Legendre-type integrands and convex integral functions

Jonathan M. Borwein* and Liangjin Yao†

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Abstract

In this paper, we study the properties of integral functionals induced on $L_E^1(S,\mu)$ by closed convex functions on a Euclidean space E. We give sufficient conditions for such integral functions to be strongly rotund (well-posed). We show that in this generality functions such as the Boltzmann-Shannon entropy and the Fermi-Dirac entropy are strongly rotund. We also study convergence in measure and give various limiting counterexample.

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1 Introduction

We assume throughout that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. The *open unit ball* and the *closed unit ball* in X is denoted respectively by $U_X := \{x \mid \|x\| < 1\}$ and $B_X := \big\{x \in X \mid \|x\| \leq 1\big\}$, $U(x,\delta) := x + \delta U_X$ and $B(x,\delta) := x + \delta B_X$ (where $\delta \geq 0$ and $x \in X$) and

*CARMA, University of Newcastle, Newcastle, New South Wales 2308, Australia. E-mail: jonathan.borwein@newcastle.edu.au. Distinguished Professor King Abdulaziz University, Jeddah.

†CARMA, University of Newcastle, Newcastle, New South Wales 2308, Australia. E-mail: liangjin.yao@newcastle.edu.au.

$\mathbb{N} = \{1, 2, 3, \ldots\}$. We also assume that $d \in \mathbb{N}$ and reserve E for the Euclidean space $\mathbb{R}^d$ with the induced norm $\|\cdot\|$.

Throughout the paper, we also assume that (S, μ) is a complete *finite* measure space (with nonzero measure μ and $S \neq \varnothing$). The Banach space $L^1_E(S, \mu)$ with $\|\cdot\|_1$ stands for the space of all (equivalence classes of) measurable functions $f : S \to \mathbb{R}^n$ such that $\int_S \|f(s)\| \mathrm{d}\mu(s) < +\infty$. The norm $\|\cdot\|_1$ on $L^1_E(S, \mu)$ and $\langle \cdot, \cdot \rangle$ on $L^1_E(S, \mu) \times \big(L^1_E(S, \mu)\big)^* \big(= L^\infty_E(S, \mu)\big)$ are respectively defined by

$$\|f\|_1 := \int_S \|f(s)\| \mathrm{d}\mu(s) \quad \text{and} \quad \langle f, g \rangle := \int_S \langle f(s), g(s) \rangle \mathrm{d}\mu(s), \quad \forall f \in L^1_E(S, \mu), g \in L^\infty_E(S, \mu).$$

The norm on $L^\infty_E(S, \mu)$ is $\|\cdot\|_\infty$.

Let $A \colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as a relation, point-to-set mapping or multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A := \big\{(x, x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of A. The *domain* of A is $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$ and $\operatorname{ran} A := A(X)$ is the *range* of A.

Recall that A is *monotone* if

$$(1) \qquad \langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*) \in \operatorname{gra} A \; \forall (y, y^*) \in \operatorname{gra} A,$$

and *maximally monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion).

We now recall some additional standard notations [8]. We denote by $\longrightarrow$ and $\rightharpoonup_{\mathrm{w}}$ respectively, the norm convergence and weak convergence of sequences. Given a subset C of X, $\operatorname{int} C$ is the *interior* of C and $\overline{C}$ is the *norm closure* of C. Let $(C_n)_{n \in \mathbb{N}}$ be a sequence of subsets in X. We define $\overline{\lim}^{\mathrm{w}} C_n$ by $\overline{\lim}^{\mathrm{w}} C_n := \big\{x \in X \mid \exists x_{n_k} \in C_{n_k} \text{ with } x_{n_k} \rightharpoonup_{\mathrm{w}} x\big\}$. Let $f \colon X \to \left]-\infty, +\infty\right]$ and $\lambda \in \mathbb{R}$. Then $\operatorname{dom} f := f^{-1}(\mathbb{R})$ is the *domain* of f. We say f is proper if $\operatorname{dom} f \neq \varnothing$. The *lower level sets* of f are the sets $\{x \in X \mid f(x) \leq \lambda\}$. The *epigraph* of f is $\operatorname{epi} f := \big\{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\big\}$. Let C be convex, we say $x \in C$ is an *extreme point* of C if $\lambda u + (1 - \lambda)v \neq x, \forall u, v \in C \backslash \{x\}, \forall \lambda \in [0, 1]$. If $x \in \operatorname{argmin} f$, then $f(x) = \inf\{f(y) \mid y \in X\}$. Let f be proper. The *subdifferential* of f is defined by

$$\partial f \colon X \rightrightarrows X^* \colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X) \; \langle y - x, x^* \rangle + f(x) \leq f(y)\big\}.$$

We say f has the *Kadec* or *Kadec–Klee* property if the following implication

$$x_n \rightharpoonup_{\mathrm{w}} x \in \operatorname{dom} f, f(x_n) \longrightarrow f(x) \quad \Rightarrow \quad x_n \longrightarrow x.$$

holds.

As in [6] we say that f is *strongly rotund* if f is strictly convex on its domain, f has weakly compact lower level sets, and f has the Kadec property. This is in effect a well-posedness condition, see [16].

Let $\phi : E \to \,]-\infty, +\infty]$ be proper lower semicontinuous and convex. We define $I_\phi : L^1_E(S,\mu) \to \,]-\infty, +\infty]$ [20, 21] by

$$x \mapsto \int_S \phi(x(s)) \mathrm{d}\mu(s).$$

Then I_ϕ is well defined. More precisely, I_ϕ is proper lower semicontinuous and convex, and hence $I_\phi(x) > -\infty,\ \forall x(\cdot) \in L^1_E(S,\mu)$ (see Fact 2.13 below or [8, §6.3]).

The integral function I_ϕ has attracted much interest, see, e.g., [20, 21, 22, 26, 3, 4, 8, 25, 9, 10] and the references given therein. In the one-dimensional case with Lebesgue measure, Borwein and Lewis presented some characterizations for the integral function I_ϕ to be strongly rotund (see [6]). In this paper, we extend their work from the line to an arbitrary Euclidean space.

1.1 Organization of the paper

The remainder of this paper is organized as follows. In Section 2, we collect preliminary results for future reference and the reader's convenience. In Section 3, we present a sufficient condition for the integral function I_ϕ to be strongly rotund in our main result (Theorem 3.8). Some examples and applications are provided in Section 4, in which we show that the Boltzmann-Shannon entropy and the Fermi-Dirac entropy defined on $L^1_E(S,\mu)$ both are strongly rotund. In Section 5 we present an enlightening illustration of failure of strong rotundity. In Section 6, we apply a lovely result due to Visintin to both strengthen Theorem 3.8 and to shed light on the Kadec property. In the final Section 7 we turn to the role of convergence in measure.

2 Preliminary results

We first introduce Vitali's covering theorem.

Fact 2.1 (Vitali) (See [13, Theorem 1, page 27].) *Let $(x_i)_{i\in I}$ be in E and $(\delta_i)_{i\in I}$ be in $]0,+\infty[$ such that $\sup_{i\in I}\delta_i < +\infty$. Then there exists a countable subset Γ of I such that $B(x_\alpha,\delta_\alpha)\cap B(x_\beta,\delta_\beta) = \varnothing$ (for every $\alpha,\beta\in\Gamma$ with $\alpha\neq\beta$) and*

$$\bigcup_{i\in I} B(x_i,\delta_i) \subseteq \bigcup_{i\in\Gamma} B(x_i, 5\delta_i).$$

Corollary 2.2 *Let U be an open subset of E and $\delta > 0$. Then there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in U and a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $]0,\delta]$ such that $B(x_n,\frac{\delta_n}{5})\cap B(x_m,\frac{\delta_m}{5}) = \varnothing$ (for every $n,m\in\mathbb{N}$*

with $n \neq m$) and

$$\bigcup_{n\in\mathbb{N}} B(x_n,\delta_n) = U. \tag{2}$$

Proof. Let $x \in U$. There exists $\beta_x \in \left]0, \frac{\delta}{5}\right]$ such that

$$B(x, 5\beta_x) \subseteq U. \tag{3}$$

Then we have

$$U \subseteq \bigcup_{x\in U} B(x,\beta_x).$$

By Fact 2.1, there exist a countable set I and $(x_i)_{i\in I}$ in U such that $B(x_i,\beta_{x_i}) \cap B(x_j,\beta_{x_j}) = \varnothing$ (for every $i, j \in I$ with $i \neq j$) and

$$U \subseteq \bigcup_{x\in U} B(x,\beta_x) \subseteq \bigcup_{i\in I} B(x_i, 5\beta_{x_i}).$$

This and (3) yield

$$U = \bigcup_{i\in I} B(x_i, 5\beta_{x_i}). \tag{4}$$

Note that I cannot be a finite set. Otherwise, $\bigcup_{i\in I} B(x_i, 5\beta_{x_i})$ is closed, which contradicts (4). Set $\alpha_i := 5\beta_{x_i}, \forall i \in I$. Thus (4) implies that (2) holds. ■

Fact 2.3 (Dunford) (See [11, Theorem 4, page 104].) *Let D be a weakly compact subset of $L^1_E(S,\mu)$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_C \big\|y(s)\big\| \mathrm{d}\mu(s) \leq \varepsilon, \quad \forall \mu(C) \leq \delta, \forall y \in D.$$

Fact 2.4 (Rockafellar) (See [18, Theorem 1] or [17, Theorem 2.28].) *Let $A : X \rightrightarrows X^*$ be monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then A is locally bounded at $x \in \operatorname{int}\operatorname{dom} A$, that is, there exist $\delta > 0$ and $K > 0$ such that*

$$\sup_{y^*\in Ay} \|y^*\| \leq K, \quad \forall y \in (x + \delta B_X) \cap \operatorname{dom} A.$$

Fact 2.5 (See [27, Theorem 2.2.1].) *Let $f : X \to \left]-\infty, +\infty\right]$ be proper convex. Then f is lower semicontinuous if and only if f is weak–lower semicontinuous.*

Fact 2.6 (Borwein and Lewis) (See [6, Lemma 2.8].) *Let $f : X \to \left]-\infty,+\infty\right]$ be proper lower semicontinuous and convex. Suppose that f^* is Fréchet differentiable on* $\operatorname{dom}\partial f^*$. *Assume that $(x_n)_{n\in\mathbb{N}}$ and $x \in \operatorname{dom}\partial f$ are such that $x_n \rightharpoonup^{\mathrm{w}} x$, $f(x_n) \longrightarrow f(x)$. Then $x_n \longrightarrow x$.*

Definition 2.7 (See [2].) *Let $f : X \to \left]-\infty,+\infty\right]$ be proper lower semicontinuous and convex. We say*

(i) *f is* essentially smooth *if ∂f is locally bounded and single-valued on its domain.*

(ii) *f is* essentially strictly convex *if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of* $\operatorname{dom}\partial f$.

(iii) *f is* Legendre *if f is essentially smooth and essentially strictly convex.*

Fact 2.8 (Rockafellar) (See [19, Theorem 26.3].) *Let $\phi : E \to \left]-\infty,+\infty\right]$ be proper lower semicontinuous and convex. Then ϕ is essentially strictly convex if and only if ϕ^* is essentially smooth.* $\Diamond$

Fact 2.9 (See [2, Theorem 5.6(ii)&(iii) and Theorem 5.11(ii)].) *Let $f : X \to \left]-\infty,+\infty\right]$ be proper lower semicontinuous and convex. Then the following hold.*

(i) *f is essentially smooth if and only if* $\operatorname{int}\operatorname{dom} f \neq \varnothing$ *and ∂f is single-valued, if and only if* $\operatorname{int}\operatorname{dom} f = \operatorname{dom}\partial f$ *and ∂f is single-valued.*

(ii) *Suppose that $X = E$. Then f is essentially strictly convex if and only if f is strictly convex on every convex subset of* $\operatorname{dom}\partial f$.

Fact 2.10 (See [8, Fact 5.3.3, page 239].) *Let $f : X \to \left]-\infty,+\infty\right]$ be proper lower semicontinuous and strictly convex. Then $(x, f(x))$ is an extreme point of* $\operatorname{epi} f$ *for every $x \in \operatorname{dom} f$.*

Lemma 2.11 *Let $A : E \rightrightarrows E$ be monotone with* $\operatorname{int}\operatorname{dom} A \neq \varnothing$. *Let C be a bounded closed subset of* $\operatorname{int}\operatorname{dom} A$. *Then there exists $M > 0$ such that*

$$\sup_{a^*\in Aa,\, a\in C} \|a^*\| \leq M.$$

Proof. Let $x \in C$. By Fact 2.4, there exist $\delta_x > 0$ and $M_x > 0$ such that

$$\sup_{a^*\in Aa,\, a\in U(x,\delta_x)} \|a^*\| \leq M_x. \tag{5}$$

Then we have

$$C \subseteq \bigcup_{x\in C} U(x,\delta_x). \tag{6}$$

Since C is compact, there exists $N \in \mathbb{N}$ such that $(x_n)_{n=1}^N$ in C and

$$C \subseteq \bigcup_{n=1}^{N} U(x_n, \delta_{x_n}). \tag{7}$$

Set $M := \max\{M_{x_n} \mid n = 1, \cdots, N\}$. Then by (5) and (7), $\sup_{a^* \in Aa,\, a \in C} \|a^*\| \leq M$. ■

Remark 2.12 If C is assumed norm compact, this proof remains valid in a general Banach space. $\Diamond$

In the following subsection we turn to properties of the function I_ϕ.

2.1 Basic properties of I_ϕ

Fact 2.13 (Rockafellar) (See [22, Theorem 3C and Theorem 3H] and [8, Exercise 6.3.7, page 306].) *Let $\phi : E \to\ ]-\infty, +\infty]$ be proper, lower semicontinuous and convex. Then I_ϕ is proper lower semicontinuous and convex, and $I_\phi^* = I_{\phi^*}$. Moreover,*

$$x^* \in \partial I_\phi(x) \Longleftrightarrow \big(x^*(s) \in \partial\phi(x(s))\, \textit{for almost all } s \textit{ in } S\big).$$

Remark 2.14 Let $\phi : E \to\ ]-\infty, +\infty]$ be proper lower semicontinuous and convex. By Fact 2.5 and Fact 2.13, I_ϕ is proper weak–lower semicontinuous and convex. $\Diamond$

The following three results were proved by Borwein and Lewis when $E = \mathbb{R}$. Their proofs can be adapted to the general space E. For the readers' convenience, we record full proofs herein.

Fact 2.15 (See [6, Lemma 3.1)].) *Let $\phi : E \to\ ]-\infty, +\infty]$ be proper, lower semicontinuous and convex. Then I_ϕ is strictly convex on its domain if and only if ϕ is strictly convex on its domain.*

Proof. "$\Rightarrow$": Let $v, w \in \operatorname{dom}\phi$ with $v \neq w$. Set $x(s) := v$ and $y(s) := w$ for every $s \in S$. Then $\{x, y\} \subseteq L^1_E(S, \mu)$ and $x \neq y$. Let $\lambda \in\ ]0, 1[$. Since I_ϕ is strictly convex on its domain,

$$\begin{aligned}
&\phi\big(\lambda v + (1-\lambda)w\big) = \frac{1}{\mu(S)} \int_S \phi\big(\lambda v + (1-\lambda)w\big) \mathrm{d}\mu(s) \\
&= \frac{1}{\mu(S)} \int_S \phi\big(\lambda x(s) + (1-\lambda)y(s)\big) \mathrm{d}\mu(s) \\
&= \frac{1}{\mu(S)} I_\phi\big(\lambda x + (1-\lambda)y\big)
\end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{\mu(S)}\lambda I_\phi(x) + \frac{1}{\mu(S)}(1-\lambda)I_\phi(y)\\
&= \frac{1}{\mu(S)}\lambda\int_S \phi(v)d\mu(s) + \frac{1}{\mu(S)}(1-\lambda)\int_S \phi(w)\mathrm{d}\mu(s)\\
&= \lambda\phi(v) + (1-\lambda)\phi(w).
\end{aligned}
$$

Hence ϕ is strictly convex on its domain.

"$\Leftarrow$": By Fact 2.13, I_ϕ is convex. Suppose to the contrary that I_ϕ is not strictly convex on its domain. Then there exists $\lambda \in \left]0,1\right[$ and $\{x,y\} \subseteq \operatorname{dom} I_\phi$ with $x \neq y$ such that

$$
I_\phi\big(\lambda x + (1-\lambda)y\big) - \lambda I_\phi(x) - (1-\lambda)I_\phi(y) = 0.
$$

Then we have

$$
\int_S \Big(\lambda\phi(x(s)) + (1-\lambda)\phi(y(s)) - \phi\big(\lambda x(s) + (1-\lambda)y(s)\big)\Big)\mathrm{d}\mu(s) = 0. \tag{8}
$$

Since ϕ is convex,

$$
g(s) := \lambda\phi(x(s)) + (1-\lambda)\phi(y(s)) - \phi\big(\lambda x(s) + (1-\lambda)y(s)\big) \geq 0. \tag{9}
$$

Thus, $\phi\big(\lambda x(s) + (1-\lambda)y(s)\big) - \lambda\phi(x(s)) - (1-\lambda)\phi(y(s)) = 0$ for all almost $s \in S$. Since ϕ is strictly convex its domain, $x(s) = y(s)$ for all almost $s \in S$. Hence x is equivalent to y and thus $x = y$, which contradicts that $x \neq y$. ■

Following [6], given a measurable set $T \subseteq S$ we denote by $T^c := \big\{s \in S \mid s \notin T\big\}$ and we denote the restriction of μ and $x \in L^1_E(S,\mu)$ to T respectively by $\mu|_T$ and $x|_T$. We define $I^T_\phi : L^1_E(T,\mu|_T) \to \left]-\infty,+\infty\right]$ by $I^T_\phi(z) := \int_T \phi(z(s))\mathrm{d}\mu(s)$.

Fact 2.16 (See [6, Lemma 3.5)].) *Let $\phi : E \to \left]-\infty,+\infty\right]$ be proper, lower semicontinuous and convex, and T be a measurable subset of S. Suppose that $(x_n)_{n\in\mathbb{N}}$ in $L^1_E(S,\mu)$ such that $x_n \rightharpoonup^{\mathrm{w}} x$. Then $x_n|_T \rightharpoonup^{\mathrm{w}} x|_T$ in $L^1_E(T,\mu|_T)$. Moreover, if $I_\phi(x_n) \longrightarrow I_\phi(x) < +\infty$, then $I^T_\phi(x_n|_T) \longrightarrow I^T_\phi(x|_T) < +\infty$.*

Proof. We first show that $x_n|_T \rightharpoonup^{\mathrm{w}} x|_T$. Let $x^* \in L^\infty_E(T,\mu)$. Then we define y^* by $y^*(s) := x^*(s)$, if $s \in T$; $y^*(s) := 0$, if $s \in T^c$. Then $y^* \in L^\infty_E(S,\mu)$ and $\langle x_n|_T, x^*\rangle = \langle x_n, y^*\rangle \longrightarrow \langle x, y^*\rangle = \langle x|_T, x^*\rangle$. Hence $x_n|_T \rightharpoonup^{\mathrm{w}} x|_T$.

Now we show that $I^T_\phi(x_n|_T) \longrightarrow I^T_\phi(x|_T) < +\infty$. Since $x_n|_T \rightharpoonup^{\mathrm{w}} x|_T$ and $x_n|_{T^c} \rightharpoonup^{\mathrm{w}} x|_{T^c}$, by Fact 2.13 and Remark 2.14,

$$
\liminf I^T_\phi(x_n|_T) \geq I^T_\phi(x|_T) \quad \text{and} \quad \liminf I^{T^c}_\phi(x_n|_{T^c}) \geq I^{T^c}_\phi(x|_{T^c}). \tag{10}
$$

Then we have

$$\begin{aligned}
&\limsup I_\phi^T(x_n|_T) = \limsup \big(I_\phi(x_n) - I_\phi^{T^c}(x_n|_{T^c})\big)\\
&= \lim I_\phi(x_n) - \liminf I_\phi^{T^c}(x_n|_{T^c})\\
&\le I_\phi(x) - I_\phi^{T^c}(x|_{T^c})\\
&= I_\phi^T(x|_T) < +\infty \quad (\text{since } I_\phi(x) < +\infty \text{ and } I_\phi^{T^c}(x|_{T^c}) > -\infty \text{ by Fact 2.13}).
\end{aligned}$$

Then by (10), $\lim I_\phi^T(x_n|_T) = I_\phi^T(x|_T)$. ■

Fact 2.17 (See [6, Lemma 3.2].) *Let $\phi : E \to]-\infty, +\infty]$ be proper, lower semicontinuous and convex. Then I_{ϕ^*} is Fréchet differentiable everywhere on $L_E^\infty(S,\mu)$ if and only if ϕ^* is differentiable everywhere on E.*

Proof. "$\Rightarrow$": Let $z \in E$ and set $w(s) := z$ for every $s \in S$. Then $w(s) \in L_E^\infty(S,\mu)$. Then we have $\phi^*(z)\mu(S) = I_{\phi^*}(w) < +\infty$ and hence $z \in \operatorname{dom}\phi^*$. Thus $\operatorname{dom}\phi^* = E$. Let $u, v \in E$. Now we show ϕ^* is differentiable at u. Set $x(s) := u$ and $y(s) := v, \forall s \in S$. Then $\{x(s), y(s)\} \subseteq L_E^\infty(S,\mu)$. Let $t > 0$. Then we have

$$\begin{aligned}
&\frac{\phi^*(u+tv) + \phi^*(u-tv) - 2\phi^*(u)}{t}\\
&= \tfrac{1}{\mu(S)} \int_S \frac{\phi^*(u+tv) + \phi^*(u-tv) - 2\phi^*(u)}{t} \mathrm{d}\mu(s)\\
&= \tfrac{1}{\mu(S)} \int_S \frac{\phi^*\big(x(s)+ty(s)\big) + \phi^*(x(s)-ty(s)) - 2\phi^*(x(s))}{t} \mathrm{d}\mu(s)\\
&= \tfrac{1}{\mu(S)} \frac{I_{\phi^*}\big(x+ty\big) + I_{\phi^*}(x-ty) - 2I_{\phi^*}(x)}{t} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 0 \quad (\text{by [17, Exercise 1.24]}).
\end{aligned}$$

By [17, Exercise 1.24] again, ϕ^* is differentiable at u.

"$\Leftarrow$": By [17, Corollary, page 20], $(\phi^*)'$ is continuous on E. Let $x^* \in L_E^\infty(S,\mu)$. Then there exists $M > 0$ such that $\|x^*(s)\| \le M$ almost everywhere. We can and do suppose that $\|x^*(s)\| \le M, \forall s \in S$. Since MB_E is compact, $(\phi^*)'$ is uniformly continuous on MB_E. Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$$\big\|(\phi^*)'(x^*(s)+v) - (\phi^*)'x^*(s)\big\| \le \frac{\varepsilon}{\mu(S)}, \quad \forall \|v\| \le \delta. \tag{11}$$

Let $y^* \in L_E^\infty(S,\mu)$ with $\|y^*\|_\infty \le \delta$. Then applying Mean Value Theorem, we have

$$\begin{aligned}
&\Big\| \int_S \phi^*\big(x^*(s)+y^*(s)\big)\mathrm{d}\mu(s) - \int_S \phi^*(x^*(s))\mathrm{d}\mu(s) - \int_S \Big\langle (\phi^*)'(x^*(s)), y^*(s)\Big\rangle \mathrm{d}\mu(s)\Big\|\\
&= \Big\| \int_S \Big[\phi^*\big(x^*(s)+y^*(s)\big) - \phi^*(x^*(s)) - \Big\langle (\phi^*)'(x^*(s)), y^*(s)\Big\rangle\Big] \,\mathrm{d}\mu(s)\Big\|
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \int_S \Big[\Big\langle (\phi^*)'\big(x^*(s) + t_s y^*(s)\big), y^*(s) \Big\rangle - \Big\langle (\phi^*)'(x^*(s)), y^*(s) \Big\rangle \Big] \, \mathrm{d}\mu(s) \Big\|, \quad \exists t_s \in \,]0,1[\\
&= \Big\| \int_S \Big[\Big\langle (\phi^*)'\big(x^*(s) + t_s y^*(s)\big) - (\phi^*)'(x^*(s)), y^*(s) \Big\rangle \Big] \, \mathrm{d}\mu(s) \Big\|, \quad \exists t_s \in \,]0,1[\\
&\leq \int_S \Big\| \Big\langle (\phi^*)'\big(x^*(s) + t_s y^*(s)\big) - (\phi^*)'(x^*(s)), y^*(s) \Big\rangle \Big\| \mathrm{d}\mu(s), \quad \exists t_s \in \,]0,1[\\
&\leq \int_S \frac{\varepsilon}{\mu(S)} \|y^*(s)\| \mathrm{d}\mu(s) \quad \text{(by (11))} \\
&\leq \frac{\varepsilon}{\mu(S)} \|y^*\|_\infty \mu(S) = \varepsilon \|y^*\|_\infty.
\end{aligned}
$$

Hence I_{ϕ^*} is Fréchet differentiable at x^*. ■

Remark 2.18 Let $\phi : E \to \,]-\infty, +\infty]$ be proper, lower semicontinuous and convex. By Fact 2.17 and $\phi^{**} = \phi$, ϕ is differentiable everywhere on E if and only if I_ϕ is Fréchet differentiable everywhere on $L^\infty_E(S, \mu)$.

2.2 Strong rotundity and stability

We may apply our results to an important optimization:

Let $(C_n)_{n\in\mathbb{N}}$ and C_∞ in X be closed convex sets, and let $f : X \to \,]-\infty, +\infty]$ be a proper convex function with weakly compact lower level sets. We consider the following sequences of optimization problems (See [6].).

$$
\begin{aligned}
&(P_n) && V(P_n) := \inf \big\{ f(x) \mid x \in C_n \big\}, \\
&(P_\infty) && V(P_\infty) := \inf \big\{ f(x) \mid x \in C_\infty \big\}.
\end{aligned}
$$

Fact 2.19 (Borwein and Lewis) (See [6, Theorem 2.9(ii)].) *Let $(C_n)_{n\in\mathbb{N}}$ and C_∞ in X be closed convex sets, and $f : X \to \,]-\infty, +\infty]$ be a proper convex function with weakly compact lower level sets. Assume that*

$$
\overline{\lim}^{\mathrm{w}} C_n \subseteq C_\infty \subseteq \bigcup_{m\geq 1} \bigcap_{n\geq m} C_n.
$$

Then $V(P_n) \longrightarrow V(P_\infty)$. If $V(P_\infty) < +\infty$ and f is strongly rotund, then (P_n) and (P_∞) respectively have unique optimal solutions with x_n and x_∞, and $x_n \longrightarrow x_\infty$.

In a typical application, $C_{n+1} \subseteq C_n$ may be nested polyhedral approximations to a convex set $C_\infty := \bigcap C_n$, that is *constructible* in the sense of [8]. We look at the failure of strong rotundity in more detail in Section 6.

3 Properties of Legendre functions and I_ϕ

Proposition 3.1 *Let $\phi : E \to \,]-\infty, +\infty]$ be proper, lower semicontinuous and convex with $\operatorname{int}\operatorname{dom}\phi \neq \varnothing$. Then ϕ is essentially smooth if and only if ∂I_ϕ is single-valued.*

Proof. "$\Rightarrow$": First, we show that ∂I_ϕ is single-valued. Let $\{x^*, y^*\} \subseteq \partial I_\phi(x)$. Then by Fact 2.13, $x^*(s) \in \partial\phi(x(s))$ for almost all $s \in S$ and $y^*(s) \in \partial\phi(x(s))$ for almost all $s \in S$. Since $\partial\phi$ is single-valued. Then $x^*(s) = y^*(s)$ for almost all $s \in S$. Hence x^* is equivalent to y^* and then $x^* = y^*$. Thus $\partial I_\phi(x)$ is single-valued.

"$\Leftarrow$": By Fact 2.9(i), it suffices to show that $\partial\phi$ is single-valued. Let $\{v_1^*, v_2^*\} \subseteq \partial\phi(v)$. Set $x(s) := v$, $x_1^*(s) := v_1^*$ and $x_2^*(s) := v_2^*$. Then $x \in L_E^1(S, \mu)$, and $\{x_1^*, x_2^*\} \subseteq L_E^\infty(S, \mu)$. Thus by Fact 2.13, $\{x_1^*, x_2^*\} \subseteq \partial I_\phi(x)$. Since ∂I_ϕ is single-valued, $x_1^* = x_2^*$ and hence $v_1^* = v_2^*$. Then $\partial\phi(x)$ is single-valued and thus $\partial\phi$ is single-valued. ■

Corollary 3.2 *Let $\phi : E \to \,]-\infty, +\infty]$ be proper, lower semicontinuous and convex with $\operatorname{int}\operatorname{dom}\phi \neq \varnothing$. Assume that I_ϕ is strictly convex on its domain and that ∂I_ϕ is single-valued. Then ϕ is Legendre.*

Proof. By Fact 2.15, ϕ is strictly convex on its domain. Fact 2.9(ii) implies that ϕ is essentially strictly convex.

Applying Proposition 3.1, ϕ is essentially smooth. Combining the above results, ϕ is Legendre. ■

Corollary 3.3 *Let $\phi : \mathbb{R} \to \,]-\infty, +\infty]$ be proper, lower semicontinuous and convex with $\operatorname{int}\operatorname{dom}\phi \neq \varnothing$. Then ϕ is Legendre if and only if I_ϕ is strictly convex on its domain and ∂I_ϕ is single-valued.*

Proof. "$\Rightarrow$": By Fact 2.9(ii), ϕ is strictly convex on $\operatorname{int}\operatorname{dom}\phi$, and then ϕ is strictly convex on $\operatorname{dom}\phi$. Hence I_ϕ is strictly convex on its domain by Fact 2.15. By Proposition 3.1, ∂I_ϕ is single-valued.

"$\Leftarrow$": Applying Corollary 3.2 directly. ■

Lemma 2.11 allows us to generalize [6, Lemma 3.3].

Lemma 3.4 *Let $\phi : E \to \,]-\infty, +\infty]$ be proper, lower semicontinuous and convex, and let $x \in L_E^1(S, \mu)$. Assume that there exists a bounded closed subset D of $\operatorname{int}\operatorname{dom}\phi$ such that $x(s) \in D$ almost everywhere on S. Then $\partial I_\phi(x) \neq \varnothing$.*

Proof. By the assumption, there exists a measurable subset T of S such that $\mu(T) = \mu(S)$ and $x(s) \in D, \forall s \in T$. By [17, Proposition 3.3 and Proposition 2.5], $\partial\phi$ is upper semicontinuous on D. Thus, for every closed set $C \subseteq E$, we have $(\partial\phi)_C := D \cap \big((\partial\phi)^{-1}C\big)$ is closed. Thus

$$\big\{s \in T \mid \partial\phi(x(s)) \cap C \neq \varnothing\big\} = \big\{s \in T \mid s \in x^{-1}\left[(\partial\phi)_C\right]\big\}$$

is measurable. Hence $s \mapsto \partial\phi(x(s))$ is measurable on T. Then by [15, Theorem 14.2.1] or [23, Corollary 14.6, page 647], there exists a measurable selection $x^*(s) \in \partial\phi(x(s))$ everywhere on T. Then $x^*(s) \in \partial\phi(x(s))$ almost everywhere on S by $\mu(T) = \mu(S)$. By Lemma 2.11, $\{x^*(s) \mid s \in T\} \subseteq \partial\phi(D)$ is bounded, and then $x^*(s)$ is bounded almost everywhere on S since $\mu(T) = \mu(S)$. Hence we have $x^* \in L^\infty_E(S,\mu)$. ■

Let $m \in \mathbb{N}$ and $x \in L^1_E(S,\mu)$, we define S_m by

$$S_m := \big\{s \in S \mid x(s) \in mB_E\big\}. \tag{12}$$

Then we have $S_m \subseteq S_{m+1}$ and $S = \bigcup_{m\geq 1} S_m$.

Remark 3.5 Assume that $x \in L^1_E(S,\mu)$. Then $\mu(S^c_m) \downarrow 0$ and $\mu(S_m) \uparrow \mu(S)$ when $m \longrightarrow \infty$.

The proof of Proposition 3.6 was inspired by that of [6, Lemma 3.6].

Proposition 3.6 *Let $\phi : E \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous and convex. Suppose that ϕ^* is differentiable on E, and that $x_n \rightharpoonup^{\mathrm{w}} x$ in $L^1_E(S,\mu)$ and $I_\phi(x_n) \longrightarrow I_\phi(x) < +\infty$. Assume that $x(s) \in \operatorname{int}\operatorname{dom}\phi$ almost everywhere. Then $\|x_n - x\|_1 \longrightarrow 0$.*

Proof. Since $x_n \rightharpoonup^{\mathrm{w}} x$, $D := (x_n)_{n\in\mathbb{N}} \cup \{x\}$ is weakly compact in $L^1_E(S,\mu)$. Let $\varepsilon > 0$. By Fact 2.3, there exists $\delta > 0$ such that

$$\int_C \big\|y(s)\big\| \mathrm{d}\mu(s) \leq \varepsilon, \quad \forall \mu(C) \leq \delta, \forall y \in D. \tag{13}$$

Set $U := \operatorname{int}\operatorname{dom}\phi$. Since E is a separable metric space (or see Corollary 2.2), there exist a sequence $(z_n)_{n\in\mathbb{N}}$ in U and a sequence $(\delta_n)_{n\in\mathbb{N}}$ in $[0,1]$ such that

$$U = \bigcup_{n\in\mathbb{N}} B(z_n, \delta_n). \tag{14}$$

Set

$$T_n := \big\{s \in S \mid x(s) \in \bigcup_{k=1}^{n} B(z_k, \delta_k)\big\}, \quad \forall n \in \mathbb{N}. \tag{15}$$

Since $x(s) \in U$ almost everywhere on S, by(14), we have $\mu({T_n}^c) \downarrow 0$ and $\mu(T_n) \uparrow \mu(S)$ when $n \longrightarrow \infty$. Set $\widetilde{S_m} := S_m \cap T_m$. Then by Remark 3.5,

$$\mu(\widetilde{S_m}^c) = \mu(S_m^c \cup T_m^c) \downarrow 0 \quad \text{and} \quad \mu(\widetilde{S_m}) \uparrow \mu(S) \quad \text{as} \quad m \longrightarrow \infty. \tag{16}$$

Then by (13), there exists $N \in \mathbb{N}$ such that

$$\mu(\widetilde{S_m}^c) \leq \delta \quad \text{and} \quad \int_{\widetilde{S_m}^c} \|x_n(s) - x(s)\| \mathrm{d}\mu(s) \leq 2\varepsilon, \quad \forall m \geq N, \forall n \in \mathbb{N}. \tag{17}$$

Then by Fact 2.16,

$$x_n|_{\widetilde{S_m}} \rightharpoonup^{\mathrm{w}} x|_{\widetilde{S_m}} \text{ in } L^1_E(\widetilde{S_m}, \mu|_{\widetilde{S_m}}) \quad \text{and} \quad I_\phi^{\widetilde{S_m}}(x_n|_{\widetilde{S_m}}) \longrightarrow I_\phi^{\widetilde{S_m}}(x|_{\widetilde{S_m}}) < +\infty. \tag{18}$$

By the definition of $\widetilde{S_m}$ and (15), we have

$$\big\{x(s) \mid s \in \widetilde{S_m}\big\} \subseteq mB_E \cap \big(\bigcup_{k=1}^m B(z_k, \delta_k)\big). \tag{19}$$

By (14), $mB_E \cap \big(\bigcup_{k=1}^m B(z_k, \delta_k)\big)$ is a bounded closed subset of $\operatorname{int}\operatorname{dom}\phi$. Then by Lemma 3.4, $\partial I_\phi^{\widetilde{S_m}}(x|_{\widetilde{S_m}}) \neq \varnothing$. Thus by Fact 2.17, Fact 2.13, Fact 2.6 and (18), we obtain that

$$\int_{\widetilde{S_m}} \|x_n(s) - x(s)\| d\mu(s) \longrightarrow 0, \quad \text{as } n \longrightarrow 0. \tag{20}$$

Then we have

$$\begin{aligned} \|x_n - x\|_1 &= \int_{\widetilde{S_m}} \|(x_n(s)) - (x(s))\| d\mu(s) + \int_{\widetilde{S_m}^c} \|(x_n(s)) - (x(s))\| d\mu(s) \\ &\leq \int_{\widetilde{S_m}} \|x_n(s) - x(s)\| d\mu(s) + 2\varepsilon, \quad \forall m \geq N \quad \text{(by (17)}. \end{aligned} \tag{21}$$

Taking $n \longrightarrow \infty$ in (21), by (20), $\limsup \|x_n - x\|_1 \leq 2\varepsilon$ and hence $\|x_n - x\|_1 \longrightarrow 0$. ■

We first prove a restrictive sufficient condition for strong rotundity.

Theorem 3.7 *Let $\phi : E \to \left]-\infty, +\infty\right]$ be proper, lower semicontinuous and convex with open domain. Suppose that ϕ^* is differentiable on E. Then I_ϕ is strongly rotund on $L^1_E(S, \mu)$.*

Proof. By Fact 2.8, ϕ is essentially strictly convex. Since $\operatorname{dom}\phi$ is open, [17, Proposition 3.3 and Proposition 1.11] implies that $\operatorname{dom}\partial\phi = \operatorname{dom}\phi$. Hence ϕ is strictly convex on $\operatorname{dom}\phi$. Then by Fact 2.15, I_ϕ is strictly convex on its domain. Since $\operatorname{dom}\phi^* = E$, by [21, Corollary 2B], I_ϕ has weakly compact lower level sets.

Now we show I_ϕ has the Kadec property. Let $x_n \rightharpoonup^{\mathrm{w}} x \in \operatorname{dom} I_\phi$ in $L^1_E(S,\mu)$ and $I_\phi(x_n) \longrightarrow I_\phi(x)$. Since $x \in \operatorname{dom} I_\phi$, $x(s) \in \operatorname{dom}\phi$ for almost all $s \in S$. Since $\operatorname{dom}\phi = \operatorname{int}\operatorname{dom}\phi$, $x(s) \in \operatorname{int}\operatorname{dom}\phi$ almost everywhere. Then by Proposition 3.6, $\|x_n - x\|_1 \longrightarrow 0$.

Hence I_ϕ has the Kadec property and consequently I_ϕ is strongly rotund. ■

When the domain of ϕ is not open we have more work to do:

Theorem 3.8 *Let $\phi_i : \mathbb{R} \to \,]-\infty,+\infty]$ be proper, lower semicontinuous and convex with* $\operatorname{int}\operatorname{dom}\phi_i \neq \varnothing$ *for every $i = 1,2,\cdots,d$. Suppose that ϕ_i^* is differentiable on $\mathbb{R}$ for every $i = 1,2,\cdots,d$. Let $\phi : E \to \,]-\infty,+\infty]$ be defined by $z := (z_n) \in E \mapsto \sum_{i=1}^d \phi_i(z_i)$. Then I_ϕ is strongly rotund on $L^1_E(S,\mu)$.*

Proof. We have ϕ is proper lower semicontinuous and convex. Let $i \in \{1,\cdots,d\}$. By Fact 2.8, ϕ_i is essentially strictly convex. Then ϕ_i is strictly convex on $\operatorname{int}\operatorname{dom}\phi_i$. Hence ϕ_i is strictly convex on its domain, so is ϕ. Then by Fact 2.15, I_ϕ is strictly convex on its domain. By the assumption, $\phi^* = \sum_{i=1}^d \phi_i^*$, hence ϕ^* is differentiable everywhere on E. Then by [21, Corollary 2B], I_ϕ has weakly compact lower level sets.

Now we show I_ϕ has the Kadec property. Let $x_n \rightharpoonup^{\mathrm{w}} x \in \operatorname{dom} I_\phi$ in $L^1_E(S,\mu)$ and $I_\phi(x_n) \longrightarrow I_\phi(x)$. Since $x \in \operatorname{dom} I_\phi$, $x(s) \in \operatorname{dom}\phi$ for almost all $s \in S$. We can and do suppose that $x(s) \in \operatorname{dom}\phi$ for all $s \in S$.

We let $x(s) := \big(x_1(s),\cdots,x_d(s)\big)$ and $x_n(s) := \big(x_{n,1}(s),\cdots,x_{n,d}(s)\big)$. Now we claim that

$$\int_S \big|x_{n,i}(s) - x_i(s)\big| \mathrm{d}\mu(s) \longrightarrow 0, \quad \forall i \in \{1,\cdots,d\}. \tag{22}$$

Fix $i \in \{1,\cdots,d\}$. Since $\operatorname{int}\operatorname{dom}\phi_i \neq \varnothing$, there exist $\alpha \in \mathbb{R}\cup\{-\infty\}$ and $\beta \in \mathbb{R}\cup\{+\infty\}$ such that $\alpha < \beta$ and $\operatorname{int}\operatorname{dom}\phi_i = \,]\alpha,\beta[$. We set

$$\begin{aligned} S_\alpha &:= \big\{s \in S \mid x_i(s) = \alpha\big\} \\ S_\beta &:= \big\{s \in S \mid x_i(s) = \beta\big\} \\ S_{\mathrm{int}} &:= \big\{s \in S \mid x_i(s) \in \operatorname{int}\operatorname{dom}\phi_i\big\}. \end{aligned}$$

Then S_α, S_β and S_{int} are measurable sets. Given $y(s) = (y_i(s))_{i=1}^d \in L^1_E(S,\mu)$. Set $\widetilde{y}$ by

$$\widetilde{y}(s) := \big(y_1(s),\cdots,y_{i-1}(s),y_{i+1}(s),\cdots,y_d(s)\big).$$

Now we show that

$$\widetilde{x_n} \rightharpoonup^{\mathrm{w}} \widetilde{x} \quad \text{in } L^1_{R^{d-1}}(S,\mu). \tag{23}$$

Let $v^*(s) \in L^\infty_{R^{d-1}}(S,\mu)$. For convenience, we write

$$v^*(s) = \big(v_1(s), \cdots, v_{i-1}(s), v_{i+1}(s), \cdots, v_d(s)\big).$$

Then we define w^* by

$$w^*(s) := \big(v_1(s), \cdots, v_{i-1}(s), 0, v_{i+1}(s), \cdots, v_d(s)\big).$$

Then $w^* \in L^\infty_E(S,\mu)$ and $\langle \widetilde{x_n}, v^*\rangle = \langle x_n, w^*\rangle \longrightarrow \langle x, w^*\rangle = \langle \widetilde{x}, v^*\rangle$. Hence $\widetilde{x_n} \overset{\mathrm{w}}{\rightharpoonup} \widetilde{x}$ and thus (23) holds.

Similarly, we have

$$x_{n,i} \overset{\mathrm{w}}{\rightharpoonup} x_i \quad \text{in } L^1_R(S,\mu). \tag{24}$$

Then by (23), (24) and Fact 2.16,

$$\widetilde{x_n}|_{S_\gamma} \overset{\mathrm{w}}{\rightharpoonup} \widetilde{x}|_{S_\gamma} \quad \text{and} \quad x_{n,i}|_{S_\gamma} \overset{\mathrm{w}}{\rightharpoonup} x_i|_{S_\gamma}, \quad \gamma \in \{\alpha, \beta, \mathrm{int}\}. \tag{25}$$

Since $I_\phi(x_n) \longrightarrow I_\phi(x) < +\infty$, we have $I_\phi(x_n) < +\infty$ and hence $x_n(s) \in \operatorname{dom}\phi$ for all almost $s \in S$ when n is larger enough. Thus, we can and do assume that $x_{n,i}(s) \in \operatorname{dom}\phi_i$ for all $n \in \mathbb{N}, s \in S$. Since $S = S_\alpha \cup S_\beta \cup S_{\mathrm{int}}$, we have

$$\begin{aligned}
&\int_S \big|x_{n,i}(s) - x_i(s)\big| \mathrm{d}\mu(s)\\
&= \int_{S_\alpha} \big|x_{n,i}(s) - x_i(s)\big| \mathrm{d}\mu(s) + \int_{S_\beta} \big|x_{n,i}(s) - x_i(s)\big| \mathrm{d}\mu(s) + \int_{S_{\mathrm{int}}} \big|x_{n,i}(s) - x_i(s)\big| \mathrm{d}\mu(s)\\
&= \int_{S_\alpha} \big(x_{n,i}(s) - x_i(s)\big) \mathrm{d}\mu(s) + \int_{S_\beta} \big(x_i(s) - x_{n,i}(s)\big) \mathrm{d}\mu(s) + \int_{S_{\mathrm{int}}} \big|x_{n,i}(s) - x_i(s)\big| \mathrm{d}\mu(s).
\end{aligned} \tag{26}$$

By (25),

$$\int_{S_\alpha} \big(x_{n,i}(s) - x_i(s)\big) \mathrm{d}\mu(s) + \int_{S_\beta} \big(x_i(s) - x_{n,i}(s)\big) \mathrm{d}\mu(s) \longrightarrow 0. \tag{27}$$

Now we show that

$$\int_{S_{\mathrm{int}}} \big|x_{n,i}(s) - x_i(s)\big| \mathrm{d}\mu(s) \longrightarrow 0. \tag{28}$$

If $\mu(S_{\mathrm{int}}) = 0$, clearly, (28) holds. Now we assume that $\mu(S_{\mathrm{int}}) \neq 0$. We define $\psi : \mathbb{R}^{d-1} \to \left]-\infty, +\infty\right]$ by $z := (z_1, z_2, \cdots, z_{i-1}, z_{i+1}, \cdots, z_d) \mapsto \sum_{j\neq i} \phi_j(z_j)$. Then by Fact 2.13, $I_\psi^{S_{\mathrm{int}}}$ and $I_{\phi_i}^{S_{\mathrm{int}}}$ are proper lower semicontinuous and convex. Then by Remark 2.14 and (25),

$$\liminf \int_{S_{\mathrm{int}}} \psi\big(\widetilde{x_n}|_{S_{\mathrm{int}}}(s)\big) \mathrm{d}\mu(s) = \liminf I_\psi^{S_{\mathrm{int}}}(\widetilde{x_n}|_{S_{\mathrm{int}}}) \geq I_\psi^{S_{\mathrm{int}}}(\widetilde{x}|_{S_{\mathrm{int}}}),$$

(29) $$\liminf I_{\phi_i}^{S_{\rm int}}(x_{n,i}|_{S_{\rm int}}) \geq I_{\phi_i}^{S_{\rm int}}(x_i|_{S_{\rm int}}).$$

We first show that

(30) $$\int_{S_{\rm int}} \phi_i\big(x_{n,i}|_{S_{\rm int}}(s)\big)\mathrm{d}\mu(s) \longrightarrow \int_{S_{\rm int}} \phi_i\big(x_i|_{S_{\rm int}}(s)\big)\mathrm{d}\mu(s) < +\infty.$$

By Fact 2.16, we have

(31) $$I_{\phi}^{S_{\rm int}}(x_n|_{S_{\rm int}}) \longrightarrow I_{\phi}^{S_{\rm int}}(x|_{S_{\rm int}}) < +\infty.$$

Then we have

$$\begin{aligned}
&\limsup I_{\phi_i}^{S_{\rm int}}(x_{n,i}|_{S_{\rm int}}) = \limsup \int_{S_{\rm int}} \phi_i\big(x_{n,i}|_{S_{\rm int}}(s)\big)\mathrm{d}\mu(s) = \limsup \Big(I_{\phi}^{S_{\rm int}}(x_n|_{S_{\rm int}}) - I_{\psi}^{S_{\rm int}}(\widetilde{x_n}|_{S_{\rm int}})\Big)\\
&= \lim I_{\phi}^{S_{\rm int}}(x_n|_{S_{\rm int}}) - \liminf I_{\psi}^{S_{\rm int}}(\widetilde{x_n}|_{S_{\rm int}}))\\
&\leq I_{\phi}^{S_{\rm int}}(x|_{S_{\rm int}}) - I_{\psi}^{S_{\rm int}}(\widetilde{x}|_{S_{\rm int}}) \quad \text{(by (31) and (29))}\\
&= I_{\phi_i}^{S_{\rm int}}(x_i|_{S_{\rm int}}) < +\infty \quad \text{(since } I_{\phi}^{S_{\rm int}}(x|_{S_{\rm int}}) < +\infty \text{ and } I_{\psi}^{S_{\rm int}}(\widetilde{x}|_{S_{\rm int}}) > -\infty \text{ by (31) and Fact 2.13).}
\end{aligned}$$

Then combining with (29), we have $\limsup I_{\phi_i}^{S_{\rm int}}(x_{n,i}|_{S_{\rm int}}) \leq I_{\phi_i}^{S_{\rm int}}(x_i|_{S_{\rm int}}) \leq \liminf I_{\phi_i}^{S_{\rm int}}(x_{n,i}|_{S_{\rm int}})$ and thus (30) holds.

By (25), (30) and Proposition 3.6, we have $\int_{S_{\rm int}} \big|x_{n,i}(s) - x_i(s)\big|\mathrm{d}\mu(s) \longrightarrow 0$ and hence (28) holds.

Combining (28), (27) and (26), we have $\int_S |x_{n,i}(s) - x_i(s)|\mathrm{d}\mu(s) \longrightarrow 0$ and hence (22) holds.

Then by (22),

$$\|x_n - x\|_1 \leq \int_S \sum_{i=1}^{d} \big|x_{n,i}(s) - x_i(s)\big|\mathrm{d}\mu(s) \longrightarrow 0.$$

Hence $x_n \longrightarrow x$ and hence I_ϕ has the Kadec property.

Combining the above results, I_ϕ is strongly rotund in $L^1_E(S,\mu)$. ■

Remark 3.9 It is noted in [6] that strongly rotund functions with points of continuity can only exist on reflexive spaces. Moreover, strongly rotund integral functions on $L^1_E(S,\mu)$ are a useful surrogate for strongly rotund renorms which always exist in the reflexive setting. $\Diamond$

4 Examples and applications

Below we use the convention that $0 \log 0 = 0$.

Example 4.1 By applying Theorem 3.7 and Theorem 3.8, we can obtain many functions ϕ such that I_ϕ is strongly rotund. Seven examples follow

(i) Let $f : \mathbb{R} \to \,]-\infty, +\infty]$ be defined by

$$f(x) = \begin{cases} x \log x - x, & \text{if } x \geq 0; \\ +\infty, & \text{otherwise} \end{cases} \quad \forall x \in \mathbb{R}.$$

Let $\phi : E \to \,]-\infty, +\infty]$ be defined by

$$\phi(x) := \sum_{i=1}^{d} f(x_i), \quad \forall x = (x_n) \in E.$$

Then I_ϕ is the *Boltzmann-Shannon entropy.*

(ii) Let $f : \mathbb{R} \to \,]-\infty, +\infty]$ be defined by

$$f(x) = \begin{cases} x \log x + (1-x)\log(1-x), & \text{if } 0 \leq x \leq 1; \\ +\infty, & \text{otherwise} \end{cases} \quad \forall x \in \mathbb{R}.$$

Let $\phi : E \to \,]-\infty, +\infty]$ be defined by

$$\phi(x) := \sum_{i=1}^{d} f(x_i), \quad \forall x = (x_n) \in E.$$

Then I_ϕ is the *Fermi-Dirac entropy*

(iii) $\phi(x) = \frac{1}{p}\|x\|^p, \quad \forall x \in E, \quad$ where $p > 1$.

(iv) $\phi(x) = \begin{cases} \sum_{i=1}^{d} -\log(\cos x_i), & \text{if } x \in \,]-\frac{\pi}{2}, \frac{\pi}{2}[\, \times \cdots \times \,]-\frac{\pi}{2}, \frac{\pi}{2}[\\ +\infty, & \text{otherwise} \end{cases} \quad \forall x = (x_n) \in E.$

(v) $\phi(x) = \sum_{i=1}^{d} \cosh x_i, \quad \forall x = (x_n) \in E.$

(vi) $\phi(x) = \begin{cases} \sum_{i=1}^{d} \left(x_i \tanh^{-1} x_i + \frac{1}{2}\log(1 - x_i^2)\right), & \text{if } |x_i| < 1, \forall i; \\ +\infty, & \text{otherwise} \end{cases} \quad \forall x = (x_n) \in E.$

(vii) $\phi(x) = \begin{cases} -\frac{1}{1-\|x\|^2}, & \text{if } \|x\| < 1; \\ +\infty, & \text{otherwise} \end{cases} \quad \forall x \in E.$

Proof. (i): Clearly, f is proper lower semicontinuous and convex. By [8, Table 2.1, pp. 45], $f^*(x) = \exp(x), \forall x \in \mathbb{R}$. Then directly apply Theorem 3.8.

(i): Clearly, f is proper lower semicontinuous and convex. By [8, Table 2.1, pp. 45], $f^*(x) = \log\big(1 + \exp(x)\big), \forall x \in \mathbb{R}$. Then directly apply Theorem 3.8.

(iii): Clearly, ϕ is continuous and convex with full domain. We have $\phi^* = \frac{1}{q}\|\cdot\|^q$, where $\frac{1}{q} + \frac{1}{p} = 1$ and $(\phi^*)' = (\|\cdot\|)^{q-2} \circ \mathrm{Id}$. Hence ϕ^* is differentiable everywhere on E. Then directly apply Theorem 3.7.

(iv): Let $f(x) := \begin{cases} -\log(\cos x), & \text{if } x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ +\infty, & \text{otherwise} \end{cases}$. By [8, Table 2.1, pp. 45], we have f is proper lower semicontinuous and convex, and $f^*(x) = x\tan^{-1} x - \frac{1}{2}\log(1 + x^2)\big), \forall x \in \mathbb{R}$. Hence f^* is differentiable everywhere on $\mathbb{R}$. Then directly apply Theorem 3.8.

(v): Let $f(x) := \cosh(x)$. By [8, Table 2.1, pp. 45], we have f is continuous and convex, and $f^*(x) = x\sinh^{-1} x - \sqrt{1 + x^2}, \forall x \in X$. Hence f^* is differentiable everywhere on $\mathbb{R}$. Then directly apply Theorem 3.8.

(vi): Let $f(x) := \begin{cases} x\tanh^{-1} x + \frac{1}{2}\log(1 - x^2)\big), & \text{if } |x| < 1; \\ +\infty, & \text{otherwise} \end{cases}$. By [8, Table 2.1, pp. 45], we have f is proper lower semicontinuous and convex, and $f^*(x) = \log(\cosh x)$. Thus f^* is differentiable everywhere on $\mathbb{R}$. Then directly apply Theorem 3.8.

(vii): Clearly, $\operatorname{dom}\phi$ is open. By [2, Example 6.4], ϕ is proper lower semicontinuous and convex, and ϕ^* is differentiable everywhere on E. Then directly apply Theorem 3.7. ■

Example 4.2 Let $(a_n^*)_{n\in\mathbb{N}}$ be a sequence in $L_E^\infty(S,\mu)$ and let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Let $\phi : X \to \left]-\infty, +\infty\right]$ be one of the functions given in Example 4.1.

We consider the following optimization problems (See [4, page 196].).

$$(P_n) \qquad \begin{cases} V(P_n) := & \inf I_\phi(x) \\ \text{subject to} & \langle a_i^*, x\rangle = b_i, \quad i = 1, \cdots, n \\ & x \in L_E^1(S,\mu) \end{cases}$$

$$(P_\infty) \qquad \begin{cases} V(P_\infty) := & \inf I_\phi(x) \\ \text{subject to} & \langle a_i^*, x\rangle = b_i, \quad i = 1, \cdots, n, n+1, \cdots \\ & x \in L_E^1(S,\mu). \end{cases}$$

Then we have $V(P_n) \longrightarrow V(P_\infty)$. If, moreover, $V(P_\infty) < +\infty$, then (P_n) and (P_∞) respectively have unique optimal solutions with x_n and x_∞, and $x_n \longrightarrow x_\infty$. $\Diamond$

Proof. Set

$$
\begin{aligned}
C_n :&= \big\{x \in L^1_E(S,\mu) \mid \langle a_i^*, x\rangle = b_i, \quad i = 1, \cdots, n\big\} \\
C_\infty :&= \big\{x \in L^1_E(S,\mu) \mid \langle a_i^*, x\rangle = b_i, \quad i = 1, \cdots, n, n+1, \cdots\big\}.
\end{aligned}
$$

Then we have $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$. Thus, $\overline{\lim}^{\mathrm{w}} C_n \subseteq C_\infty$ and $C_\infty = \bigcap_{n\geq 1} C_n \subseteq \bigcup_{m\geq 1}\bigcap_{n\geq m} C_n$. We finish with a direct application of Example 4.1 and Fact 2.19. $\blacksquare$

We next revisit a function ϕ given in [3] such that I_ϕ is not strongly rotund but ϕ is everywhere strictly convex.

Example 4.3 (Borwein and Lewis) Let $\phi(x) := \begin{cases} -\log x, & \text{if } x > 0; \\ +\infty, & \text{otherwise} \end{cases}, \forall x \in \mathbb{R}$. Let $S = [0,1]$ and μ be Lebesgue measure.

Then I_ϕ is the *Burg entropy*, and $\phi^*(x) = \begin{cases} -1 - \log(-x), & \text{if } x < 0; \\ +\infty, & \text{otherwise} \end{cases}, \forall x \in \mathbb{R}$. However, I_ϕ does not have weakly compact lower level sets (See [3, page 258].). Hence I_ϕ is not strongly rotund. $\Diamond$

5 Watson integral and Burg entropy nonattainment

Let $S = [0,1] \times [0,1] \times [0,1]$ and μ be the Lebesgue measure, and let ϕ be defined as in Example 4.3. Consider the perturbed *Burg entropy* minimization problem

$$
\begin{cases}
 & \inf I_\phi(x) \\
\text{subject to} & \int_S x(s)\mathrm{d}\mu(s) = 1 \\
 & \int_S x(s)\cos(2\pi s_k)\,\mathrm{d}\mu(s) = \alpha, \quad k = 1,2,3 \\
 & x \in L^1_{\mathbb{R}}(S,\mu),
\end{cases}
$$

where $s := (s_1, s_2, s_3)$ and $d\mu(s) := ds_1 ds_2 ds_3$. Then the above problem is equivalent to the following.

$$
v(\alpha) := \sup_{0\leq p \in L^1_{\mathbb{R}}(S,\mu)} \left\{ \int_S \log\left(p(x_1,x_2,x_3)\right) \,\middle|\, \int_S p(x_1,x_2,x_3)dx_1dx_2dx_3 = 1, \right.
$$

$$\text{and for } k=1,2,3,\ \int_S p(x_1,x_2,x_3)\cos(2\pi x_k)\,dx_1dx_2dx_3=\alpha\Bigg\},$$

maximizing the log of a density p with given mean, and with the first three cosine moments fixed at a parameter value $0 \le \alpha < 1$. It transpires that there is a parameter value $\overline{\alpha}$ such that below and at that value $v(\alpha)$ is attained, while above it is finite but unattained. This is interesting, because:

The general method—maximizing $\int_S \log(p(s))\,\mathrm{d}\mu(s)$ subject to a finite number of trigonometric moments—is frequently used. In one or two dimensions, such spectral problems are always attained when feasible.

There is no easy way to see that this problem qualitatively changes at $\overline{\alpha}$, (by [5, Eqs. (5.8)&(5.10)]) but we can get an idea by considering

$$\overline{p}\,(x_1,x_2,x_3)=\frac{1/W_1}{3-\sum_1^3\cos(2\pi x_i)},$$

and checking that this is feasible for

$$\overline{\alpha}=1-1/(3W_1)\approx 0.340537329550999142833$$

in terms of the first Watson integral, $W_1 := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{3-\cos(x_1)-\cos(x_2)-\cos(x_3)}dx_1dx_2dx_3$ (See [7, Item 20, page 117] and [14] for more information about W_1.). By using Fenchel duality [8] one can show that this $\overline{p}$ is optimal.

Indeed, for all $\alpha \ge 0$ the only possible optimal solution is of the form

$$\overline{p}_\alpha\,(x_1,x_2,x_3)=\frac{1}{\lambda_\alpha^0-\sum_1^3\lambda_\alpha^i\cos(2\pi x_i)},$$

for some real numbers λ_α^i. Note that we have four coefficients to determine; using the four constraints we can solve for them. Let $W_1(w)$ be the generalized Watson integral, i.e., $W_1(w) := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{3-w\left(\cos(x_1)+\cos(x_2)+\cos(x_3)\right)}dx_1dx_2dx_3$ (See [7, Item 21(e), page 120] and [14] for more information about $W_1(w)$.).

For $0 \le \alpha \le \overline{\alpha}$, the precise form is parameterized by the generalized Watson integral:

$$\overline{p}_\alpha\,(x_1,x_2,x_3)=\frac{1/W_1(w)}{3-\sum_1^3 w\cos(2\pi x_i)},$$

and $\alpha = 1-1/(3W_1(w))$, as w ranges from zero to one.

Note also that $W_1(w) = \pi^3\int_0^\infty I_0^3(w\,t)\,e^{-3t}\,dt$ allows one to quickly obtain w from α numerically. For $\alpha > \overline{\alpha}$, no feasible reciprocal polynomial can stay positive. Full details are given in [5, Example 4, pp. 264-265].

6 Applications of Visintin's Theorem

Visintin's Theorem [26, Theorem 3(i)] on norm convergence of sequences converging weakly to an extreme point, allows for a very efficient proof of the Kadec property for integral functionals. Indeed, using Fact 2.10, we arrive at the following.

Fact 6.1 (Visintin) (See [26, Theorem 3(i)].) *Let $\phi : E \to\ ]-\infty, +\infty]$ be proper, lower semicontinuous and strictly convex. Then I_ϕ has the Kadec property.*

Remark 6.2 In the proofs of Theorem 3.7 and Theorem 3.8, we can also apply Visintin Theorem (see Fact 6.1) to show that I_ϕ has the Kadec property.

Example 6.3 Let ϕ be defined as in Example 4.3. Then I_ϕ has the Kadec property. Indeed, since ϕ is proper, lower semicontinuous and strictly convex, it follows from Fact 6.1 that I_ϕ has the Kadec property. $\diamondsuit$

Theorem 6.4 (Strong rotundity) *Let $\phi : E \to\ ]-\infty, +\infty]$ be proper, lower semicontinuous and convex. Suppose that ϕ is strictly convex on its domain and ϕ^* is differentiable on E. Then I_ϕ is strongly rotund on $L^1_E(S, \mu)$.*

Proof. By Fact 2.15, I_ϕ is strictly convex on its domain. Since $\operatorname{dom} \phi^* = E$, by [21, Corollary 2B], I_ϕ has weakly compact lower level sets. Visintin Theorem (see Fact 6.1) implies that I_ϕ has the Kadec property. Hence I_ϕ is strongly rotund. ■

Remark 6.5 We cannot remove the assumption of strict convexity of ϕ in Theorem 6.4. For example, let $\phi : \mathbb{R}^2 \to\ ]-\infty, +\infty]$ be defined by

$$(x, y) \mapsto \begin{cases} -(xy)^{\frac{1}{4}}, & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1; \\ +\infty, & \text{otherwise} \end{cases}.$$

Then ϕ is proper lower semicontinuous and convex. By [8, Exercise 5.3.10, page 249], ϕ is not strictly convex on its domain although ϕ^* is differentiable everywhere on $\mathbb{R}^2$. Hence I_ϕ is not strongly rotund.

Remark 6.6 Let $\phi : E \to\ ]-\infty, +\infty]$ be proper, lower semicontinuous and convex. Suppose that ϕ^* is differentiable on E. Assume that E is one-dimensional or $\operatorname{dom} \phi = \operatorname{dom} \partial\phi$ (for example, $\operatorname{dom} \phi$ is open), by Fact 2.8 and Fact 2.9(ii), the differentiability of ϕ^* implies that the strictly convexity of ϕ. Thus we can remove the assumption of the strictly convexity of ϕ in Theorem 6.4 under this constraint.

Example 6.7 Let F be the Euclidean space that consists of all symmetric $d \times d$ matrices with the inner product $\langle M, N\rangle = \mathbf{tr}(MN)$ (for every $M, N \in F$), where $\mathbf{tr}(M)$ is the trace of the matrix M. Let F_{++} be the set of symmetric $d \times d$ positive definite matrices. We define ϕ on F by $M \mapsto \phi(M) := \begin{cases} -\log\det(M), & \text{if } M \in F_{++}; \\ +\infty, & \text{otherwise} \end{cases}$, where $\det(M)$ is the determinant of the matrix M. Then I_ϕ has the Kadec property in $L^1_F(S,\mu)$.

Proof. By [8, Proposition 3.2.3, page 100], ϕ is proper lower semicontinuous and strictly convex. Then by Fact 6.1, I_ϕ has the Kadec property in $L^1_F(S,\mu)$. ■

7 Convergence in measure

Recall that (S,μ) is a complete finite measure space (with nonzero measure μ). Let $(x_n)_{n\in\mathbb{N}}$ and x be in $L^1_E(S,\mu)$. We say $(x_n)_{n\in\mathbb{N}}$ *converges to x in measure* if for every $\eta > 0$, $\lim \mu\big\{s \in S \mid \|x_n(s) - x(s)\| \geq \eta\big\} = 0$. We say $(x_n)_{n\in\mathbb{N}}$ *converges to x μ– uniformly* if for every $\varepsilon > 0$, there exists a measurable subset T of S such that $\mu(T) < \varepsilon$ and $(x_n)_{n\in\mathbb{N}}$ converges uniformly to x on T^c.

Let $(x_n)_{n\in\mathbb{N}}$ and x be in $L^1_E(S,\mu)$. Then $(x_n)_{n\in\mathbb{N}}$ strongly converges to x if and only if $(x_n)_{n\in\mathbb{N}}$ converges to x in measure and $(x_n)_{n\in\mathbb{N}}$ also weakly converges to x (see [26, Lemma 1 and Lemma 2]). Thus, for a strictly convex integrand, Theorem 6.4 shows that weak convergence must fail whenever measure convergence holds and strong convergence does not follow.

The following is another sufficient condition for a sequence convergent in measure to be strongly convergent.

Fact 7.1 See ([12, Theorem 3.6, page 122]) *Let $(x_n)_{n\in\mathbb{N}}$ be in $L^1_E(S,\mu)$ and $x : S \to E$. Then $x \in L^1_E(S,\mu)$ and $x_n \longrightarrow x$ if and only if the following conditions hold:*

(i) *$(x_n)_{n\in\mathbb{N}}$ converges to x in measure.*

(ii) $\lim_{\mu(E)\to 0} \int_E \|x_n(s)\| \mathrm{d}\mu(s) = 0$ *uniformly in n.*

See [1] for more information on the relationships between weak, measure and strong convergence.

Fact 7.2 (See [12, Corollary 3.3, page 145].) *Let $(x_n)_{n\in\mathbb{N}}$ and x be in $L^1_E(S,\mu)$. Assume that $(x_n)_{n\in\mathbb{N}}$ converges to x in measure. Then there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to x μ–uniformly.*

Let $f : X \to]-\infty, +\infty]$ be lower semicontinuous at $x_0 \in \operatorname{dom} f$. Then the *Clarke-Rockafellar directional derivative* of f at x_0 is defined

$$f^{\uparrow}(x_0; v) := \sup_{\varepsilon>0} \limsup_{t\downarrow 0,\, x\to_f x_0} \inf_{\|u-v\|\le\varepsilon} \tfrac{f(x+tu)-f(x)}{t}, \quad \forall v \in X,$$

where $x \to_f x_0$ means that $x \longrightarrow x_0$ and $f(x) \longrightarrow f(x_0)$. Then the *Clarke subdifferential* of f at x_0 is defined by

$$\partial_C f(x_0) := \big\{x^* \in X^* \mid \langle x^*, v\rangle \le f^{\uparrow}(x_0; v),\, \forall v \in X\big\}.$$

If f is also convex, then $\partial f = \partial_C f$ (see [27, Theorem 3.2.4(ii)]).

We are now ready for two results showing when convergence in measure of a sequence $(x_n)_{n\in\mathbb{N}}$ allows us to deduce convergence of $\big(I_\phi(x_n)\big)_{n\in\mathbb{N}}$. This is useful if one thinks of I_ϕ as a measurement of a reconstruction x_n for a member of a sequence which may not be norm convergent to the underlying signal x.

Theorem 7.3 (Preservation of convergence in measure) *Let $\phi : E \to]-\infty, +\infty]$ be proper such that $\phi|_{\operatorname{dom}\phi}$ is continuous on $\operatorname{dom}\phi$. Assume that there exists $M > 0$ such that $|\phi(v)| \le M$ for all $v \in \operatorname{dom}\phi$. Let $(x_n)_{n\in\mathbb{N}}$ and x be in $\operatorname{dom} I_\phi$ such that $(x_n)_{n\in\mathbb{N}}$ converges to x in measure. Suppose that one of the following conditions holds.*

(i) *$x \in L^\infty_E(S, \mu)$; or*

(ii) *$\phi|_{\operatorname{dom}\phi}$ is uniformly continuous on $\operatorname{dom}\phi$, in particular, when $\phi|_{\operatorname{dom}\phi}$ is globally Lipschitz on $\operatorname{dom}\phi$.*

Then $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \longrightarrow 0$. Consequently, $I_\phi(x_n) \longrightarrow I_\phi(x)$.

Proof. Since $(x_n)_{n\in\mathbb{N}}$ and x are in $\operatorname{dom} I_\phi$, we can and do assume that $x_n(s) \in \operatorname{dom}\phi$ for all $n \in \mathbb{N}, s \in S$ and $x(s) \in \operatorname{dom}\phi$ for all $s \in S$.

We first assume that $x \in L^\infty_E(S, \mu)$. Suppose to the contrary that $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \not\to 0$. Then there exist $\varepsilon_0 > 0$ and a subsequence of $(x_n)_{n\in\mathbb{N}}$, for convenience, still denoted by $(x_n)_{n\in\mathbb{N}}$, such that

$$\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \ge \varepsilon_0, \quad \forall n \in \mathbb{N}. \tag{32}$$

Since $x \in L^\infty_E(S, \mu)$, there exists $L > 0$ such that $\|x(s)\| \le L$ for almost all $s \in S$. We can and do suppose that

$$\|x(s)\| \le L, \quad \forall s \in S. \tag{33}$$

Let $\varepsilon > 0$. Since $\phi|_{\operatorname{dom}\phi}$ is continuous on $\operatorname{dom}\phi$, then $\phi|_{\operatorname{dom}\phi}$ is uniformly continuous on $\operatorname{dom}\phi \cap (L+1)B_E$. Then there exists $\delta > 0$ such that

$$|\phi(u) - \phi(v)| \leq \varepsilon, \quad \forall \|u - v\| \leq \delta, \quad \forall u, v \in \operatorname{dom}\phi \cap (L+1)B_E. \tag{34}$$

By Fact 7.2, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(x_{n_k})_{k\in\mathbb{N}}$ converges to x μ–uniformly. Then there exist $N_1 \in \mathbb{N}$ and a measurable subset T of S such that $\mu(T) < \varepsilon$ and

$$\|x_{n_k}(s) - x(s)\| \leq \min\{\delta, 1\}, \quad \forall k \geq N_1, \forall s \in T^c. \tag{35}$$

Then by (33),

$$x_{n_k}(s) \in \operatorname{dom}\phi \cap (L+1)B_E, \quad \forall k \geq N_1, \forall s \in T^c. \tag{36}$$

Then by assumption, we have

$$\begin{aligned}
&\int_S \big|\phi\big(x_{n_k}(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s)\\
&= \int_{T^c} \big|\phi\big(x_{n_k}(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) + \int_T \big|\phi\big(x_{n_k}(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s)\\
&\leq \int_{T^c} \varepsilon \mathrm{d}\mu(s) + \int_T \big|\phi\big(x_{n_k}(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \quad \text{(by (36), (33),(34) and (35))}\\
&\leq \int_{T^c} \varepsilon \mathrm{d}\mu(s) + \int_T 2M \mathrm{d}\mu(s) \quad (\text{since } |\phi(v)| \leq M \text{ for all } v \in E)\\
&\leq \varepsilon\mu(T^c) + 2M\varepsilon, \quad \forall k \geq N_1.
\end{aligned}$$

Then $\int_S \big|\phi\big(x_{n_k}(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \longrightarrow 0$, which contradicts (32). Hence $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \longrightarrow 0$. Consequently, $I_\phi(x_n) \longrightarrow I_\phi(x)$.

The proof is similar when ϕ is assumed uniformly continuous but x is allowed to lie in $L^1_E(S,\mu)$. ■

The following brief proof of Corollary 7.4 is due to the referee.

Corollary 7.4 *Let $\phi : E \to\,]-\infty, +\infty]$ be proper lower semicontinuous. Let $(x_n)_{n\in\mathbb{N}}$ and x be in $L^1_E(S,\mu)$. Assume that there exists $\delta > 0$ such that*

$$\sup_{(x,x^*)\in \operatorname{gra}\partial_C\phi} \|x^*\| \leq \delta. \tag{37}$$

Suppose that $(x_n)_{n\in\mathbb{N}}$ converges to x in measure and there exists $M > 0$ such that $|\phi(v)| \leq M$ for all $v \in \operatorname{dom}\phi$. Then $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \longrightarrow 0$. Consequently, $I_\phi(x_n) \longrightarrow I_\phi(x)$.

Proof. By [24, Theorem 2.1], ϕ is $\delta-$ Liptschtiz on E. Then we directly apply Theorem 7.3(ii). ■

While non-trivial convex integrands will not satisfy (37) there are many simple examples which do.

Example 7.5 (Nonconvex integrands) Let $\phi(x) := \min\{\|x\|, 1\}$ for every $x \in E$. Let $(x_n)_{n\in\mathbb{N}}$ and x be in $L^1_E(S, \mu)$. Suppose that $(x_n)_{n\in\mathbb{N}}$ converges to x in measure. Then $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \longrightarrow 0$. Consequently, $I_\phi(x_n) \longrightarrow I_\phi(x)$.

Proof. Clearly, ϕ is continuous (actually Lipschitz) and $\sup_{(x,x^*)\in \operatorname{gra} \partial_C \phi} \|x^*\| \leq 1$. By the definition of ϕ, we have $(x_n)_{n\in\mathbb{N}}$ and x are in $\operatorname{dom} I_\phi$. Then directly apply Corollary 7.4. ■

Now we give an example of a useful convex integrand.

Example 7.6 (Convex integrands) Let ϕ be defined as in Example 4.1(ii) (i.e., I_ϕ is the Fermi-Dirac entropy). Let $(x_n)_{n\in\mathbb{N}}$ and x be in $\operatorname{dom} I_\phi$. Suppose that $(x_n)_{n\in\mathbb{N}}$ converges to x in measure. Then $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \longrightarrow 0$. Consequently, $I_\phi(x_n) \longrightarrow I_\phi(x)$.

Proof. Since $\operatorname{dom} \phi$ is compact, $\phi|_{\operatorname{dom}\phi}$ is uniformly continuous on $\operatorname{dom} \phi$. We have $\sup_{v\in \operatorname{dom}\phi} |\phi(v)| \leq d\ln(2)$. Then apply Theorem 7.3(ii) directly. ■

To use such value convergence results, it behoves us to provide an example of integrands such that $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \,\mathrm{d}\mu(s) \longrightarrow 0$ implies $x_n \to x$ in measure.

Example 7.7 Let $\phi(x) := \begin{cases} -\log x, & \text{if}\, x > 0; \\ +\infty, & \text{otherwise} \end{cases}, \forall x \in \mathbb{R}$. Let $S = [0, 1]$ and let μ be the Lebesgue measure. Let $(x_n)_{n\in\mathbb{N}}$ and x be in $\operatorname{dom} I_\phi$. Suppose that $x \in L^\infty_E(S, \mu)$ and $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \longrightarrow 0$. Then $x_n \to x$ in measure.

Proof. By the assumption, we can and do assume that $x_n(s) \in \operatorname{dom}\phi$ for all $n \in \mathbb{N}, s \in S$ and $x(s) \in \operatorname{dom}\phi$ for all $s \in S$. Since $x \in L^\infty_E(S, \mu)$, there exists $L > 0$ such that $|x(s)| \leq L$ for almost all $s \in S$. We can and do suppose that

$$|x(s)| \leq L, \quad \forall s \in S. \tag{38}$$

Suppose to the contrary that $(x_n)_{n\in\mathbb{N}}$ does not converge to x in measure. Then there exist $\eta > 0$, $\varepsilon_0 > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\mu(T_k) \geq \varepsilon_0, \quad \forall k \in \mathbb{N}, \tag{39}$$

where $T_k := \{s \in S \mid |x_{n_k}(s) - x(s)| \geq \eta\}$. Then we have

$$\begin{aligned}
&\int_S |\phi(x_{n_k}(s)) - \phi(x(s))| \mathrm{d}\mu(s) \\
&= \int_{T_k} |\phi(x_{n_k}(s)) - \phi(x(s))| \mathrm{d}\mu(s) + \int_{(T_k)^c} |\phi(x_{n_k}(s)) - \phi(x(s))| \mathrm{d}\mu(s) \\
&\geq \int_{T_k} |\phi(x_{n_k}(s)) - \phi(x(s))| \mathrm{d}\mu(s) \\
&= \int_{T_k} |\langle \phi'(y_{k_s}), x_{n_k}(s) - x(s)\rangle| \mathrm{d}\mu(s), \quad \exists y_{k_s} \in [x_{n_k}(s), x(s)] \quad \text{(by Mean Value Theorem)} \\
&= \int_{T_k} \tfrac{1}{|x(s)+t_{k_s}(x_{n_k}(s)-x(s))|} \cdot |x_{n_k}(s) - x(s)| \mathrm{d}\mu(s), \quad \exists t_{k_s} \in [0,1] \\
&\geq \int_{T_k} \tfrac{1}{|x(s)|+|x_{n_k}(s)-x(s)|} \cdot |x_{n_k}(s) - x(s)| \mathrm{d}\mu(s) \\
&\geq \int_{T_k} \tfrac{\eta}{L+\eta} \geq \tfrac{\eta\varepsilon_0}{L+\eta} \quad \text{(by (38) and (39))}, \forall k \in \mathbb{N},
\end{aligned}$$

which contradicts that $\int_S |\phi(x_n(s)) - \phi(x(s))| \mathrm{d}\mu(s) \longrightarrow 0$. Hence $x_n \to x$ in measure. ■

Sadly, in Example 7.7, we cannot replace $\int_S |\phi(x_n(s)) - \phi(x(s))| \mathrm{d}\mu(s) \longrightarrow 0$ by $I_\phi(x_n) \longrightarrow I_\phi(x)$. We use the following example to show that.

Example 7.8 Let $\phi(x) := \begin{cases} -\log x, & \text{if } x > 0; \\ +\infty, & \text{otherwise} \end{cases}, \forall x \in \mathbb{R}$, and let S, μ be defined as in Example 7.7. We define $x_n : S \to \mathbb{R}$ (for every $n \in \mathbb{N}$) by $x_n(s) := \begin{cases} n, & \text{if } s \in \left[0, \frac{1}{1+\log n}\right]; \\ 1, & \text{otherwise} \end{cases}, \forall s \in S$. Set $x(s) := \exp(1), \forall s \in S$.

Then $(x_n)_{n\in\mathbb{N}}$ and x are in $\operatorname{dom} I_\phi$, $x \in L^\infty_{\mathbb{R}}(S, \mu)$ and $I_\phi(x_n) \longrightarrow I_\phi(x) = -1$ but $(x_n)_{n\in\mathbb{N}}$ does not converge to x in measure.

Proof. Clearly, $x \in L^1_{\mathbb{R}}(S, \mu) \cap L^\infty_{\mathbb{R}}(S, \mu)$. Now we show $(x_n)_{n\in\mathbb{N}}$ is in $L^1_{\mathbb{R}}(S, \mu)$. Fix $n \in \mathbb{N}$. Then x_n is a bounded and measurable function. Thus, $x_n \in L^1_{\mathbb{R}}(S, \mu)$.

Now we show that $I_\phi(x_n) \longrightarrow I_\phi(x)$. Clearly, $I_\phi(x) = \int_S -\log\Big(\exp(1)\Big) \mathrm{d}\mu(s) = -1$.

$$\begin{aligned}
I_\phi(x_n) &= \int_S \phi(x_n(s)) \mathrm{d}\mu(s) \\
&= \int_{\left[0, \frac{1}{1+\log n}\right]} \phi(x_n(s)) \mathrm{d}\mu(s) + \int_{\left]\frac{1}{1+\log n}, 1\right]} \phi(x_n(s)) \mathrm{d}\mu(s)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\left[0,\frac{1}{1+\log n}\right]} -\log n \mathrm{d}\mu(s) + \int_{\left]\frac{1}{1+\log n},1\right]} -\log 1 \mathrm{d}\mu(s) \\
&= -\tfrac{\log n}{1+\log n} \longrightarrow -1 = I_\phi(x).
\end{aligned}
$$

Hence $I_\phi(x_n) \longrightarrow I_\phi(x)$.

On the other hand,

$$
\begin{aligned}
&\mu\big\{s \in S \mid |x_n(s) - x(s)| \geq 1\big\} = \mu\big\{s \in S \mid |x_n(s) - \exp(1)| \geq 1\big\} \\
&\geq \mu\Big\{ \left]\tfrac{1}{1+\log n}, 1\right] \Big\} = 1 - \tfrac{1}{1+\log n} \geq \tfrac{1}{2}, \quad \forall n \geq 3.
\end{aligned}
$$

Hence $(x_n)_{n\in\mathbb{N}}$ does not converge to x in measure. ∎

The converse of Example 7.7 cannot hold either.

Example 7.9 Let ϕ, S, μ and $(x_n)_{n\in\mathbb{N}}$ be all defined as in Example 7.8. Let $x(s) := 1, \forall s \in S$. Then $(x_n)_{n\in\mathbb{N}}$ and x are in $\operatorname{dom} I_\phi$, $x \in L^\infty_{\mathbb{R}}(S,\mu)$ and $x_n \to x$ in measure but $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \nrightarrow 0$.

Proof. Example 7.8 shows that $(x_n)_{n\in\mathbb{N}}$ is in $\operatorname{dom} I_\phi$.

Clearly, $x \in \operatorname{dom} I_\phi$ and $x \in L^\infty_{\mathbb{R}}(S,\mu)$. Now we show that $x_n \to x$ in measure. Let $\eta > 0$. Then we have

$$
\mu\big\{s \in S \mid |x_n(s) - x(s)| \geq \eta\big\} = \mu\big\{s \in S \mid |x_n(s) - 1| \geq \eta\big\} \leq \mu\Big\{ \left[0, \tfrac{1}{1+\log n}\right] \Big\} = \tfrac{1}{1+\log n} \longrightarrow 0.
$$

Hence $x_n \to x$ in measure.

We have

$$
\begin{aligned}
&\lim \int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) = \lim \int_S \big|\phi\big(x_n(s)\big)\big)\big| \mathrm{d}\mu(s) \\
&= \lim \int_S -\phi\big(x_n(s)\big)\big) \mathrm{d}\mu(s) = \lim -I_\phi(x_n) = 1 \neq 0 \quad (\text{ by Example 7.8}).
\end{aligned}
$$

Hence $\int_S \big|\phi\big(x_n(s)\big) - \phi\big(x(s)\big)\big| \mathrm{d}\mu(s) \nrightarrow 0$. ∎

Let $\phi : E \to \left]-\infty, +\infty\right]$ be proper lower semicontinuous and strictly convex. Let $(x_n)_{n\in\mathbb{N}}$ and x be in $\operatorname{dom} I_\phi$. Assume that $x \in \operatorname{argmin} I_\phi$. The results so far given provoke the following question:

If $x_n \longrightarrow x$ in measure, is it necessarily true that $I_\phi(x_n) \longrightarrow I_\phi(x)$?

The following example shows that the above statement cannot be true without imposing extra conditions.

Example 7.10 (Incompatibility of measure and value convergence) Let $\phi(x) := \begin{cases} -\log x + x, & \text{if } x > 0; \\ +\infty, & \text{otherwise} \end{cases}, \forall x \in \mathbb{R}$. Let $S = [0,1]$ and let μ be the Lebesgue measure. Set (for every $n \in \mathbb{N}$) $x_n(s) := \begin{cases} n, & \text{if } s \in \left[0, \frac{1}{n}\right]; \\ 1, & \text{otherwise} \end{cases}, \forall s \in S$. Then ϕ is proper lower semicontinuous and strictly convex. Let $x : S \to \mathbb{R}$ be given by $x(s) := 1, \forall s \in S$.

Then $(x_n)_{n\in\mathbb{N}}$ and x are in $L^1_{\mathbb{R}}(S, \mu)$,

$$\operatorname{argmin} I_\phi = \{x\} \ \text{ and } x_n \to x \text{ in measure}$$

but $I_\phi(x_n) \nrightarrow I_\phi(x)$. In particular, $(x_n)_{n\in\mathbb{N}}$ does not converge weakly to x.

Proof. Clearly, $x \in L^1_{\mathbb{R}}(S, \mu)$. First we show $(x_n)_{n\in\mathbb{N}}$ is in $L^1_{\mathbb{R}}(S, \mu)$. Let $n \in \mathbb{N}$. Then x_n is a measurable function. Let $n \in \mathbb{N}$. Then x_n is a bounded and measurable function. Thus, $x_n \in L^1_{\mathbb{R}}(S, \mu)$. We have

$$\begin{aligned} \int_S |x_n(s)| \mathrm{d}\mu(s) &= \int_S x_n(s) \mathrm{d}\mu(s) = \int_{\left[0,\frac{1}{n}\right]} x_n(s) \mathrm{d}\mu(s) + \int_{\left]\frac{1}{n},1\right]} x_n(s) \mathrm{d}\mu(s) \\ &= \int_{\left[0,\frac{1}{n}\right]} n \mathrm{d}\mu(s) + \int_{\left]\frac{1}{n},1\right]} 1 \mathrm{d}\mu(s) \\ &= 1 + (1 - \tfrac{1}{n}). \end{aligned} \tag{40}$$

Since $\operatorname{argmin} \phi = \{1\}$, $I_\phi(x) = \int_S \phi(1) \mathrm{d}\mu(s) \leq \int_S \phi\big(z(s)\big) \mathrm{d}\mu(s) = I_\phi(z), \forall z \in L^1_{\mathbb{R}}(S, \mu)$. Then $x \in \operatorname{argmin} I_\phi$. By Fact 2.15, I_ϕ has unique minimizer and hence $\operatorname{argmin} I_\phi = \{x\}$.

Now we show that $x_n \to x$ in measure. Let $\eta > 0$. Then we have

$$\mu\big\{s \in S \mid |x_n(s) - x(s)| \geq \eta\big\} = \mu\big\{s \in S \mid |x_n(s) - 1| \geq \eta\big\} \leq \mu\big\{ \left[0, \tfrac{1}{n}\right] \big\} = \tfrac{1}{n} \longrightarrow 0.$$

Hence $\lim \mu\big\{s \in S \mid |x_n(s) - x(s)| \geq \eta\big\} = 0$ and thus $x_n \to x$ in measure.

By (40), $\|x_n\|_1 \nrightarrow \|x\|_1 = 1$. Then $x_n \nrightarrow x$. Since $x_n \to x$ in measure, [26, Lemma 2] implies that $(x_n)_{n\in\mathbb{N}}$ does not converge weakly to x.

We claim that $I_\phi(x_n) \nrightarrow I_\phi(x)$. We have

$$I_\phi(x_n) = \int_S \phi\big(x_n(s)\big) \mathrm{d}\mu(s)$$

$$
\begin{aligned}
&= \int_{\left[0,\frac{1}{n}\right]} \phi\big(x_n(s)\big)\mathrm{d}\mu(s) + \int_{\left]\frac{1}{n},1\right]} \phi\big(x_n(s)\big)\mathrm{d}\mu(s)\\
&= \int_{\left[0,\frac{1}{n}\right]} -\log n + n\mathrm{d}\mu(s) + \int_{\left]\frac{1}{n},1\right]} -\log 1 + 1\mathrm{d}\mu(s)\\
&= -\tfrac{\log n}{n} + 1 + (1-\tfrac{1}{n}) \longrightarrow 2.
\end{aligned}
$$

However,

$$
I_\phi(x) = \int_S \phi\big(x(s)\big)\mathrm{d}\mu(s) = \int_S -\log 1 + 1\mathrm{d}\mu(s) = 1.
$$

Combining the results above, $I_\phi(x_n) \not\to I_\phi(x)$. ■

Remark 7.11 Let $(C_n)_{n\in\mathbb{N}}$ and C_∞ in $L^1_E(S,\mu)$ be closed convex sets, and let $\phi : E \to]-\infty,+\infty]$ be proper lower semicontinuous and convex. When, as in [6], we consider the following sequences of optimization problems

$$
\begin{aligned}
&(P_n) \qquad\qquad && V(P_n) := \inf\big\{I_\phi(x) \mid x \in C_n\big\},\\
&(P_\infty) \qquad\qquad && V(P_\infty) := \inf\big\{I_\phi(x) \mid x \in C_\infty\big\},
\end{aligned}
$$

the above results indicate that one cannot significantly weaken the conditions of Fact 2.19 (such as, replacing weak convergence by measure convergence).

To conclude, we observe that the examples of this section indicate the limited use of convergence in measure in the absence of weak compactness conditions.

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