar{x}) = +\infty$, then $\partial f(\bar{x})$ is defined to be empty. Moreover, if $\phi \in \partial f(\bar{x})$, then ϕ is said to be a *subgradient* of f at $\bar{x}$. Note that, trivially but importantly, $0 \in \partial f(x)$ – and we call x a *critical point* – if and only if x is a minimizer of f.

While it is possible for the subdifferential to be empty, we will see below that very often it is not. An important consideration for this is whether $\bar{x}$ is in the boundary of the domain of f or in its interior, and in fact, in finite dimensions, the *relative interior*

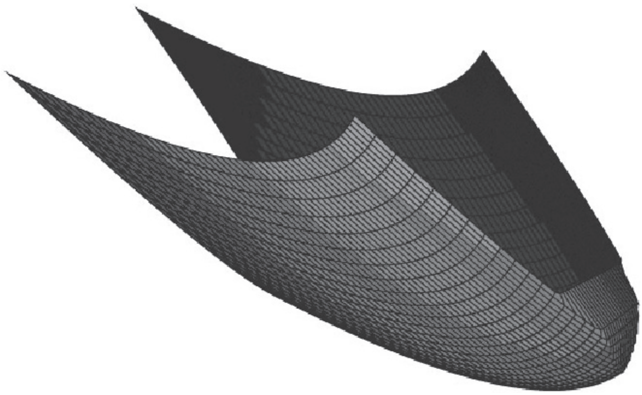

Figure 1.1 A subtle two-dimensional function from Chapter 6.

(i.e. the interior relative to the affine hull of the set) plays an important role. The function f is Fréchet differentiable at $\bar{x}$ with Fréchet derivative $f'(\bar{x})$ if

$$\lim_{t\to 0}\frac{f(\bar{x}+th)-f(\bar{x})}{t}=\langle f'(\bar{x}),h\rangle$$

exists uniformly for all h in the unit sphere. If the limit exists only pointwise, f is *Gâteaux differentiable*. With these terms in mind we are now ready for the next theorem.

Theorem 1.2.2. *In Banach space, the following are central properties of convexity:*

(a) Global minima and local minima coincide for convex functions.
(b) Weak and strong closures coincide for convex functions and convex sets.
(c) A convex function is locally Lipschitz if and only if it is continuous if and only if it is locally bounded above. A finite lsc convex function is continuous; in finite dimensions lower-semicontinuity is not automatic.
(d) In finite dimensions, say n=dim E, the following hold.
  (i) The relative interior of a convex set always exists and is nonempty.
  (ii) A convex function is differentiable if and only if it has a unique subgradient.
  (iii) Fréchet and Gâteaux differentiability coincide.
  (iv) 'Finite' if and only if '$n+1$' or 'n' (e.g. the theorems of Radon, Helly, Carathéodory, and Shapley–Folkman stated below in Theorems 1.2.3, 1.2.4, 1.2.5, and 1.2.6). These all say that a property holds for all finite sets as soon as it holds for all sets of cardinality of order the dimension of the space.

Proof. For (a) see Proposition 2.1.14; for (c) see Theorem 2.1.10 and Proposition 4.1.4. For the purely finite-dimensional results in (d), see Theorem 2.4.6 for (i); Theorem 2.2.1 for (ii) and (iii); and Exercises 2.4.13, 2.4.12, 2.4.11, and 2.4.15, for Helly's, Radon's, Carathéodory's and Shapley–Folkman theorems respectively. □

Theorem 1.2.3 (Radon's theorem). *Let* $\{x_1, x_2, \ldots, x_{n+2}\} \subset \mathbb{R}^n$. *Then there is a partition* $I_1 \cup I_2 = \{1, 2, \ldots, n+2\}$ *such that* $C_1 \cap C_2 \neq \emptyset$ *where* $C_1 = \operatorname{conv}\{x_i : i \in I_1\}$ *and* $C_2 = \operatorname{conv}\{x_i : i \in I_2\}$.

Theorem 1.2.4 (Helly's theorem). *Suppose* $\{C_i\}_{i \in I}$ *is a collection of nonempty closed bounded convex sets in* $\mathbb{R}^n$, *where* I *is an arbitrary index set. If every subcollection consisting of* $n+1$ *or fewer sets has a nonempty intersection, then the entire collection has a nonempty intersection.*

In the next two results we observe that when positive as opposed to convex combinations are involved, '$n+1$' is replaced by 'n'.

Theorem 1.2.5 (Carathéodory's theorem). *Suppose* $\{a_i : i \in I\}$ *is a finite set of points in* E. *For any subset* J *of* I, *define the cone*

$$C_J = \left\{ \sum_{i \in J} \mu_i a_i \ : \ \mu_i \in [0, +\infty), \ i \in J \right\}.$$

(a) The cone C_I *is the union of those cones* C_J *for which the set* $\{a_j : j \in J\}$ *is linearly independent. Furthermore, any such cone* C_J *is closed. Consequently, any finitely generated cone is closed.*
(b) If the point x *lies in* $\operatorname{conv}\{a_i : i \in I\}$ *then there is a subset* $J \subset I$ *of size at most* $1 + \dim E$ *such that* $x \in \operatorname{conv}\{a_i : i \in J\}$. *It follows that if a subset of* E *is compact, then so is its convex hull.*

Theorem 1.2.6 (Shapley–Folkman theorem). *Suppose* $\{S_i\}_{i \in I}$ *is a finite collection of nonempty sets in* $\mathbb{R}^n$, *and let* $S := \sum_{i \in I} S_i$. *Then every element* $x \in \operatorname{conv} S$ *can be written as* $x = \sum_{i \in I} x_i$ *where* $x_i \in \operatorname{conv} S_i$ *for each* $i \in I$ *and moreover* $x_i \in S_i$ *for all except at most* n *indices.*

Given a nonempty set $F \subset E$, the *core* of F is defined by $x \in \operatorname{core} F$ if for each $h \in E$ with $\|h\| = 1$, there exists $\delta > 0$ so that $x + th \in F$ for all $0 \leq t \leq \delta$. It is clear from the definition that the *interior* of a set F is contained in its core, that is, $\operatorname{int} F \subset \operatorname{core} F$. Let $f : E \to (-\infty, +\infty]$. We denote the set of points of continuity of f is denoted by $\operatorname{cont} f$. The *directional derivative* of f at $\bar{x} \in \operatorname{dom} f$ in the direction h is defined by

$$f'(\bar{x}; h) := \lim_{t \to 0^+} \frac{f(\bar{x} + th) - f(\bar{x})}{t}$$

if the limit exists – and it always does for a convex function. In consequence one has the following simple but crucial result.

Theorem 1.2.7 (First-order conditions)**.** *Suppose* $f : E \to (-\infty, +\infty]$ *is convex. Then for any* $x \in \operatorname{dom} f$ *and* $d \in E$,

$$f'(x; d) \leq f(x + d) - f(x). \tag{1.2.1}$$

In consequence, f *is minimized (locally or globally) at* x_0 *if and only if* $f'(x_0; d) \geq 0$ *for all* $d \in E$ *if and only if* $0 \in \partial f(x_0)$.

The following fundamental result is also a natural starting point for the so-called *Fenchel duality/Hahn–Banach theorem circle*. Let us note, also, that it directly relates differentiability to the uniqueness of subgradients.

Theorem 1.2.8 (Max formula)**.** *Suppose* $f : E \to (-\infty, +\infty]$ *is convex (and lsc in the infinite-dimensional setting) and that* $\bar{x} \in \operatorname{core}(\operatorname{dom} f)$. *Then for any* $d \in E$,

$$f'(\bar{x}; d) = \max\{\langle \phi, d\rangle : \phi \in \partial f(\bar{x})\}. \tag{1.2.2}$$

In particular, the subdifferential $\partial f(\bar{x})$ *is nonempty at all core points of* $\operatorname{dom} f$.

Proof. See Theorem 2.1.19 for the finite-dimensional version and Theorem 4.1.10 for infinite-dimensional version. □

Building upon the Max formula, one can derive a quite satisfactory calculus for convex functions and linear operators. Let us note also, that for $f : E \to [-\infty, +\infty]$, the *Fenchel conjugate* of f is denoted by f^* and defined by $f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$. The conjugate is always convex (as a supremum of affine functions) while $f = f^{**}$ exactly if f is convex, proper and lsc. A very important case leads to the formula $\delta_C^*(x^*) = \sup_{x\in C}\langle x^*, x\rangle$, the *support function* of C which is clearly continuous when C is bounded, and usually denoted by σ_C. This simple conjugate formula will play a crucial role in many places, including Section 6.6 where some duality relationships between Asplund spaces and those with the Radon–Nikodým property are developed.

Theorem 1.2.9 (Fenchel duality and convex calculus)**.** *Let* E *and* Y *be Euclidean spaces, and let* $f : E \to (-\infty, +\infty]$ *and* $g : Y \to (-\infty, +\infty]$ *and a linear map* $A : E \to Y$, *and let* $p, d \in [-\infty, +\infty]$ *be the primal and dual values defined respectively by the Fenchel problems*

$$p := \inf_{x\in E}\{f(x) + g(Ax)\} \tag{1.2.3}$$

$$d := \sup_{\phi\in Y}\{-f^*(A^*\phi) - g^*(-\phi)\}. \tag{1.2.4}$$

Then these values satisfy the weak duality inequality $p \geq d$. If, moreover, f and g are convex and satisfy the condition

$$0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f) \tag{1.2.5}$$

or the stronger condition

$$A \operatorname{dom} f \cap \operatorname{cont} g \neq \emptyset \tag{1.2.6}$$

then $p = d$ and the supremum in the dual problem (1.2.4) is attained if finite.

At any point $x \in E$, the subdifferential sum rule,

$$\partial(f + g \circ A)(x) \supset \partial f(x) + A^* \partial g(Ax) \tag{1.2.7}$$

holds, with equality if f and g are convex and either condition (1.2.5) or (1.2.6) holds.

Proof. The proof for Euclidean spaces is given in Theorem 2.3.4; a version in Banach spaces is given in Theorem 4.4.18. □

A nice application of Fenchel duality is the ability to obtain primal solutions from dual ones; this is described in Exercise 2.4.19.

Corollary 1.2.10 (Sandwich theorem)**.** *Let $f : E \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be convex, and let $A : E \to Y$ be linear. Suppose $f \geq -g \circ A$ and $0 \in \operatorname{core}(\operatorname{dom} g - A \operatorname{dom} f)$ (or $A \operatorname{dom} f \cap \operatorname{cont} g \neq \emptyset$). Then there is an affine function $\alpha : E \to \mathbb{R}$ satisfying $f \geq \alpha \geq -g \circ A$.*

It is sometimes more desirable to symmetrize this result by using a concave function g, that is a function for which $-g$ is convex, and its *hypograph*, hyp g, as in Figure 1.2.

Using the sandwich theorem, one can easily deduce Hahn–Banach extension theorem (2.1.18) and the Max formula to complete the so-called Fenchel duality/Hahn–Banach circle.

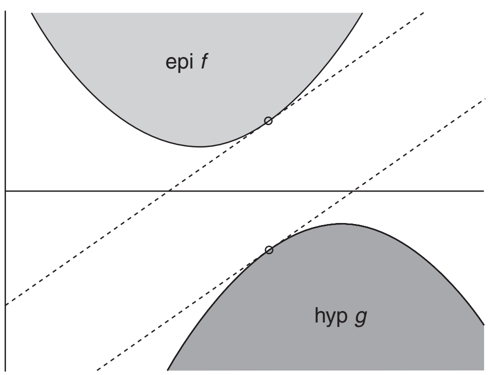

Figure 1.2 A sketch of the sandwich theorem.

A final key result is the capability to reconstruct a convex set from a well defined set of boundary points, just as one can reconstruct a convex polytope from its corners (extreme points). The basic result in this area is:

Theorem 1.2.11 (Minkowski)**.** *Let E be a Euclidean space. Any compact convex set $C \subset E$ is the convex hull of its extreme points. In Banach space it is typically necessary to take the closure of the convex hull of the extreme points.*

Proof. This theorem is proved in Euclidean spaces in Theorem 2.7.2. □

With these building blocks in place, we use the following sections to illustrate some diverse examples where convex functions and convexity play a crucial role.

1.3 Some mathematical illustrations

Perhaps the most forcible illustration of the power of convexity is the degree to which the theory of *best approximation,* i.e. existence of *nearest points* and the study of nonexpansive mappings, can be subsumed as a convex optimization problem. For a closed set S in a Hilbert space X we write $d_S(x) := \inf_{x \in S} \|x - s\|_2$ and call d_S the (metric) *distance function* associated with the set S. A set C in X such that each $x \in X$ has a unique nearest point in C is called a *Čebyšev set*.

Theorem 1.3.1. *Let X be a Euclidean (resp. Hilbert) space and suppose C is a nonempty (weakly) closed subset of X. Then the following are equivalent.*

(a) C is convex.
(b) C is a Čebyšev set.
(c) d_C^2 is Fréchet differentiable.
(d) d_C^2 is Gâteaux differentiable.

Proof. See Theorem 4.5.9 for the proof. □

We shall use the necessary condition for $\inf_C f$ to deduce that the projection on a convex set is nonexpansive; this and some other properties are described in Exercise 2.3.17.

Example 1.3.2 (Algebra)**.** *Birkhoff's theorem* [57] says the doubly stochastic matrices (those with nonnegative entries whose row and column sum equal one) are convex combinations of permutation matrices (their extreme points).

A proof using convexity is requested in Exercise 2.7.5 and sketched in detail in [95, Exercise 22, p. 74].

Example 1.3.3 (Real analysis)**.** The following very general construction links convex functions to nowhere differentiable continuous functions.

Theorem 1.3.4 (Nowhere differentiable functions [145])**.** *Let $a_n > 0$ be such that $\sum_{n=1}^{\infty} a_n < \infty$. Let $b_n < b_{n+1}$ be integers such that $b_n | b_{n+1}$ for each n, and the* *sequence $a_n b_n$ does not converge to 0. For each index $j \geq 1$, let f_j be a continuous function mapping the real line onto the interval* $[0, 1]$ *such that* $f_j = 0$ *at each even integer and* $f_j = 1$ *at each odd integer. For each integer k and each index j, let f_j be convex on the interval* $(2k, 2k+2)$.

Then the continuous function $\sum_{j=1}^{\infty} a_j f_j(b_j x)$ has neither a finite left-derivative nor a finite right-derivative at any point.

In particular, for a convex nondecreasing function f mapping $[0, 1]$ to $[0, 1]$, define $f(x) = f(2-x)$ for $1 < x < 2$ and extend f periodically. Then $F_f(x) := \sum_{j=1}^{\infty} 2^{-j} f(2^j x)$ defines a continuous nowhere differentiable function.

Example 1.3.5 (Operator theory)**.** The Riesz–Thorin convexity theorem informally says that if T induces a bounded linear operator between Lebesgue spaces L^{p_1} and L^{p_2} and also between L^{q_1} and L^{q_2} for $1 < p_1, p_2 < \infty$ and $1 < q_1, q_2 < \infty$ then it also maps L^{r_1} to L^{r_2} whenever $(1/r_1, 1/r_2)$ is a convex combination of $(1/p_1, 1/p_2)$ and $(1/p_1, 1/p_2)$ (all three pairs lying in the unit square).

A precise formulation is given by Zygmund in [451, p. 95].

Example 1.3.6 (Real analysis)**.** The Bohr–Mollerup theorem characterizes the gamma-function $x \mapsto \int_0^\infty t^{x-1} \exp(-t)\, \mathrm{d}t$ as the unique function f mapping the positive half line to itself such that (a) $f(1) = 1$, (b) $xf(x) = f(x+1)$ and (c) $\log f$ is convex function

A proof of this is outlined in Exercise 2.1.24; Exercise 2.1.25 follows this by outlining how this allows for computer implementable proofs of results such as $\beta(x, y) = \Gamma(x)\Gamma(y)/\,\Gamma(x, y)$ where β is the classical beta-function. A more extensive discussion of this topic can be found in [73, Section 4.5].

Example 1.3.7 (Complex analysis)**.** *Gauss's theorem* shows that the roots of the derivative of a polynomial lie inside the convex hull of the zeros.

More precisely one has the *Gauss–Lucas theorem*: For an arbitrary not identically constant polynomial, the zeros of the derivative lie in the smallest convex polygon containing the zeros of the original polynomial. While Gauss originally observed: *Gauss's theorem*: The zeros of the derivative of a polynomial P that are not multiple zeros of P are the positions of equilibrium in the field of force due to unit particles situated at the zeros of P, where each particle repels with a force equal to the inverse distance. *Jensen's sharpening* states that if P is a real polynomial not identically constant, then all nonreal zeros of P' lie inside the Jensen disks determined by all pairs of conjugate nonreal zeros of P. See Pólya–Szegő [273].

Example 1.3.8 (Levy–Steinitz theorem (combinatorics))**.** The rearrangements of a series with values in Euclidean space always is an affine subspace (also called a flat).

Riemann's rearrangement theorem is the one-dimensional version of this lovely result. See [382], and also *Pólya-Szẽgo* [272] for the complex (planar) case.

We finish this section with an interesting example of a convex function whose convexity, established in [74, §1.9], seems hard to prove directly (a proof is outlined in Exercise 4.4.10):

Example 1.3.9 (Concave reciprocals)**.** Let $g(x) > 0$ for $x > 0$. Suppose $1/g$ is concave (which implies $\log g$ and hence g are convex) then

$$(x,y) \mapsto \frac{1}{g(x)} + \frac{1}{g(y)} - \frac{1}{g(x+y)},$$

$$(x,y,z) \mapsto \frac{1}{g(x)} + \frac{1}{g(y)} + \frac{1}{g(z)} - \frac{1}{g(x+y)} - \frac{1}{g(y+z)} - \frac{1}{g(x+z)} + \frac{1}{g(x+y+z)}$$

and all similar n-fold alternating combinations are reciprocally concave on the strictly positive orthant. The foundational case is $g(x) := x$. Even computing the Hessian in a computer algebra system in say six dimensions is a Herculean task.

1.4 Some more applied examples

Another lovely advertisement for the power of convexity is the following reduction of the classical *Brachistochrone problem* to a tractable convex equivalent problem. As Balder [29] recalls

> 'Johann Bernoulli's famous 1696 brachistochrone problem asks for the optimal shape of a metal wire that connects two fixed points A and B in space. A bead of unit mass falls along this wire, without friction, under the sole influence of gravity. The shape of the wire is defined to be optimal if the bead falls from A to B in as short a time as possible.'

Example 1.4.1 (Calculus of variations)**.** *Hidden convexity* in the Brachistochrone problem. The standard formulation, requires one to minimize

$$T(f) := \int_0^{x_1} \frac{\sqrt{1+f'^2(x)}}{\sqrt{gf(x)}}\, \mathrm{d}x \tag{1.4.1}$$

over all positive smooth arcs f on $(0,x_1)$ which extend continuously to have $f(0) = 0$ and $f(x_1) = y_1$, and where we let $A = (0,0)$ and $B := (x_1,y_1)$, with $x_1 > 0, y_1 \geq 0$. Here g is the gravitational constant.

A priori, it is not clear that the minimum even exists – and many books slough over all of the hard details. Yet, it is an easy exercise to check that the substitution $\phi := \sqrt{f}$ makes the integrand *jointly convex*. We obtain

$$S(\phi) := \sqrt{2g T}(\phi^2) = \int_0^{x_1} \sqrt{1/\phi^2(x) + 4\phi'^2(x)}\, \mathrm{d}x. \tag{1.4.2}$$

One may check elementarily that the solution ψ on $(0,x_1)$ of the differential equation

$$\left(\psi'(x)\right)^2 \psi^2(x) = C/\psi(x)^2 - 1, \qquad \psi(0) = 0,$$

where C is chosen to force $\psi(x_1) = \sqrt{y_1}$, exists and satisfies $S(\phi) > S(\psi)$ for all other feasible ϕ. Finally, one unwinds the transformations to determine that the original problem is solved by a cardioid.

It is not well understood when one can make such convex transformations in variational problems; but, when one can, it always simplifies things since we have immediate access to Theorem 1.2.7, and need only verify that the first-order necessary condition holds. Especially for hidden convexity in quadratic programming there is substantial recent work, see e.g. [50, 440].

Example 1.4.2 (Spectral analysis)**.** There is a beautiful *Davis–Lewis theorem* characterizing convex functions of eigenvalues of symmetric matrices. We let $\lambda(S)$ denote the (real) eigenvalues of an n by n symmetric matrix S in nonincreasing order. The theorem shows that if $f : E \to (-\infty, +\infty]$ is a symmetric function, then the 'spectral function' $f \circ \lambda$ is (closed) and convex if and only if f is (closed) and convex. Likewise, differentiability is inherited.

Indeed, what Lewis (see Section 3.2 and [95, §5.2]) established is that the convex conjugate which we shall study in great detail satisfies

$$(f \circ \lambda)^* = f^* \circ \lambda,$$

from which much more actually follows. Three highly illustrative applications follow.

I. (Log determinant) Let $\mathrm{lb}(x) := -\log(x_1 x_2 \cdots x_n)$ which is clearly symmetric and convex. The corresponding spectral function is $S \mapsto -\log\det(S)$.

II. (Sum of eigenvalues) Ranging over permutations π, let

$$f_k(x) := \max_\pi \{x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(k)}\} \text{ for } k \le n.$$

This is clearly symmetric, continuous and convex. The corresponding spectral function is $\sigma_k(S) := \lambda_1(S) + \lambda_2(S) + \cdots + \lambda_k(S)$. In particular the largest eigenvalue, σ_1, is a continuous convex function of S and is differentiable if and only if the eigenvalue is simple.

III. (k-th largest eigenvalue) The k-th largest eigenvalue may be written as

$$\mu_k(S) = \sigma_k(S) - \sigma_{k-1}(S).$$

In particular, this represents μ_k as the difference of two convex continuous, hence locally Lipschitz, functions of S and so we discover the very difficult result that for each k, $\mu_k(S)$ is a locally Lipschitz function of S. Such *difference convex functions* appear at various points in this book (e.g. Exercises 3.2.11 and 4.1.46) Sometimes, as here, they inherit useful properties from their convex parts.

Harder analogs of the Davis–Lewis theorem exists for *singular values, hyperbolic polynomials, Lie algebras*, and the like.

Lest one think most results on the real line are easy, we challenge the reader to prove the empirical observation that

$$p \mapsto \sqrt{p} \int_0^\infty \left| \frac{\sin x}{x} \right|^p dx$$

is difference convex on $(1, \infty)$.

Another lovely application of modern convex analysis is to the theory of two-person zero-sum games.

Example 1.4.3 (Game theory)**.** The seminal result due to von Neumann shows that

$$\mu := \min_C \max_D \langle Ax, y\rangle = \max_D \min_C \langle Ax, y\rangle, \tag{1.4.3}$$

where $C \subset E$ and $D \subset F$ are compact convex sets (originally sets of finite probabilities) and $A : E \mapsto F$ is an arbitrary *payoff matrix*. The common value μ is called the *value* of the game.

Originally, Equation (1.4.3) was proved using fixed point theory (see [95, p. 201]) but it is now a lovely illustration of the power of Fenchel duality since we may write $\mu := \inf_E \left\{\delta_D^*(Ax) + \delta_C(x)\right\}$; see Exercise 2.4.21.

One of the most attractive extensions is due to Sion. It asserts that

$$\min_C \max_D f(x, y) = \max_D \min_C f(x, y)$$

when C, D are compact and convex in Banach space while $f(\cdot, y)$, $-f(x, \cdot)$ are required only to be lsc and *quasi-convex* (i.e. have convex lower level sets). In the convex-concave proof one may use compactness and the Max formula to achieve a very neat proof. We shall see substantial applications of reciprocal concavity and log convexity to the construction of barrier functions in Section 7.4.

Next we turn to entropy:

> 'Despite the narrative force that the concept of entropy appears to evoke in everyday writing, in scientific writing entropy remains a thermodynamic quantity and a mathematical formula that numerically quantifies disorder. When the American scientist Claude Shannon found that the mathematical formula of Boltzmann defined a useful quantity in information theory, he hesitated to name this newly discovered quantity entropy because of its philosophical baggage. The mathematician John von Neumann encouraged Shannon to go ahead with the name entropy, however, since "no one knows what entropy is, so in a debate you will always have the advantage.'[3]

Example 1.4.4 (Statistics and information theory)**.** The function of finite probabilities

$$\overrightarrow{p} \mapsto \sum_{i=1}^{n} p_i \log(p_i)$$

defines the (negative of) *Boltzmann–Shannon entropy*, where $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$, and where we set $0 \log 0 = 0$. (One maximizes entropy and minimizes convex functions.)

I. (Extended entropy.) We may extend this function (minus 1) to the nonnegative orthant by

$$\overrightarrow{x} \mapsto \sum_{i=1}^{n} \left(x_i \log(x_i) - x_i\right). \tag{1.4.4}$$

[3] *The American Heritage Book of English Usage*, p. 158.

(See Exercise 2.3.25 for some further properties of this function.) It is easy to check that this function has Fenchel conjugate

$$\vec{y} \mapsto \sum \exp(y_i),$$

whose conjugate is given by (1.4.4) which must therefore be convex – of course in this case it is also easy to check that $x \log x - x$ has second derivative $1/x > 0$ for $x > 0$.

II. (Divergence estimates.) The function of two finite probabilities

$$(\vec{p}, \vec{q}) \mapsto \sum_{i=1}^{n} \left\{ p_i \log \left(\frac{p_i}{q_i} \right) - (p_i - q_i) \right\},$$

is called the Kullback–Leibler *divergence* and measures how far $\vec{q}$ deviates from $\vec{p}$ (care being taken with $0 \div 0$). Somewhat surprisingly, this function is jointly convex as may be easily seen from Lemma 1.2.1 (d), or more painfully by taking the second derivative. One of the many attractive features of the divergence is the beautiful inequality

$$\sum_{i=1}^{n} p_i \log \left(\frac{p_i}{q_i} \right) \geq \frac{1}{2} \left(\sum_{i=1}^{n} |p_i - q_i| \right)^2, \tag{1.4.5}$$

valid for any two finite probability measures. Note that we have provided a lower bound in the 1-norm for the divergence (see Exercise 2.3.26 for a proof and Exercise 7.6.3 for generalizations). Inequalities bounding the divergence (or generalizations as in Exercise 7.6.3) below in terms of the 1-norm are referred to as *Pinsker-type* inequalities [228, 227].

III. (Surprise maximization.) There are many variations on the current theme. We conclude this example by describing a recent one. We begin by recalling the *Paradox of the Surprise Exam*:

> 'A teacher announces in class that an examination will be held on some day during the following week, and moreover that the examination will be a surprise. The students argue that a surprise exam cannot occur. For suppose the exam were on the last day of the week. Then on the previous night, the students would be able to predict that the exam would occur on the following day, and the exam would not be a surprise. So it is impossible for a surprise exam to occur on the last day. But then a surprise exam cannot occur on the penultimate day, either, for in that case the students, knowing that the last day is an impossible day for a surprise exam, would be able to predict on the night before the exam that the exam would occur on the following day. Similarly, the students argue that a surprise exam cannot occur on any other day of the week either. Confident in this conclusion, they are of course totally surprised when the exam occurs (on Wednesday, say). The announcement is vindicated after all. Where did the students' reasoning go wrong?' ([151])

This paradox has a grimmer version involving a hanging, and has a large literature [151]. As suggested in [151], one can leave the paradox to philosophers and ask, more pragmatically, the information-theoretic question *what distribution of events will*

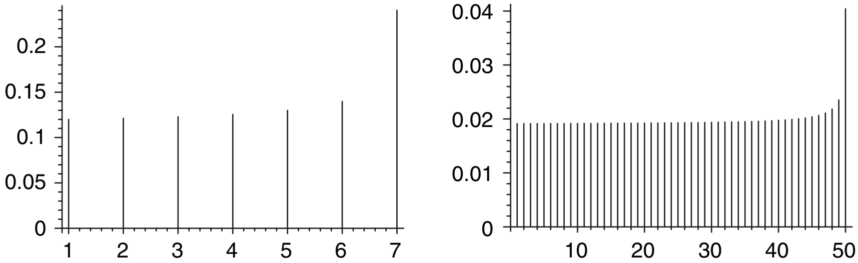

Figure 1.3 Optimal distributions: $m = 7$ (L) and $m = 50$ (R).

maximize group surprise? This question has a most satisfactory resolution. It leads naturally (see [95, Ex. 28, p. 87]) to the following optimization problem involving S_m, the *surprise function*, given by

$$S_m(\overrightarrow{p}) := \sum_{j=1}^{m} p_j \log\left(\frac{p_j}{\frac{1}{m}\sum_{i\geq j} p_i}\right),$$

with the explicit constraint that $\sum_{j=1}^{m} p_j = 1$ and the implicit constraint that each $p_i \geq 0$.

From the results quoted above the reader should find it easy to show S_m is convex. Remarkably, the optimality conditions for maximizing surprise can be solved beautifully recursively as outlined in [95, Ex. 28, p.87]. Figure 1.3 shows examples of optimal probability distributions, for $m = 7$ and $m = 50$.

1.4.1 Further examples of hidden convexity

We finish this section with two wonderful 'hidden convexity' results.

I. (Aumann integral) The integral of a multifunction $\Omega : T \mapsto E$ over a finite measure space T, denoted $\int_T \Omega$, is defined as the set of all points of the form $\int_T \phi(t)\mathrm{d}\mu$, where μ is a finite positive measure and $\phi(\cdot)$ is an integrable measurable *selection* $\phi(t) \in \Omega(t)$ a.e. We denote by $\operatorname{conv}\Omega$ the multifunction whose value at t is the convex hull of $\Omega(t)$. Recall that Ω is *measurable* if $\{t : \Omega(t) \cap W \neq \emptyset\}$ is measurable for all open sets W and is *integrably bounded* if $\sup_{\sigma\in\Omega} \|\sigma(t)\|$ is integrable; here σ ranges over all integrable selections.

Theorem 1.4.5 (Aumann convexity theorem)**.** *Suppose (a) E is finite-dimensional and μ is a nonatomic probability measure. Suppose additionally that (b) Ω is measurable, has closed nonempty images and is integrably bounded. Then*

$$\int_T \Omega = \int_T \operatorname{conv}\Omega,$$

and is compact.

In the fine survey by Zvi Artstein [9] compactness follows from the Dunford–Pettis criterion (see §5.3); and the exchange of convexity and integral from an extreme point argument plus some measurability issues based on Filippov's lemma. We refer the reader to [9, 155, 156, 157] for details and variants.[4]

In particular, since the right-hand side of Theorem 1.4.5 is clearly convex we have the following weaker form which is easier to prove – directly from the Shapley–Folkman theorem (1.2.6) – as outlined in Exercise 2.4.16 and [415]. Indeed, we need not assume (b).

Theorem 1.4.6 (Aumann convexity theorem (weak form)). *If E is finite-dimensional and μ is a nonatomic probability measure then*

$$\int_T \Omega = \operatorname{conv} \int_T \Omega.$$

The simplicity of statement and the potency of this result (which predates Aumann) means that it has attracted a large number of alternative proofs and extensions, [155]. An attractive special case – originating with Lyapunov – takes

$$\Omega(t) := \{-f(t), f(t)\}$$

where f is any continuous function. This is the genesis of so-called 'bang-bang' control since it shows that in many settings control mechanisms which only take extreme values will recapture all behaviour. More generally we have:

Corollary 1.4.7 (Lyapunov convexity theorem). *Suppose E is finite-dimensional and μ is a nonatomic finite vector measure $\mu := (\mu_1, \mu_2, \ldots, \mu_n)$ defined on a sigma-algebra, Σ, of subsets of T, and taking range in E. Then $R(\mu) := \{\mu(A) : A \in \Sigma\}$ is convex and compact.*

We *sketch* the proof of convexity (the most significant part). Let $\nu := \sum |\mu_k|$. By the Radon–Nikodým theorem, as outlined in Exercise 6.3.6, each μ_i is absolutely continuous with respect to ν and so has a Radon–Nikodým derivative f_k. Let $f := (f_1, f_2, \ldots, f_n)$. It follows, with $\Omega(t) := \{0, f(t)\}$ that we may write

$$R(\mu) = \int_T \Omega \, d\nu.$$

Then Theorem 1.4.5 shows the convexity of the range of the vector measure. (See [260] for another proof.)

II. (Numerical range) As a last taste of the ubiquity of convexity we offer the beautiful hidden convexity result called the *Toeplitz–Hausdorff theorem* which establishes the convexity of the *numerical range*, $W(A)$, of a complex square matrix A (or indeed of a bounded linear operator on complex Hilbert space). Precisely,

$$W(A) := \{\langle Ax, x\rangle : \langle x, x\rangle = 1\},$$

[4] [156] discusses the general statement.

so that it is not at all obvious that $W(A)$ should be convex, though it is clear that it must contain the spectrum of A.

Indeed much more is true. For example, for a normal matrix the numerical range is the convex hull of the eigenvalues. Again, although it is not obvious there is a tight relationship between the Toeplitz–Hausdorff theorem and Birkhoff's result (of Example 1.3.2) on doubly stochastic matrices.

Conclusion Another suite of applications of convexity has not been especially highlighted in this chapter but will be at many places later in the book. Wherever possible, we have illustrated a convexity approach to a piece of pure mathematics. Here is one of our favorite examples.

Example 1.4.8 (Principle of uniform boundedness)**.** The principle asserts that *pointwise bounded families of bounded linear operators between Banach spaces are uniformly bounded*. That is, we are given bounded linear operators $T_\alpha : X \to Y$ for $\alpha \in \mathcal{A}$ and we know that $\sup_{\alpha\in A} \|T_\alpha(x)\| < \infty$ for each x in X. We wish to show that $\sup_{\alpha\in A} \|T_\alpha\| < \infty$. Here is the convex analyst's proof:

Proof. Define a function f_A by

$$f_A(x) := \sup_{\alpha\in A} \|T_\alpha(x)\|$$

for each x in X. Then, as observed in Lemma 1.2.1, f_A is convex. It is also closed since each mapping $x \mapsto \|T_\alpha(x)\|$ is (see also Exercise 4.1.5). Hence f_A is a finite, closed convex (actually sublinear) function. Now Theorem 1.2.2 (c) (Proposition 4.1.5) ensures f_A is continuous at the origin. Select $\varepsilon > 0$ with $\sup\{f_A(x) : \|x\| \le \varepsilon\} \le 1$. It follows that

$$\sup_{\alpha\in A} \|T_\alpha\| = \sup_{\alpha\in A} \sup_{\|x\|\le 1} \|T_\alpha(x)\| = \sup_{\|x\|\le 1} \sup_{\alpha\in A} \|T_\alpha(x)\| \le 1/\varepsilon.$$

□

We give a few other examples:

- The Lebesgue–Radon–Nikodým decomposition theorem viewed as a convex optimization problem (Exercise 6.3.6).
- The Krein–Šmulian or Banach–Dieudonné theorem derived from the von Neumann minimax theorem (Exercise 4.4.26).
- The existence of Banach limits for bounded sequences illustrating the Hahn–Banach extension theorem (Exercise 5.4.12).
- Illustration that the full axiom of choice is embedded in various highly desirable convexity results (Exercise 6.7.11).
- A variational proof of Pitt's theorem on compactness of operators in ℓ_p spaces (Exercise 6.6.3).

- The whole of Chapter 9 in which convex Fitzpatrick functions are used to attack the theory of maximal monotone operators – not to mention Chapter 7.

Finally we would be remiss not mentioned the many lovely applications of convexity in the study of partial differential equations (especially elliptic) see [195] and in the study of control systems [157]. In this spirit, Exercises 3.5.17, 3.5.18 and Exercise 3.5.19 make a brief excursion into differential inclusions and convex Lyapunov functions.

References

[1] S. Adly, E. Ernst, and M. Théra. On the closedness of the algrebraic difference of closed convex sets. *J. Math. Pures Appl.*, **9**:1219–1249, 2003.

[2] A. D. Alexandrov. Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it. *Leningrad State Univ. Ann. Math Ser.*, **6**:3–35, 1939.

[3] M. Alves and B. F. Svaiter. Brøndsted-Rockafellar property and maximality of monotone operators representable by convex functions in non-reflexive Banach spaces. *J. Convex Anal.*, **15**:673–706, 2008.

[4] M. Alves and B. F. Svaiter. Maximal monotone operators with a unique extension to the bidual. arXiv:0805.4604v1, 2008.

[5] M. Alves and B. F. Svaiter. A new proof for maximal monotonicity of subdifferential operators. *J. Convex Anal.*, **15**:345–348, 2008.

[6] M. Alves and B. F.. Svaiter. On gossez type (D) maximal monotone operators. arXiv:0903.5332v1, 2009.

[7] Dan Amir. *Characterizations of inner product spaces*, volume 20 of *Operator Theory: Advances and Applications*. Birkhäuser, Boston, 1986.

[8] N. Aronszajn. Differentiability of Lipschitzian mappings between Banach spaces. *Studia Math.*, **57**:147–190, 1976.

[9] Z. Artstein. Discrete and continuous bang-bang and facial spaces or: look at the extreme points. *SIAM Review*, **22**:172–185, 1980.

[10] S. Artstein-Avidan and V. Milman. A characterization of the concept of duality. Preprint, 2008.

[11] S. Artstein-Avidan and V. Milman. A characterization of the concept of duality in asymptotic geometric analysis and the characterization of the Legendre transform. Preprint, 2008.

[12] E. Asplund. Averaged norms. *Israel J. Math.*, **5**:227–233, 1967.

[13] E. Asplund. Fréchet differentiability of convex functions. *Acta Math.*, **128**:31–47, 1968.

[14] E. Asplund. Cebysev sets in Hilbert spaces. *Trans. Amer. Math. Soc.*, **144**:235–240, 1969.

[15] E. Asplund. Topics in the theory of convex functions. In *Theory and Applications of Monotone Operators*, pages 1–33. Gubbio, 1969.

[16] E. Asplund. A monotone convergence theorem for sequences of nonlinear mappings. *Proc. Symposia Pure Math.*, **18**:1–9, 1970.

[17] E. Asplund and R. T. Rockafellar. Gradients of convex functions. *Trans. Amer. Math. Soc.*, **139**:443–467, 1969.

[18] H. Attouch. Convergences de fonctions convexes, des sous-différentiels et semi-groupes associés. *C. R. Acad. Sci. Paris*, **284**:539–542, 1977.

[19] H. Attouch. *Variational Convergence for Functions and Operators*. Ptiman, New York, 1984.

[20] H. Attouch and G. Beer. On the convergence of subdifferentials of convex functions. *Arch. Math.*, **60**:389–400, 1993.

[21] J.-P. Aubin and A. Cellina. *Differential Inclusions*. Springer-Verlag, Berlin, 1984.

[22] J.-P. Aubin and H. Frankowska. *Set-Valued Analysis*, volume 2 of *Systems & Control: Foundations & Applications*. Birkhäuser, Boston, 1990.

[23] A. Auslender and J. P. Crouzeiz. Well behaved asymptotically convex functions. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, **S6**:101–121, 1989.

[24] A. Auslender and M. Teboulle. *Asymptotic Cones and Functions in Optimization and Variational Inequalities*. Springer-Verlag, New York, 2003.

[25] M. Avriel, W. Diewert, S. Schaible, and I. Zang. *Generalized concavity*, volume 36 of *Mathematical Concepts and Methods in Science and Engineering*. Plenum Press, New York, 88.

[26] D. Azagra and J. Ferrera. Every closed convex set is the set of minimizers of some C^{∞}-smooth convex function. *Proc. Amer. Math. Soc.*, **130**:3687–3892, 2002.

[27] D. Azé and J.-P. Penot. Uniformly convex and uniformly smooth convex functions. *Ann. Fac. Sci. Toulouse Math.*, **4**:705–730, 1995.

[28] G. Bachman and N. Narici. *Functional Analysis*. Dover Books, 1966.

[29] E. Balder. The Brachistochrone problem made elementary. Available at: http://www.math.uu.nl/people/balder/talks/vanbeek.pdf, 2002.

[30] V. Bangert. Analytische Eigenschaften knovexer Funktionen auf Riemannschen Mannigfaltigkeiten. *J. Reine Angew. Math*, **307**:309–324, 1979.

[31] S. Bartz, H. H. Bauschke, J. M. Borwein, S. Reich, and X. Wang. Fitzpatrick functions, cyclic monotonicity and Rockafellar's antiderivative. *Nonlinear Anal.*, **66**:1198–1223, 2007.

[32] H. H. Bauschke and J. M. Borwein. Legendre functions and the method of random Bregman projections. *J. Convex Anal.*, **4**(1):27–67, 1997.

[33] H. H. Bauschke and J. M. Borwein. Maximal monotonicity of dense type, local maximal monotonicity, and monotonicity of the conjugate are all the same for continuous linear operators. *Pacific J. Math.*, **189**:1–20, 1999.

[34] H. H. Bauschke and J. M. Borwein. Joint and separate convexity of the Bregman distance. In D. Butnariu, Y. Censor, and S. Reich, editors, *Inherently Parallel Algorithms in Feasibility and Optimization and their Applications (Haifa 2000)*, pages 23–36. Elsevier, 2001.

[35] H. H. Bauschke, J. M. Borwein, and P. L. Combettes. Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces. *Commun. Contemp. Math.*, **3**:615–647, 2001.

[36] H. H. Bauschke, J. M. Borwein, and P. L. Combettes. Bregman monotone optimization algorithms. *SIAM J. Control and Optim.*, **42**:596–636, 2003.

[37] H. H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. CMS Books in Mathematics. Springer-Verlag, to appear.

[38] H. H. Bauschke, D. A. McLaren, and H. S. Sendov. Fitzpatrick functions: inequalities, examples and remarks on a problem of S. Fitzpatrick. *J. Convex Analysis*, to appear.

[39] H. H. Bauschke and X. Wang. The kernel average for two convex functions and its applications to the extension and representation of monotone operators. *Trans. Amer. Math. Soc.*, to appear.

[40] H. H. Bauschke, X. Wang, and L. Yao. Monotone linear relations: maximality and Fitzpatrick functions. Preprint, 2008.

[41] H. H. Bauschke, X. Wang, X. Yuan, and J. Ye. Bregman proximity operators and Chebyshev sets. Preprint, 2008.

[42] G. Beer. Conjugate convex functions and the epi-distance topology. *Proc. Amer. Math. Soc.*, **108**:117–126, 1990.

[43] G. Beer. The slice topology: a viable alternative to Mosco convergence in nonreflexive spaces. *Nonlinear Anal.*, **19**:271–290, 1992.

[44] G. Beer. *Topologies on Closed and Closed Convex Sets*, volume 268 of *Mathematics and Its Applications*. Kluwer Academic Publishers, Dordrecht, 1993.

[45] G. Beer. Lipschitz regularization and the convergence of convex functions. *Numer. Funct. Anal. Optim.*, **15**:31–46, 1994.

[46] G. Beer and J. M. Borwein. Mosco convergence and reflexivity. *Proc. Amer. Math. Soc.*, **109**:427–436, 1990.

[47] E. Behrends. New proofs of Rosenthal's ℓ_1-theorem and the Josefson–Nissenzweig theorem. *Bull. Polish Acad. Sci. Math*, **43**:283–295, 1995.

[48] J. L. Bell and D. H. Fremlin. A geometric form of the axiom of choice. *Fund. Math.*, **77**:167–170, 1972.

[49] R. Bellman. *Introduction to Matrix Analysis*. SIAM, Philadelphia, 1997.

[50] A. Ben-Tal and M. Teboulle. Hidden convexity in some nonconvex quadratically constrained quadratic programming. *Math. Programming*, **72**:51–63, 1996.

[51] J. Benoist, J. M. Borwein, and N. Popovici. Q characterization of quasiconvex vector-valued functions. *Proc. Amer. Math. Soc.*, **121**:1109–1103, 2002.

[52] Y. Benyamini and J. Lindenstrauss. *Geometric Nonlinear Functional Analysis*, volume 48 of *AMS Colloquium Publications*. American Mathematical Society, Providence, Rhode Island, 2000.

[53] M. Berger. Convexity. *Amer. Math. Monthly*, **97**:650–678, 1990.

[54] D. S. Bernstein. *Matrix Mathematics: Theory, Facts, and Formulas with Application to Linear Systems Theory*. Princeton University Press, Princeton, NJ, 2005.

[55] B. N. Bessis and F. H. Clarke. Partial subdifferentals, derivatives and Rademacher's theorem. *Trans. Amer. Math. Soc.*, **351**:2899–2926, 1999.

[56] B. Bhattacharya. An iterative procedure for general probability measures to obtain 1-projections onto intersections of convex sets. *Ann. Statist.*, **34**:878–902, 2006.

[57] G Birkhoff. Tres observaciones sobre el algebra lineal. *Nacionale Tucamán Revista*, **5**:147–151, 1946.

[58] E. Bishop and R. R. Phelps. A proof that every Banach space is subreflexive. *Bull. Amer. Math. Soc.*, **67**:97–98, 1961.

[59] E. Bishop and R. R. Phelps. Topics in the theory of convex functions. In V.L. Klee, editor, *Proc. Sympos. Pure Math.*, volume VII, pages 27–35. Amer. Math. Soc., Providence, R.I., 1963.

[60] B. Bollobás. An extension to the theorem of Bishop and Phelps. *Bull. London Math. Soc.*, **2**:181–182, 1970.

[61] R. Bonic and J. Frampton. Smooth functions on Banach manifolds. *J. Math. Mech.*, **15**:877–898, 1966.

[62] J. M. Borwein. A Lagrange multiplier theorem and a sandwich theorem for convex relations. *Math. Scand.*, **48**:198–204, 1981.

[63] J. M. Borwein. A note on the existence of subgradients. *Math. Programming*, **24**:225–228, 1982.

[64] J. M. Borwein. On the Hahn–Banach extension property. *Proc. Amer. Math. Soc.*, **86**:42–46, 1982.

[65] J. M. Borwein. On the existence of pareto efficient points. *Math. Oper. Res.*, **8**:64–73, 1983.

[66] J. M. Borwein. An integral characterization of Euclidean space. *Bull. Austral. Math. Soc.*, **29**:357–364, 1984.

[67] J. M. Borwein. Epi-Lipschitz-like sets in Banach space: Theorems and examples. *Nonlinear Anal.*, **11**:1207–1217, 1987.

[68] J. M. Borwein. Minimal cuscos and subgradients of Lipschitz functions. In J.B. Baillon and M. Thera, editors, *Fixed Point Theory and its Applications*, Pitman Research Notes in Mathematics, pages 57–82. Longman, Essex, 1991.

[69] J. M. Borwein. Maximal monotonicity via convex analysis. *J. Convex Anal.*, 13/**14**:561–586, 2006.

[70] J. M. Borwein. Asplund decompositions of monotone operators. *Proc. Control, Set-Valued Analysis and Applications, ESAIM: Proceedings, A. Pietrus & M. H. Geoffroy, Eds.*, **17**:19–25, 2007.

[71] J. M. Borwein. Maximality of sums of two maximal monotone operators in general Banach space. *Proc. Amer. Math. Soc.*, **135**:3917–3924, 2007.

[72] J. M. Borwein. Proximality and Chebyshev sets. *Optimization Letters*, **1**:21–32, 2007.

[73] J. M. Borwein and D. Bailey. *Mathematics by Experiment*. A K Peters, Natick, MA, 2004.

[74] J. M. Borwein, D. Bailey, and R. Girgensohn. *Experimentation in Mathematics*. A K Peters, Natick, MA, 2004.

[75] J. M. Borwein and M. A. H. Dempster. The linear order-complementarity problem. *Math. Oper. Res.*, **14**:534–558, 1989.

[76] J. M. Borwein and A. Eberhard. Banach spaces of type (NI) and maximal monotone operators on reflexive Banach spaces. Preprint, 2008.

[77] J. M. Borwein and M. Fabian. On convex functions having points of Gateaux differentiability which are not points of Fréchet differentiability. *Canad. J. Math.*, **45**:1121–1134, 1993.

[78] J. M. Borwein, M. Fabian, and J. Vanderwerff. Characterizations of Banach spaces via convex and other locally Lipschitz functions. *Acta. Vietnamica*, **22**:53–69, 1997.

[79] J. M. Borwein and S. Fitzpatrick. Local boundedness of monotone operators under minimal hypotheses. *Bull. Austral. Math. Soc.*, **39**:339–341, 1988.

[80] J. M. Borwein and S. Fitzpatrick. Existence of nearest points in Banach spaces. *Canad. J. Math.*, **61**:702–720, 1989.

[81] J. M. Borwein and S. Fitzpatrick. Mosco convergence and the Kadec property. *Proc. Amer. Math. Soc.*, **106**:843–851, 1989.

[82] J. M. Borwein and S. Fitzpatrick. A weak Hadamard smooth renorming of $L_1(\omega, \mu)$. *Canad. Math. Bull.*, **36**:407–413, 1993.

[83] J. M. Borwein, S. Fitzpatrick, and R. Girgensohn. Subdifferentials whose graphs are not bounded$\times$weak-star closed. *Canad. Math. Bulletin*, **46**:538–545, 2003.

[84] J. M. Borwein, S. Fitzpatrick, and P. Kenderov. Minimal convex uscos and monotone operators on small sets. *Canad. J. Math.*, **43**:461–476, 1991.

[85] J. M. Borwein, S. Fitzpatrick, and J. Vanderwerff. Examples of convex functions and classifications of normed spaces. *J. Convex Anal.*, **1**:10–18, 1994.

[86] J. M. Borwein, A. Guirao, P. Hájek, and J. Vanderwerff. Uniformly convex functions on Banach spaces. *Proc. Amer. Math. Soc.*, **137**:1081–1091, 2009.

[87] J. M. Borwein and C. Hamilton. Symbolic convex analysis: Algorithms and examples. *Math. Programing*, **116**:17–35, 2009.

[88] J. M. Borwein, M. Jiménez Sevilla, and Moreno J. P. Antiproximinal norms in Banach spaces. *J. Approx. Theory*, **114**:57–69, 2002.

[89] J. M. Borwein and A. S. Lewis. On the convergence of moment problems. *Tran. Amer. Math. Soc.*, **325**:249–271, 1991.

[90] J. M. Borwein and A. S. Lewis. Partially-finite convex programming (I). *Math. Prog.*, **57**:15–48, 1992.

[91] J. M. Borwein and A. S. Lewis. Partially-finite convex programming (II). *Math. Prog.*, **57**:49–84, 1992.

[92] J. M. Borwein and A. S. Lewis. Convergence of decreasing sequences of convex sets in non-reflexive Banach spaces. *Set-Valued Analysis*, **4**:355–364, 1993.

[93] J. M. Borwein and A. S. Lewis. Partially-finite convex programming in l^1: entropy maximization. *SIAM J. Optimization*, **3**:248–267, 1993.

[94] J. M. Borwein and A. S. Lewis. Strong rotundity and optimization. *SIAM J. Optimization*, **4**:146–158, 1994.

[95] J. M. Borwein and A. S. Lewis. *Convex Analysis and Nonlinear Optimization: Theory and Examples*, volume 3 of *CMS Books in Mathematics*. Springer, New York, 2000.

[96] J. M. Borwein and A. S. Lewis. *Convex Analysis and Nonlinear Optimization: Theory and Examples*, volume 3 of *CMS Books in Mathematics*. Springer, New York, 2nd edition, 2005.

[97] J. M. Borwein, A. S. Lewis, N. N. Limber, and D. Noll. Fast heuristic methods for function reconstruction using derivative information. *Appl. Anal.*, **58**:241–261, 1995.

[98] J. M. Borwein, A. S. Lewis, N. N. Limber, and D. Noll. Maximum entropy reconstruction using derivative information. II. computational results. *Numer. Math.*, **69**:243–256, 1995.

[99] J. M. Borwein, A. S. Lewis, and D. Noll. Maximum entropy reconstruction using derivative information. *Math. Oper. Res.*, **21**:442–468, 1996.

[100] J. M. Borwein, A. S. Lewis, and R. Nussbaum. Entropy minimization, DAD problems and doubly-stochastic kernels. *J. Funct. Anal.*, **123**:264–307, 1994.

[101] J. M. Borwein, A. S. Lewis, and Q. J. Zhu. Convex spectral functions of compact operators, Part II: lower semicontinuity and rearrangement invariance. In *Proceedings of the Optimization Miniconference VI*, pages 1–18. Kluwer Academic, Dordrecht, 2000.

[102] J. M. Borwein, P. Maréchal, and D. Naugler. A convex dual approach to the computation of NMR complex spectra. *Math. Oper. Res.*, **51**:91–102, 2000.

[103] J. M. Borwein, V. Montesinos, and J. Vanderwerff. Differentiability and extensions of convex functions. *J. Convex Anal.*, **18**:587–602, 2006.

[104] J. M. Borwein and W. B. Moors. Essentially smooth Lipschitz functions. *J. Funct. Anal.*, **49**:305–351, 1997.

[105] J. M. Borwein and W. B. Moors. Null sets and essentially smooth Lipschitz functions. *SIAM J. Opt.*, **8**:309–323, 1998.

[106] J. M. Borwein and W. B. Moors. Stability of closedness of cones under linear mappings. *J. Convex Anal.*, 2008.

[107] J. M. Borwein and D. Noll. Second order differentiability of convex functions in Banach spaces. *Tran. Amer. Math. Soc.*, **342**:43–81, 1994.

[108] J. M. Borwein and D. Preiss. A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions. *Trans. Amer. Math. Soc.*, **303**:517–527, 1987.

[109] J. M. Borwein, J. Read, A. S. Lewis, and Zhu Q. J. Convex spectral functions of compact operators. *J. Nonlinear Convex Anal.*, **1**:17–35, 2000.

[110] J. M. Borwein and H. M. Strójwas. Directionally Lipschitzian mappings on Baire spaces. *Canad. J. Math.*, **36**:95–130, 1984.

[111] J. M. Borwein and J. Vanderwerff. Dual Kadec–Klee norms and the relationships between Wijsman, slice and Mosco convergence. *Michigan Math. J.*, **41**:371–387, 1994.

[112] J. M. Borwein and J. Vanderwerff. Convergence of Lipschitz regularizations of convex functions. *J. Funct. Anal.*, **128**:139–162, 1995.

[113] J. M. Borwein and J. Vanderwerff. Epigraphical and uniform convergence of convex functions. *Trans. Amer. Math. Soc.*, **348**:1617–1631, 1996.

[114] J. M. Borwein and J. Vanderwerff. Convex functions on Banach spaces not containing ℓ_1. *Canad. Math. Bull.*, **40**:10–18, 1997.

[115] J. M. Borwein and J. Vanderwerff. Convex functions of Legendre type in general Banach spaces. *J. Convex Anal.*, **8**:569–581, 2001.

[116] J. M. Borwein and J. Vanderwerff. On the continuity of biconjugate convex functions. *Proc. Amer. Math. Soc.*, **130**:1797–1803, 2001.

[117] J. M. Borwein and J. Vanderwerff. Constructible convex sets. *Set-Valued Anal.*, **12**:61–77, 2004.

[118] J. M. Borwein and H. Wiersma. Decompositions of monotone operators. *SIAM J. Opt.*, **18**:946–960, (2007).

[119] J. M. Borwein and D. A. Yost. Absolute norms on vector lattices. *Proc. Edinburgh Math. Soc.*, **27**:215–222, 19894.

[120] J. M. Borwein and Q. J. Zhu. Multifunctional and functional analytic techniques in nonsmooth analysis. In F.H. Clarke and R.J. Stern, editors, *Nonlinear Analysis, differential equations and control*, NATO Sci. Ser. C Math. Phys. Sci., pages 61–157. Kluwer Academic, Dordrecht, 1999.

[121] J. M. Borwein and Q. J. Zhu. *Techniques of Variational Analysis*, volume 20 of *CMS Books in Mathematics*. Springer, New York, 2005.

[122] J. M. Borwein and Q.J. Zhu. A survey of subdifferential calculus with applications. *Nonlinear Anal., Theory, Methods, Appl.*, **38**:687–773, 1999.

[123] J. M. Borwein and D. Zhuang. On Fan's minimax theorem. *Math. Prog.*, **34**:232–234, 1986.

[124] C. J. Bose and R. Murray. Duality and the computation of approximately invariant measures for non-singular transformations. *SIAM J. Optim.*, **18**:691–709, 2007.

[125] R. M. Bot, S.-I. Grad, and G. Wanka. Maximal monotonicity for the precomposition with a linear operator. Preprint, 2005.

[126] R. D. Bourgin. *Geometric Aspects of Convex Sets with the Radon–Nikodym Property*, volume 993 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[127] S. Boyd, L. El Ghaoui, E. Feron, and V. Balikrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, Philadelphia, 1994.

[128] S. Boyd and L. Vandenberghe. *Convex Optimization*, volume 317 of *A Series of Comprehensive Studies in Mathematics*. Cambridge University Press, New York, 2004.

[129] L. M. Bregman. A relaxation method of finding a common point of convex sets and its application to the solution of problems in convex programming. *U.S.S.R. Comput. Math. Math. Phys.*, **7**:200–217, 1967.

[130] J Brimberg. The Fermat–Weber location problem revisited. *Math. Programming*, **71**:71–76, 1995.

[131] F. E. Browder. Fixed point theory and nonlinear problems. *Bull. Amer. Math. Soc.*, **9**:1–39, 1983.

[132] A. L. Brown. A rotund, reflexive space having a subspace of co-dimention 2 with a discontinuous metric projection. *Mich. Math. J.*, **21**:145–151, 1974.

[133] A. M. Bruckner, Brukner J. B., and B. S. Thomson. *Real Analysis*. Prentice-Hall, Inc., Upper Saddle River, New Jersey, 1997.

[134] H. D. Brunk. An inequality for convex functions. *Proc. Amer. Math. Soc.*, **7**:817–824, 1956.

[135] R. S. Burachik and S. Fitzpatrick. On a family of convex functions attached to a subdifferential. *J. Nonlinear Convex Anal.*, **6**:165–172, 2005.

[136] R. S. Burachik and V. Jeyakumar. A new geometric condition for Fenchel's duality in infinite dimensional spaces. *Math. Prog.*, **104**:229–233, 2005.

[137] R. S. Burachik and B. F. Svaiter. Maximal monotone operators, convex functions and a special family of enlargements. *Set-Valued Anal.*, **10**:297–316, 2002.

[138] H. Busemann and W. Feller. Krümmungseigenschaften konvexer Flächen. *Acta Math.*, **66**:1–47, 1936.

[139] D. Butnariu, Y. Censor, P. Gurfil, and E. Hadar. On the behavior of subgradient projections methods for convex feasibility problems in Euclidean spaces. *SIAM J. Optim.*, **19**:786–807, 2008.

[140] D. Butnariu and A. N. Iusem. *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*. Kluwer, Dordrecht, 2000.

[141] D. Butnariu and E. Resmerita. Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces. *Abstract Appl. Anal.*, **39** pages, 2006.

[142] H. Cabezas* and A. T. Karunanithi. Fisher information, entropy, and the second and third laws of thermodynamics. *ASAP Ind. Eng. Chem. Res.*, ASAP Article, 10.1021/ie7017756, Web published 09-July-2008.

[143] A. K. Cakrabarty, P. Shummugaraj, and C. Zălinescu. Continuity properties for the subdifferential and ε-subdifferential of a convex function and its conjugate. *J. Convex Anal.*, **14**:479–514, 2007.

[144] H. Cartan. *Differential Calculus*. Hermann/Houghton Mifflin, Paris/Boston, 1967/1971.

[145] F. S. Cater. Constructing nowhere differentiable functions from convex functions. *Real Anal. Exchange*, **28**:617–621, 2002/2003.

[146] Y. Censor and A. Lent. An iterative row-action method for interval convex programming. *J. Optim. Theory Appl.*, **34**(3):321–353, 1981.

[147] Y. Censor and S. A. Zenios. *Parallel Optimization: Theory, Algorithms, and Applications*. Oxford University Press, 1997.

[148] M. Cepedello-Bosio. Approximation of Lipschitz functions by δ-convex functions in Banach spaces. *Israel J. Math.*, **106**:269–284, 1998.

[149] M. Cepedello-Bosio and P. Hájek. Analytic approximations of uniformly continuous functions in real Banach spaces. *J. Math. Anal. Appl.*, **256**:80–98, 2001.

[150] L. Cheng and Y. Teng. A strong optimization theorem in locally convex spaces. *Chin. Ann. Math.*, **24B**:395–402, 2003.

[151] T. Y. Chow. The suprise examination or unexpected hanging paradox. *Amer. Math. Monthly*, **105**:41–51, 1998.

[152] J. P. R. Christensen. On sets of Haar measure zero in Abelian Polish groups. *Israel J. Math.*, **13**:255–260, 1972.

[153] J. P. R. Christensen. Measure theoretic zero-sets in infinite dimensional spaces and applications to differentiability of Lipschitz functions, II. *Colloq. Analyse Fonctionnel (Bordeaux)*, pages 29–39, 1973.

[154] V. Chvátal. *Linear Programming*. Freeman, New York, 1983.

[155] F. H. Clarke. A variational proof of Aumann's theorem. *Appl. Math. Optim.*, **7**:373–378, 1981.

[156] F. H. Clarke. *Optimization and Nonsmooth Analysis*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley, New York, 1983.

[157] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. *Nonsmooth analysis and control theory*, volume 178 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[158] M. Coban and P. S. Kenderov. Dense Gâteaux differentiablity of the supremum norm in $C(T)$ and topological properties of T. *C. R. Acad. Bulgare Sci.*, **64**:103–106, 1976.

[159] J. B. Collier. The dual of a space with the Radon Nikodym property. *Pacific J. Math.*, **64**:103–106, 1976.

[160] J. Conway. *A Course in Functional Analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 2nd edition, 1990.

[161] R. Correa, P. Gajardo, and L. Thibault. Subdifferential representation formula and subdifferential criteria for the behavior of nonsmooth functions. *Nonlinear Anal.*, **65**:864–891, 2006.

[162] M. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second-order partial differential equations. *Bull. Amer. Math. Soc.*, **27**:1–67, 1992.

[163] J. P. Crouziex. Duality between direct and indirect utility functions. *J. Math. Econ.*, **12**:149–165, 1983.

[164] D. Cudia. The geometry of Banach spaces: Smoothness. *Trans. Amer. Math. Soc.*, **110**:284–314, 1964.

[165] A. Daniilidis, A. Lewis, J. Malick, and H. Sendov. Prox regularity of spectral functions and sets. *J. Convex Analysis*, **15**:547–560, 2008.

[166] G. B. Dantzig. *Linear Programming and its Extensions*. Princeton University Press, Princeton, 1963.

[167] F. K. Dashiell and J. Lindenstrauss. Some examples concerning strictly convex norms on $C(K)$ spaces. *Israel J. Math.*, **16**:329–342, 1973.

[168] C. Davis. All convex invariant functions of hermitian matrices. *Archiv der Mathematik*, **8**:276–278, 1957.

[169] W. J. Davis, N. Ghoussoub, and J. Lindenstrauss. A lattice renorming theorem and applications to vector-valued processes. *Trans. Amer. Math. Soc.*, **263**:531–540, 1981.

[170] W. J. Davis and W. B. Johnson. A renorming of nonreflexive Banach spaces. *Proc. Amer. Math. Soc.*, **37**:486–488, 1973.

[171] M. M. Day. Strict convexity and smoothness of normed spaces. *Trans. Amer. Math. Soc.*, **78**:516–528, 1955.

[172] M. M. Day. *Normed Linear Spaces*, volume Ed. 3 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1973.

[173] C. De Bernardi and L. Veselý. On support points and support functionals of convex sets. *Israel J. Math.*, to appear.

[174] H. Debrunner and P. Flor. Ein Erweiterungssatz für monotone Mengen. *Arch. Math.*, **15**:445–447, 1964.

[175] F. Deutsch. *Best Approximation in Inner Product Spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 7. Springer-Verlag, New York, 2001.

[176] R. Deville, V. Fonf, and P. Hájek. Analytic and C^k approximations of norms in separable Banach spaces. *Studia Math.*, **120**:61–74, 1996.

[177] R. Deville, V. Fonf, and P. Hájek. Analytic and polyhedral approximation of convex bodies in separable polyhedral Banach spaces. *Israel J. Math.*, **105**:139–154, 1998.

[178] R. Deville, G. Godefroy, and V. Zizler. Smooth bump functions and the geometry of Banach spaces. *Mathematika*, **40**:305–321, 1993.

[179] R. Deville, G. Godefroy, and V. Zizler. A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions. *J. Funct. Anal.*, **111**:197–212, 1993.

[180] R. Deville, G. Godefroy, and V. Zizler. *Smoothness and Renormings in Banach spaces*, volume 64 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1993.

[181] R. Deville and J. Revalski. Porosity of ill-posed problems. *Proc. Amer. Math. Soc.*, **128**:1117–1124, 2000.

[182] J. Diestel. *Geometry of Banach Spaces – Selected Topics*, volume 485 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1975.

[183] J. Diestel. *Sequences and Series in Banach spaces*, volume 92 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1984.

[184] J. Diestel and J. J. Uhl. *Vector Measures*, volume 15 of *AMS Mathematical Surveys*. American Mathematical Society, Providence, 1977.

[185] J. Dieudonné. *Foundations of Modern Analysis*, volume 10 of *Pure and Applied Mathematics*. Academic Press, New York, 1960.

[186] W. E. Diewert. Duality approaches to microeconomic theory. In K. J. Arrow and M. D. Intriligator, editors, *Handbook of Mathematical Economics*, volume II, pages 535–599. North-Holland, Amsterdam, 1982.

[187] T. Domíngues Benavides. A renorming of some nonseparable Banach spaces with the fixed point property. *J. Math. Anal. Appl.*, **350**:525–530, 2009.

[188] P. N. Dowling, W. B. Johnson, C. J. Lennard, and B. Turett. The optimality of James's distortion theorems. *Proc. Amer. Math. Soc.*, **125**:167–174, 1997.

[189] J. Duda, L. Veselý, and L. Zajíček. On d.c. functions and mappings. *Atti Sem. Mat. Fis. Univ. Modena*, **51**(1):111–138, 2003.

[190] A. C. Eberhard and J. M. Borwein. Second order cones for maximal monotone operators via representative functions. *Set-Valued Anal.*, (e-published, March 5, 2008.

[191] A. C. Eberhard and R. Wenczel. On the calculus of limiting subhessians. *Set-Valued Anal.*, **15**:377–424, 2007.

[192] N. V. Effimov and S. B. Stechkin. Support properties of sets in Banach spaces and Chebyshev sets. *Dokl. Akad. Nauk SSSR*, **127**:254–257, 1959.

[193] I. Ekeland. On the variational principle. *J. Math. Anal. Appl.*, **47**:324–353, 1974.

[194] I. Ekeland and G. Lebourg. Generic Fréchet differentiability and perturbed optimization problems. *Trans. Amer. Math. Soc.*, **224**:193–216, 1976.

[195] I. Ekeland and R Temam. *Convex Analysis and Variational Problems*. North-Holland, New York, 1976.

[196] D. Ellis. A modification of the parallelogram law characterization of Hilbert spaces. *Math. Z.*, **59**:94–96, 53.

[197] P. Enflo. Banach spaces which can be given an equivalent uniformly convex norm. In *Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972)*, volume 13, pages 281–288 (1973), 1972.

[198] M. Fabian *et al. Functional Analysis and Infinite-Dimensional Geometry*, volume 8 of *CMS Books in Mathematics*. Springer-Verlag, Berlin, 2001.

[199] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. CRC Press, Boca Raton, 1992.

[200] M. Fabian. Lipschitz smooth points of convex functions and isomorphic characterizations of Hilbert spaces. *Proc. London Math. Soc.*, **51**:113–126, 1985.

[201] M. Fabian. *Gâteaux Differentiability of Convex Functions and Topology: Weak Asplund Spaces*. CMS Series of Monographs and Advanced Texts. John Wiley, New York, 1997.

[202] M. Fabian, D. Preiss, J. H. M. Whitfield, and V. Zizler. Separating polynomials on Banach spaces. *Quart. J. Math. Oxford (2)*, **40**:409–422, 1989.

[203] M. Fabian, J. H. M. Whitfield, and V. Zizler. Norms with locally Lipschitzian derivatives. *Israel J. Math*, **44**:262–276, 1983.

[204] M. Fabian, L. Zajíček, and V. Zizler. On residuality of the set of rotund norms on a Banach space. *Math. Ann.*, **258**:349–351, 1981/82.

[205] M. Fabian and V. Zizler. An elementary approach to some questions in higher order smoothness in Banach spaces. *Extracta Math.*, **14**:295–327, 1999.

[206] M. Fabian and V. Zizler. On uniformly Gâteaux smooth $C^{(n)}$-smooth norms on separable Banach spaces. *Czech. Math. J.*, **49**:657–672, 1999.

[207] M. Fabian and V. Zizler. A “nonlinear” proof of Pitt’s compactness theorem. *Proc. Amer. Math. Soc.*, **131**:3693–3694, 2003.

[208] K. Fan. Fixed point and minimax theorems in locally convex topological linear spaces. *Proc. Nat. Acad. Sci. U.S.A.*, **38**:431–437, 1952.

[209] J. Farkas. Theorie der einfachen Ungleichungen. *Journal für die reine und angewandte Mathematik*, **124**:1–27, 1902.

[210] L. Faybusovich. Infinite-dimensional semidefinite programming: regularized determinants and self-concordant barriers. In *Topics in semidefinite and interior-point methods (Toronto, ON, 1996)*, volume 18 of *Fields Inst. Commun.*, pages 39–49. Amer. Math. Soc., Providence, RI, 1998.

[211] W. Fenchel. On conjugate convex functions. *Canad. J. Math.*, **1**:73–77, 1949.

[212] W. Fenchel. Convexity through the Ages. In P.M. Gruber and J. M. Wills, editors, *Convexity and its Applications*. Birkhäuser-Verlag, Boston, MA, 1983.

[213] T. Figiel. On the moduli of convexity and smoothness. *Studia Math.*, **56**(2):121–155, 1976.

[214] P. A. Fillmore and J. P. Williams. Some convexity theorems for matrices. *Glasgow Math. J.*, **12**:110–117, 1971.

[215] S. Fitzpatrick. Representing monotone operators by convex functions. *Workshop/Miniconference on Functional Analysis and Optimization , Proc. Centre Math. Anal. Austral. Nat. Univ., 20, Austral. Nat. Univ., Canberra*, pages 59–65, 1988.

[216] S. Fitzpatrick and R. R. Phelps. Differentiability of the metric projection in Hilbert space. *Trans. Amer. Math. Soc.*, **270**:483–501, 1982.

[217] S. Fitzpatrick and R. R. Phelps. Bounded approximants to monotone operators on Banach spaces. *Ann. Inst. Henri Poincaré, Analyse non linéaire*, **9**:573–595, 1992.

[218] S. Fitzpatrick and R. R. Phelps. Some properties of maximal monotone operators on nonreflexive Banach spaces. *Set-Valued Anal.*, **3**:51–69, 1995.

[219] R. Fletcher. A new variational result for quasi-Newton formulae. *SIAM Journal on Optimization*, **1**:18–21, 1991.

[220] V. P. Fonf. On supportless convex sets in incomplete normed spaces. *Proc. Amer. Math. Soc.*, **120**:1173–1176, 1994.

[221] J. Frontisi. Smooth partitions of unity in Banach spaces. *Rocky Mount. J. Math.*, **25**:1295–1304, 1995.

[222] J. Frontisi. Lissité et dualité dans les espaces de Banach. *Thèse de Doctorat de Mathématiques de l’Université Paris 6*, 1996.

[223] R. Fry. Analytic approximation on c_0. *J. Funct. Anal.*, **158**:509–520, 1998.

[224] R. Fry. Approximation by C^p-smooth, Lipschitz functions on Banach spaces. *J. Math. Anal. Appl.*, **315**:599–605, 2006.

[225] R. Fry. Corrigendum to “Approximation by C^p-smooth, Lipschitz functions on Banach spaces”. *J. Math. Anal. Appl.*, **348**:571, 2008.

[226] N. Ghoussoub. Renorming dual Banach lattices. *Proc. Amer. Math. Soc.*, **84**:521–524, 1982.

[227] G. Gilardoni. On Pinsker’s type inequalities and Csiszár’s f-divergences. Part I: Second and fourth-order inequalities. arXiv:cs/0603097v2, 2006.

[228] G. Gilardoni. Sur la f-divergence minimale pour variation totale donné. *C. R. Acad. Sci. Paris*, **343**:763–766, 2006.

[229] J. R. Giles. *Convex analysis with application in the differentiation of convex functions*, volume 58 of *Research Notes in Mathematics*. Pitman, Boston, Mass., 1982.

[230] K. Gobel and J. Wosko. Making a hole in the space. *Proc. Amer. Math. Soc.*, **114**:475–476, 1992.

[231] M.A. Goberna and M.A. López. *Linear Semi-Infinite Optimization*. Wiley Series in Mathematical Methods in Practice. John Wiley, New York, 1998.

[232] G. Godefroy. Prolongement de fonctions convexes définies sur un espace de Banach E au bidual E''. *C. R. Acad. Sci. Paris*, **292**:371–374, 1981.

[233] R Goebel and R. T. Rockafellar. Local strong convexity and local lipschitz continuity of the gradient of convex functions. *J. Convex Analysis*, **15**:179–190, 2008.

[234] R. Goebel, A. R. Teel, T. Hu, and Z. Lin. Conjugate convex lyapunov functions for dual linear differential equations. *IEEE Trans. Automat. Control*, **51**:661–666, 2006.

[235] M. X. Goemans. Semidefinite programming and combinatorial optimization. *Doc. Math. J. DMV*, ICM 1998 III:657–666, 1998.

[236] G. Golub and C. F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, Baltimore, 3rd edition, 1996.

[237] P. Gordan. Uber die Auflösung linearer Gleichungen mit reelen Coefficienten. *Mathematische Annalen*, **6**:23–28, 1873.

[238] J.-P. Gossez. On the extensions to the bidual of a maximal monotone operator. *Proc. Amer. Math. Soc.*, **62**:67–71, 1977.

[239] J.-P. Gossez and E. Lami Dozo. Some geometric properties related to the fixed point theory for nonexpansive mappings. *Pacific J. Math.*, **40**:565–573, 1972.

[240] J.-P. Gossez. Opérateurs monotones non linéaires dans les espaces de Banach non réflexifs. *J. Math. Anal. Appl.*, **34**:371–395, 1971.

[241] J.-P. Gossez. On the range of a coercive maximal monotone operator in a nonreflexive Banach space. *Proc. Amer. Math. Soc.*, **35**:88–92, 1972.

[242] B. Grone, C.R. Johnson, E. Marques de Sá, and H. Wolkowicz. Positive definite completions of partial Hermitian matrices. *Linear Algebra and Its Applications*, **58**:109–124, 1984.

[243] B. Grunbaum. *Polyhedron Models*. Graduate Texts in Mathematics. Springer-Verlag, New York, 2003.

[244] R. Grzaślewicz. A universal convex set in Euclidean space. *Colloq. Math.*, **45**:41–44, 1982.

[245] A. Guirao and P. Hájek. On the moduli of convexity. *Proc. Amer. Math. Soc.*, **135**:3233–3240, 2007.

[246] V. I. Gurari. Differential properties of the convexity moduli of Banach spaces. *Mat. Issled.*, **2**(vyp. 1):141–148, 1967.

[247] P. Hájek, V. Montesinos, J. Vanderwerff, and V. Zizler. *Biorthogonal Systems in Banach Spaces*. CMS Books in Mathematics. Springer-Verlag, New York, 2008.

[248] M. P. Hanna. Generalized overrelaxation and Gauss–Seidel convergence on Hilbert space. *Proc. Amer. Math. Soc.*, **35**:524–530, 1972.

[249] A. Hantoute, M López, and C. Zălinescu. Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions. *SIAM J. Opim.*, **19**:863–882, 2008.

[250] P. Hartman. On functions representable as a difference of convex functions. *Pacific J. Math.*, **9**:707–713, 1959.

[251] R. Haydon. A non-reflexive Grothendieck space that does not contain ℓ_∞. *Israel J. Math.*, **40**:65–73, 1981.

[252] R. Haydon. Trees in renorming theory. *Proc. London Math. Soc.*, **78**:541–584, 1999.

[253] Robin D. Hill. Dual periodicity in l_1-norm minimisation problems. *Systems Control Lett.*, **57**:489–496, 2008.

[254] J.-B. Hiriart-Urruty. What conditions are satisfied at points minimizing the maximum of a finite number of differentiable functions. In *Nonsmooth Optimization: Methods and Applications*. Gordan and Breach, New York, 1992.

[255] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms*. Springer-Verlag, New York, 1996.

[256] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Grundlehren der mathematien Wissenschaften. Springer-Verlag, New York, 2001.

[257] J.-B. Hiriart-Urruty and R. R. Phelps. Subdifferential calculus using ε-subdifferentials. *J. Funct. Anal.*, **188**:154–156, 1993.

[258] J.-B. Hiriart-Urruty and A. Seeger. The second-order subdifferential and the Dupin indicies of a non-differentiable convex function. *Proc. London Math. Soc.*, **58**:351–365, 1989.

[259] P. Holický, O. Kalenda, L. Veselý, and L. Zajíček. Quotients of continuous convex functions on nonreflexive Banach spaces. arXiv:0706.0633, 2007.

[260] R. B. Holmes. *Geometric Functional Analysis and its Applications*, volume 24 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1975.

[261] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[262] A. Ioffe. Three theorems on subdifferentiation of convex integral functionals. *J. Convex Anal.*, **13**:759–772, 2006.

[263] A. D. Ioffe and J.-P. Penot. Limiting subhessians, limiting subjets and their calculus. *Trans. Amer. Math. Soc*, **349**:789–807, 1997.

[264] R. C. James. Superreflexive Banach spaces. *Canad. J. Math.*, **24**:896–904, 1972.

[265] R. C. James. Superreflexive spaces with bases. *Pacific J. Math.*, **41**/**42**:409–419, 1972.

[266] G. J. O. Jameson. *Topology and Normed Spaces*. Chapman and Hall, London, 1974.

[267] M. Jiang. On Johnson's example of a nonconvex Chebyshev set. *J. Approx. Theory*, **74**:152–158, 1993.

[268] K. John and V. Zizler. A short proof of a version of Asplund's norm averaging theorem. *Proc. Amer. Math. Soc.*, **73**:277–278, 1979.

[269] G. Johnson. A nonconvex set which has the unique nearest point property. *J. Approx. Theory*, **51**:289–332, 1987.

[270] G. Johnson. Chebyshev foam. *Topol. Appl.*, **20**:1–9, 1998.

[271] B. Josefson. Weak sequential convergence in the dual of a Banach space does not imply norm convergence. *Ark. Mat.*, **13**:79–89, 1975.

[272] G. Pólya and G. Szegő. *Problems and Theorems in Analysis, I.* Springer Verlag, New York, 2004.

[273] G. Pólya and G. Szegő. *Problems and Theorems in Analysis, II.* Springer Verlag, New York, 2004.

[274] S. Kakutani. A generalization of Brouwer's fixed point theorem. *Duke Math. J.*, **8**:457–459, 1941.

[275] A. G. Kartsatos and I. V. Skrypnik. On the eigenvalue problem for perturbed nonlinear maximal monotone operators in reflexive Banach spaces. *Trans. Amer. Math. Soc.*, **358**:3851–3881, 2006.

[276] N. Kato. On the second derivatives of convex functions on Hilbert spaces. *Proc. Amer. Math. Soc.*, **106**:697–705, 1989.

[277] V. Klee. Convexity of Cebysev sets. *Math. Annalen*, **142**:291–304, 1961.

[278] V. L. Klee. Some new results on smoothness and rotundity in normed linear spaces. *Math. Ann.*, **139**:51–63, 1959.

[279] E. Krauss. A representation of arbitrary maximal monotone operators via subgradients of skew-symmetric saddle functions. *Nonlinear Anal.*, **9**:1381–1399, 1985.

[280] H. W. Kuhn. Steiner's problem revisited. *MAA Studies in Optimization*, **10**:52–70, 1974.

[281] O. Kurka. Reflexivity and sets of Fréchet subdifferentiability. *Proc. Amer. Math. Soc.*, electronically published June 17, 2008.

[282] J. Kurzwiel. On approximation in real Banach spaces. *Studia Math.*, **14**:214–231, 1955.

[283] S. Kwapien. Isomorphic characterizations of inner product space by orthogonal series with vector valued coefficients. *Studia Math.*, **44**:583–595, 1972.

[284] J. M. Lasry and P. L. Lions. A remark on regularization in Hilbert spaces. *Israel J. Math*, **55**:257–266, 1986.

[285] M. Lassonde. Asplund spaces, Stegal variational principle and the RNP. To appear.

[286] M. Lassonde and J. Revalski. Fragmentability of sequences of set-valued mappings with applications to variational principles. *Proc. Amer. Math. Soc.*, **133**:2637–2646, 2005.

[287] E. S. Levitin and B. T. Polyak. Convergence of minimizing sequences in the conditional-extremum problem. *Dolk. Akad. Nauk. SSSR*, **168**:997–1000, 1966.

[288] A. S. Lewis. Convex analysis on the Hermitian matrices. *SIAM J. Optim.*, **6**:164–177, 1996.

[289] A. S. Lewis. The mathematics of eigenvalue optimization. *Math. Programming*, **97**:155–176, 2003.

[290] A. S. Lewis and M. L. Overton. Eigenvalue optimization. *Acta Numerica*, **5**:149–190, 1996.

[291] F. Liese and I. Vajda. *Convex statistical distance*, volume 95 of *Teubner Texts in Mathematics*. Teubner, Leipzig, 1987.

[292] J. Lindenstrauss. On the modulus of smoothness and divergent series in Banach spaces. *Mich. Math. J.*, **10**:241–252, 1963.

[293] J. Lindenstrauss, D. Preiss, and J. Tišer. Fréchet differentiability of Lipschitz maps and porous sets in Banach spaces. *Banach Spaces and their Applications in Analysis*, pages 111–123, 2007.

[294] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces*, volume I & II. Springer-Verlag, New York, 1977.

[295] R. Lipschitz. *Lehrbuch der Analysis*. Cohen und Sohn, Bonn, 1877.

[296] P. Loewen and X. Wang. A generalized variational principle. *Canad. J. Math*, **53**:1174–1193, 2001.

[297] R. Lucchetti. *Convexity and Well-Posed Problems*, volume 22 of *CMS Books in Mathematics*. Springer-Verlag, New York, 2006.

[298] Y. Lucet. What shape is your conjugate? A survey of computational convex analysis and its applications. Available at: www.optimization-online.org/DB_FILE/2007/12/1863.pdf, 2007.

[299] D. G. Luenberger. *Optimization by Vector Space Methods*. John Wiley, New York, 1969.

[300] D. G. Luenberger. Benefit functions and duality. *J. Math. Econom.*, **21**:461–481, 1992.

[301] H. Mankiewicz. On the differentiability of Lipschitz mappings in Fréchet spaces. *Studia Math.*, **45**:15–29, 1973.

[302] P. Maréchal. On the convexity of multiplicative potential and penalty functions and related topics. *Math. Program., Ser. A*, **89**:505–516, 2001.

[303] A. W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and its Applications*. Academic Press, New York, 1979.

[304] M. Matić, C. E. M. Pearce, and J. Pečarić. Improvements on some bounds on entropy measures in information theory. *Math. Inequal. Appl.*, **1**:295–304, 1998.

[305] J. Matoušek and E. Matoušková. A highly nonsmooth norm on Hilbert space. *Israel J. Math.*, **112**:1–28, 1999.

[306] E. Matoušková. One counterexample concerning the Fréchet differentiability of convex functions on closed sets. *Acta Univ. Carolin. Math. Phys.*, **34**:97–105, 1993.

[307] E. Matoušková. An almost Fréchet smooth norm on superreflexive spaces. *Studia Math.*, **133**:93–99, 1999.

[308] E. Matoušková. Translating finite sets into convex sets. *Bull London Math Soc.*, **33**:711–714, 2001.

[309] E. Matoušková and C. Stegall. A characterization of reflexive Banach spaces. *Proc. Amer. Math. Soc.*, **124**:1083–1090, 1996.

[310] E. Matoušková and L. Zajíček. Second order differentiability and Lipschitz smooth points of convex functionals. *Czech. Math. J.*, **48**:617–640, 1998.

[311] S. Mazur. Über konvexe Mengen in linearen normierten Räumen. *Studia Math.*, **4**:70–84, 1933.

[312] D. McLaughlin, R. Poliquin, J. Vanderwerff, and V. Zizler. Second-order Gâteaux differentiable bump functions and approximations in Banach spaces. *Canad. J. Math.*, **45**:612–625, 1993.

[313] D. McLaughlin and J. Vanderwerff. Higher order Gateaux smooth bump functions on Banach spaces. *Bull. Austral. Math. Soc.*, **51**:291–300, 1995.

[314] V. Z. Meshkov. Smoothness properties in Banach spaces. *Studia Math.*, **63**:111–123, 1978.

[315] E. Michael. Continuous selections I. *Ann. Math.*, **63**:361–382, 1956.

[316] F. Mignot. Contrôle dans les inéquations variationelles elliptiques. *J. Funct. Anal.*, **22**:130–185, 1976.

[317] D. P. Milman. On some criteria for the regularity of spaces of the type (B). *Dokl. Akad. Nauk SSSR*, **20**:243–246, 1938.

[318] H. Minkowski. *Geometrie der Zahlen*. Teubner, Leipzig, 1910.

[319] W. B. Moors. The relationship between Goldstine's theorem and the convex point of continuity propety. *J. Math. Anal. Appl.*, **188**:819–832, 1994.

[320] W. B. Moors and S. Somasundaram. Usco selections of densely defined set-valued mappings. *Bull. Aust. Math. Soc.*, **65**:307–313, 2002.

[321] W. B. Moors and S. Somasundaram. A weakly Stegall space that is not a Stegall space. *Proc. Amer. Math. Soc.*, **131**:647–654, 2003.

[322] W. B. Moors and S. Somasundaram. A Gâteaux differentiability space that is not weak Asplund. *Proc. Amer. Math. Soc.*, **134**:2745–2754, 2006.

[323] B. Mordukhovich. *Variational Analysis and Generalized Differentiation I & II*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, New York, 2005.

[324] B. Mordukhovich and M. N. Nam. Limiting subgradients of minimal time functions in Banach spaces. Preprint, 2008.

[325] J.-J. Moreau. Sur la polaire d'une fonctionelle semi-continue superieurment. *C. R. Acad. Sci.*, **258**:1128–1130, 1964.

[326] U. Mosco. Convergence of convex sets and solutions of variational inequalities. *Adv. Math.*, **3**:510–585, 1969.

[327] U. Mosco. On the continuity of the Young-Fenchel transform. *J. Math. Anal. Appl.*, **35**:518–535, 1971.

[328] M. Moussaoui and A. Seeger. Second-order subgradients of convex integral functions. *Trans. Amer. Math. Soc.*, **351**:3687–3711, 1999.

[329] A. Nekvinda and L. Zajíček. A simple proof of the Rademacher theorem. *Časopis pro pěstování matematiky*, **113**:337–341, 1988.

[330] I. Nemeth. Characterization of a Hilbert vector lattice by the metric projection onto its positive cone. *J. Approx. Theory*, **123**:295–299, 2003.

[331] Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. SIAM, 1994.

[332] C. Niculescu and L.-E. Perrson. *Convex Functions and Their Applications*, volume 23 of *CMS Books in Mathematics*. Springer-Verlag, New York, 2006.

[333] A. Nissenzweig. w^* sequential convergence. *Israel J. Math.*, **22**:266–272, 1975.

[334] E. Odell and T. Schlumprecht. The distortion of Hilbert space. *Geom. Funct. Anal.*, **3**:202–207, 1993.

[335] E. Odell and T. Schlumprecht. The distortion problem. *Acta Math.*, **173**:259–281, 1994.

[336] E. Odell and T. Schlumprecht. Asymptotic properties of Banach spaces under renormings. *J. Amer. Math. Soc.*, **11**:1759–188, 1998.

[337] A. Ohara. Geometry of distributions associated with Tsallis statistics and properties of relative entropy minimization. *Phys. Lett. A*, **370**:184–193, 2007.

[338] P. Ørno. On J. Borwein's concept of sequentially reflexive Banach spaces. *Banach Space Bulletin Board*, 1991.

[339] J. O'Rourke. *Computational Geometry in C*. Cambridge University Press, 2nd edition, 1998.

[340] G Pataki. Cone-lp's and semidefinite programs: geometry and a simplex-type method. *Proceedings of IPCO V*, **71**:1–13, 1996.

[341] J. Pechanec, J. H. M. Whitfield, and V. Zizler. Norms locally dependent on finitely many coordinates. *An. Acad. Brasil Cienc.*, **53**:415–417, 1981.

[342] G. K. Pedersen. *Analysis Now*, volume 118 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.

[343] J.-P. Penot. The relevance of convex analysis for the study of monotonicity. Preprint, 2002.

[344] J.-P. Penot. Is convexity useful for the study of monotonicity. In R.P. Agarwal and D. O'Regan, editors, *Nonlinear Analysis and Applications*, pages 807–822. Kluwer Academic, Dordrecht, 2003.

[345] B. J. Pettis. A proof that every uniformly convex space is reflexive. *Duke Math. J.*, **5**:249–253, 1939.

[346] R. R. Phelps. Convex sets and nearest points. *Proc. Amer. Math. Soc.*, **8**:790–797, 1957.

[347] R. R. Phelps. Gaussian null sets and differentiability of Lipschitz maps on Banach spaces. *Pacific J. Math.*, **77**:523–521, 1978.

[348] R. R. Phelps. Counterexample concerning support theorems for convex sets in Hilbert space. *Canad. Math. Bull.*, **31**:121–128, 1988.

[349] R. R. Phelps. *Convex Functions, Monotone Operators and Differentiability*, volume 1364 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, 1st edition, 1989.

[350] R. R. Phelps. *Convex Functions, Monotone Operators and Differentiability*, volume 1364 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, 2nd edition, 1993.

[351] R. R. Phelps. Lectures on maximal monotone operators. arXiv:math.FA/9302209, 1993.

[352] R. S. Phillips. On linear transformations. *Tran. Amer. Math. Soc.*, **48**:516–548, 1940.

[353] G. Pisier. Martingales with values in uniformly convex spaces. *Israel J. Math.*, **20**(3-4):326–350, 1975.

[354] D. Preiss. Fréchet derivatives of Lipschitz functions. *J. Funct. Anal.*, **91**:312–345, 1990.

[355] D. Preiss, R. R. Phelps, and I. Namioka. Smooth Banach spaces, weak Asplund spaces and monotone or usco mappings. *Israel J. Math.*, **72**:257–279, 1990.

[356] D. Preiss and L. Zajíček. Stronger estimates of smallness of sets of Fréchet nondifferentiability of convex functions. *Proc. 11th Winter School, Suppl. Rend. Circ. Mat. di Palermo, Ser. II*, pages 219–223, 1984.

[357] B. Pshenichnii. *Necessary Conditions for an Extremum*. Marcel Dekker, New York, 1971.

[358] F. Pukelsheim. *Optimal Design of Experiments*. Wiley Series in Probability and Mathematical Statistics. John Wiley, New York, 1993.

[359] F. Pukelsheim and D. M. Titterington. General differential and Lagrangian theory for optimal experimental design. *Ann. Stat.*, **11**:1060–1068, 1982.

[360] M. Raja. Locally uniformly rotund norms. *Mathematika*, **46**:343–358, 1999.

[361] M. Raja. On dual loally uniformly rotund norms. *Israel J. Math.*, **129**:77–91, 2002.

[362] T Rapcsák. *Smooth Nonlinear Optimization in* $\mathbb{R}^n$, volume 19 of *Series in Nonconvex Optimization and its Applications*. Kluwer Academic, Dordrecht, 1997.

[363] S. Reich. The range of sums of accretive and monotone operators. *J. Math. Anal. Appl.*, **68**:310–317, 1979.

[364] S. Reich. A weak convergence theorem for the alternating method with Bregman distances. In *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, pages 313–318. Dekker, New York, 1996.

[365] R. M. Rifkin and R. A. Lippert. Value regularization and Fenchel duality. *J. Machine Learn.*, **8**:441–479, 2007.

[366] A. R. Roberts and D. E. Varberg. *Convex Functions*. Academic Press, New York, 1973.

[367] R. T. Rockafellar. Duality and stability in extremum problems involving convex functions. *Pacific J. Math.*, **21**:167–187, 1967.

[368] R. T. Rockafellar. Local boundedness of nonlinear monotone operators. *Mich. Math. J.*, **16**:397–407, 1969.

[369] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.

[370] R. T. Rockafellar. On the maximality of sums of nonlinear monotone operators. *Trans. Amer. Math. Soc.*, **149**:75–88, 1970.

[371] R. T. Rockafellar. *Conjugate Duality and Optimization*, volume 16 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. SIAM, Philadephia, 1974.

[372] R. T. Rockafellar. Integral functionals, normal integrals and measurable selections. In A. Dold and B. Eckmann, editors, *Nonlinear Operators and the Calculus of Variations*, volume 543 of *Lecture Notes in Mathematics*, pages 157–207. Springer-Verlag, New York, 1976.

[373] R. T. Rockafellar. *The Theory of Subgradients and its Application to Problems of Optimizatin of Convex and Non-convex Functions*. Heldermann-Verlag, Berlin, 1981.

[374] R. T. Rockafellar. Maximal monotone relations and the second derivatives of nonsmooth functions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **2**:167–184, 1985.

[375] R. T. Rockafellar. Generalized second derivatives of convex functions and saddle functions. *Trans. Amer. Math. Soc.*, **322**:51–77, 1990.

[376] R. T. Rockafellar. On a special class of convex functions. *J. Opt. Appl.*, **71**:619–621, 1991.

[377] R. T. Rockafellar. Second-order convex analysis. *J. Nonlinear Convex Anal.*, **1**:1–16, 1999.

[378] R. T. Rockafellar and R. J-B. Wets. *Variational Analysis*, volume 317 of *A Series of Comprehensive Studies in Mathematics*. Springer-Verlag, New York, 1998.

[379] B. Rodrigues and S. Simons. A minimax proof of the Krein–Smulian theorem. *Arch. Math.*, **51**:570–572, 1988.

[380] H. P. Rosenthal. On injective Banach spaces and the spaces $L_\infty(\mu)$ for finite measures μ. *Acta. Math.*, **124**:205–248, 1970.

[381] H. P. Rosenthal. The complete separable extension property. *J. Operator Theory*, **43**:329–374, 2000.

[382] P. Rosenthal. The remarkable theorem of Lévy and Steinitz. *Amer. Math. Monthly*, **94**(4):342–351, 1987.

[383] W. Rudin. *Functional Analysis*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill, New York, 1973.

[384] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, New York, 1987.

[385] A. Ruszczyński and A. Shapiro. Optimization of convex risk functions. *Math. Oper. Res.*, 31(3):433–452, 2006.

[386] H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer-Verlag, New York, 1974.

[387] S. O. Schönbeck. On the extension of Lipschitz maps. *Ark. Mat.*, **7**:201–209, 1967.

[388] K. Schulz and B. Schwartz. Finite extensions of convex functions. *Math. Oper. Statist. Ser. Optim.*, **10**:501–509, 1979.

[389] T. Sellke. Yet another proof of the Radon–Nikodym theorem. *Amer. Math. Monthly*, **108**:74–76, 2002.

[390] S. Simons. The range of a monotone operator. *J. Math. Anal. App.*, **199**:176–201, 1996.

[391] S. Simons. *Minimax and Monotonicity*, volume 1693 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.

[392] S. Simons. Maximal monotone multifunctions of Brøndsted–Rockafellar type. *Set-Valued Analysis*, **7**:255–294, 1999.

[393] S. Simons. Five kinds of maximal monotonicity. *Set-Valued Analysis*, **9**:391–403, 2001.

[394] S. Simons. A new version of the Hahn-Banach theorem. Available at: www.math.ucsb.edu/$\sim$simons/preprints/, 2003.

[395] S. Simons. Dualized and scaled Fitzpatrick functions. *Proc. Amer. Math. Soc.*, **134**:2983–2987, 2006.

[396] S. Simons. *From Hahn–Banach to Monotonicity*, volume 1693 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, New York, expanded 2nd edition, 2008.

[397] S. Simons. An improvement of the Fitzpatrick inequality for maximally montone multifunctions of type (NI). Available at: www.math.ucsb.edu/∼simons/preprints/, 2008.

[398] S. Simons and C. Zălinescu. A new proof for Rockafellar's characterization of maximal monotone operators. *Proc. Amer. Math. Soc.*, **132**:2969–2972, 2004.

[399] S. Simons and C. Zălinescu. Fenchel duality, Fitzpatrick functions and maximal monotonicity. *J. Nonlinear Convex Anal.*, **6**:1–22, 2005.

[400] B. Sims. Banach space geometry and the fixed point property. In *Recent advances on metric fixed point theory*, volume 48 of *Ciencias*. Univ. Sevilla, Sevilla, Spain, 1996.

[401] I. Singer. *Duality for nonconvex approximation and optimization*, volume 24 of *CMS Books in Mathematics/Ouvrages de Mathḿatiques de la SMC*. Springer, New York, 2006.

[402] M. Smith. Rotundity and smoothness in conjugate spaces. *Proc. Amer. Math. Soc.*, **61**:232–234, 1976.

[403] M. Smith. Some examples concerning rotundity in Banach spaces. *Math. Ann.*, **233**:155–161, 1978.

[404] V. L. Šmulian. Sur la structure de la sphere unitair dans l'espace de Banach. *Mat. Sb.*, **51**:545–561, 1941.

[405] V. Soloviov. Duality for nonconvex optimization and its applications. *Anal. Math.*, **19**:297–315, 1993.

[406] V. Soloviov. Dual extremal problems and their applications to minimax estimation problems. *Russian Math. Surveys*, **52**:685–720, 1997.

[407] C. Stegall. The duality between Asplund spaces and spaces with the Radon–Nikodym property. *Israel J. Math.*, **29**:408–412, 1978.

[408] C. Stegall. Optimization of functions on certain subsets of Banach spaces. *Math. Ann.*, **236**:171–176, 1978.

[409] J. Stoer and A. Witzgall. *Convexity and Optimization in Finite Dimensions. I.* Springer-Verlag, New York, 1970.

[410] K. R. Stromberg. *An Introduction to Classical Real Analyis*. Wadsworth International Mathematics Series. Wadsworth, Belmont, California, 1981.

[411] M. Talagrand. Renormages de quelques $C(K)$. *Israel J. Math.*, **54**:327–334, 1986.

[412] W. K. Tang. On the extension of rotund norms. *Manuscripta Math.*, **91**:73–82, 1996.

[413] W. K. Tang. Uniformly differentiable bump functions. *Arch. Math.*, **68**:55–59, 1997.

[414] W. K. Tang. On Asplund functions. *Comment. Math. Univ. Carolinae*, **40**:121–132, 1999.

[415] F. Tardella. A new proof of the Lyapunov convexity theorem. *Siam J. Control and Optimization*, **28**:478–481, 1990.

[416] S. L. Troyanski. On locally convex and differentiable norms in certain nonseparable Banach spaces. *Studia Math.*, **37**:173–180, 1971.

[417] L. Tuncel and H. Wolkowicz. Strong duality and minimal representations for cone optimization. Preprint, 2008.

[418] J. Van Tiel. *Convex Analysis: An Introductory Text*. John Wiley, New York, 1984.

[419] J. Vanderwerff. Smooth approximation in Banach spaces. *Proc. Amer. Math. Soc.*, **115**:113–120, 1992.

[420] J. Vanderwerff. Second-order Gâteaux differentiability and an isomorphic characterization of Hilbert spaces. *Quart. J. Oxford. Math*, **44**:249–255, 1993.

[421] A. Verona and M. E. Verona. Regular maximal monotone operators and the sum rule. *J. Convex Analysis*, **7**:115–128, 2000.

[422] L. Veselý. A short proof of a theorem on composition of DC functions. *Proc. Amer. Math. Soc.*, **87**:685–686, 1987.

[423] L. Veselý. A parametric smooth variational principle and support properties of convex sets and functions. *J. Math. Anal. Appl.*, **350**:550–561, 2009.

[424] L. Veselý and L. Zajíček. Delta-convex mappings between Banach spaces and applications. *Dissertationes Math. (Pozprawy Mat.)*, volume 289, 1989.

[425] L. Veselý and L. Zajíček. On composition of d.c. functions and mappings. arXiv:0706.0624, 2007.

[426] L. Veselý and L. Zajíček. On extensions of d.c. functions and convex functions. arXiv:0810.1433v1, 2008.

[427] A. Visintin. Strong convergence results related to strict convexity. *Comm. Partial Differential Equations*, **9**:439–466, 1984.

[428] L. P. Vlasov. Almost convex and Chebyshev subsets. *Math. Notes Acad. Sci. USSR*, **8**:776–779, 1970.

[429] L. P. Vlasov. Several theorems on Chebyshev sets. *Mat. Zametki*, **11**:135–144, 1972.

[430] L. P. Vlasov. Approximative properties of sets in normed linear spaces. *Russian Math. Surveys*, **28**:1–93, 1973.

[431] M. D. Voisei. A maximality theorem for the sum of maximal monotone operators in non-reflexive Banach spaces. *Math. Sci. Res. J.*, **10**:36–41, 2006.

[432] M. D. Voisei and C. Zălinescu. Linear monotone subspaces of locally convex spaces. arXiv:0809.5287v1, 2008.

[433] M. D. Voisei and C. Zălinescu. Strongly-representable operators. arXiv:0802.3640v1, 2008.

[434] M. J. Wenninger. *Polyhedron Models*. Cambridge University Press, 1989.

[435] J. Westfall and U. Frerking. On a property of metric projections onto closed subsets of Hilbert spaces. *Proc. Amer. Math. Soc.*, **105**:644–651, 1989.

[436] H. Weyl. Elementare Theorie der konvexen Polyeder. *Commentarii Math. Helvetici*, **7**:290–306, 1935.

[437] R. Wijsman. Convergence of sequences of convex sets, cones and functions, ii. *Trans. Amer. Math. Soc.*, **123**:32–45, 1966.

[438] H. Wolkowicz, R. Saigal, and L. Vandenberghe (Eds.). *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications*, volume 27 of *International Series in Operations Research & Management Science*. Springer-Verlag, Berlin, 2000.

[439] S. J. Wright. *Primal-Dual Interior-Point Methods*. SIAM, Philadelphia, 1997.

[440] Z. Y. Wu, D. Li, L. S. Zhang, and X. M. Yang. Peeling off a nonconvex cover of an actual convex problem: hidden convexity. *SIAM J. Optim.*, **18**:507–536, 2007.

[441] D. Yost. M-Ideals, the strong 2-ball property and some renorming theorems. *Proc. Amer. Math. Soc.*, **81**:299–303, 1981.

[442] L. Zajíček. On the differentiabilty of convex functions in finite and infinite dimensional spaces. *Czech. Math. J.*, **29**:340–348, 1979.

[443] L. Zajíček. Porosity and σ-porosity. *Real Anal. Exchange*, **13**:314–350, 1987/88.

[444] C. Zălinescu. On uniformly convex functions. *J. Math. Anal. Appl.*, **95**:344–374, 1983.

[445] C. Zălinescu. *Convex Analysis in General Vector Spaces*. World Scientific, New Jersey, 2002.

[446] C. Zălinescu. A new proof of the maximal monotonicity of the sum using the Fitzpatrick function. In F. Giannessi and Eds. A. Maugeri, editors, *Variational Analysis and Appls.*, volume 79, pages 1159–1172. Springer, New York, 2005.

[447] C. Zălinescu. A new convexity property for monotone operators. *J. Convex Anal.*, **13**:883–887, 2006.

[448] L. Zhou. A simple proof of the Shaply–Folkman theorem. *Econ. Theory*, **3**:371–372, 1993.

[449] M. Zippin. Extension of bounded linear operator. In W. B. Johnson and J. Lindenstrauss, editors, *Handbook of the Geometry of Banach Spaces*, volume II, pages 1703–1741. Elsevier, Amsterdam, 2003.

[450] V. Zizler. Smooth extensions of norms and complementability of subspaces. *Arch. Math.*, **53**:585–589, 1989.

[451] A. Zygmund. *Trigonometric Series*, volume I,II. Cambridge University Press, New York, 2nd edition, 1959.