Exploration and Discovery in Inverse Scattering

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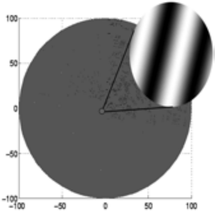
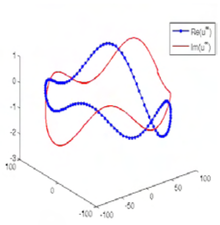
A Physical Experiment



Experimental domain: $\mathbb{D} \subset \mathbb{R}^2$
Illuminating source: $u^i(x; \widehat{\eta}, k) := e^{ik\widehat{\eta}\cdot x}, \quad x \in \mathbb{R}^2$,
a plane wave where $\widehat{\eta} \in \mathbb{S} := \left\{ d \in \mathbb{R}^2 \mid |d| = 1 \right\}$ is the incident direction and $k > 0$ is the **wavenumber.**

A Physical Experiment

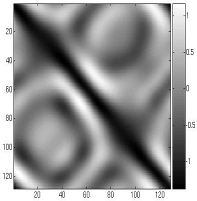


Measured data: far field pattern for the scattered field, denoted by $u^\infty(\cdot, \widehat{\eta}, k) : \partial\mathbb{D} \to \mathbb{C}$ at points $\widehat{x}$ uniformly distributed around $\partial\mathbb{D}$, the boundary of $\mathbb{D}$.

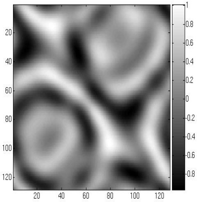
A Physical Experiment

Repeat at N incident directions $\widehat{\eta}_n$ equally distributed on the interval $[-\pi, \pi]$. For each incident direction $\widehat{\eta}_n$, collect N far field measurements at points $\widehat{x}_n \in \partial\mathbb{D}$.

(a)



(b)



Far field data, real (a) and imaginary (b) parts, from 128 experiments differing in the direction of the incident field. Each experiment is at the same incident wavenumber $k = 2$.

A Physical Experiment

Goal

Determine as much as possible about the scatterer(s) that produced this data, e.g. *where is it?*, *what kind of scatterer is it?* and *what is its shape?*

A Mathematical Experiment

Use the far field data, u^∞, as the kernel of an integral operator, the far field operator, $\mathcal{F}$, defined by

$$\mathcal{F}g(z,-\widehat{\eta}) := \int_{\mathbb{S}} u^\infty(\widehat{x},-\widehat{\eta})g(-\widehat{x};z)\; ds(\widehat{x}).$$

A Mathematical Experiment

For a given $0 < N_\epsilon < \overline{N}$ and $x \in \mathbb{R}^2$ fixed, we construct an incident field as a superposition of incident plane waves

$$v_g^i(x) = \sum_{n=N_\epsilon}^{\overline{N}} |v_n^i(x)|$$

where v_n^i is given by

$$v_n^i(x) = \int_{\mathbb{S}} \xi_n(\widehat{\eta}) u^i(x, \widehat{\eta}) \, ds(\widehat{\eta})$$

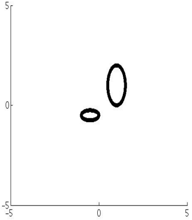
and ξ_n are singular functions of the *far field operator* $\mathcal{F}$ that is

$$\mathcal{F}\xi_n = \sigma_n \psi_n, \quad \text{and} \quad \mathcal{F}^* \psi_n = \sigma_n \xi_n$$

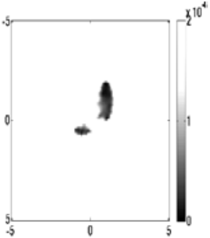
A Mathematical Experiment

The numerical far field patterns are sampled at 128 points for 128 incident plane wave directions evenly distributed on $\mathbb{S}$. We use 12 singular vectors corresponding to the 12 smallest singular values of $\mathcal{F}$ for our constructed incident field.

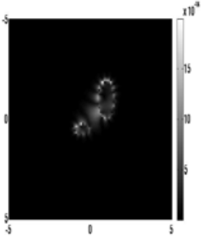
A Mathematical Experiment



Obstacle



$|v_g^i|$



$|v_g^s|$

Outline

Outline

Let $\mathbb{D} \subset \mathbb{R}^2$, be the support of one or more scattering obstacles, each with connected, piecewise C^2 boundaries $\partial\mathbb{D}_j$.

Model

Find $u : \mathbb{R}^2 \to \mathbb{C}$ that satisfies

$$(*) \qquad [\triangle + n(x)k^2]u(x) = 0, \quad x \in \mathbb{R}^2$$

where

$$n(x) := \frac{c_0^2}{c^2(x)} + i\sigma(x), \qquad \textbf{(index of refraction)}$$

with background sound speed $c_0 > 0$, scatterer soundspeed $c : \mathbb{R}^2 \to \mathbb{R}_+ \setminus \{0\}$, and absorption $\sigma : \mathbb{R}^2 \to \mathbb{R}_+$.

This models the propogation of waves in an **inhomogeneous medium.**

If the absorption or the scatterer soundspeed take on extreme values of ∞ or 0 respectively, we model the scatterers as impenetrable **obstacles** with one of the boundary conditions

$$\begin{cases} u(x) = 0, \quad x \in \partial\mathbb{D}_j & \text{(a)} \\ \frac{\partial u}{\partial n}(x) = 0, \quad x \in \partial\mathbb{D}_j & \text{(b)} \\ \frac{\partial u}{\partial n}(x) + \mathrm{i}k\lambda u(x) = 0, \quad x \in \partial\mathbb{D}_j & \text{(c)} \end{cases}$$

on $x \in \partial\mathbb{D} = \cap_{j=1}^{J} \partial\mathbb{D}_j$ with unit outward normal n.

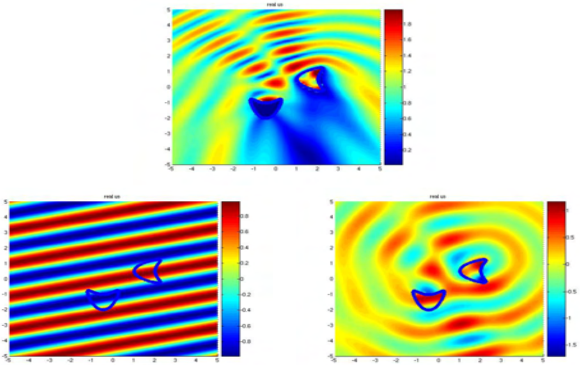
The boundary value problem with boundary conditions (a), (b) or (c) are limiting cases, we focus on the case of an **inhomogeneous medium.**

Write $u = u^i + u^s$ where

- the total field $u : \mathbb{R}^2 \to \mathbb{C}$
- the incident field $u^i : \mathbb{R}^2 \to \mathbb{C}$
- the scattered field $u^s : \mathbb{R}^2 \to \mathbb{C}$ satisfies the radiation condition

$$r^{\frac{1}{2}}\left(\frac{\partial}{\partial r} - ik\right)u^s(x) \to 0, \quad r = |x| \to \infty$$

Mathematical Model



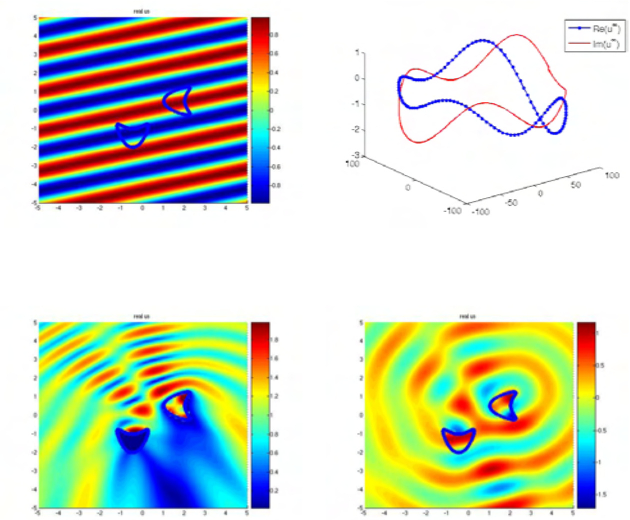
If $|x|$ is very large (the far field) then

$$u^s(x,\widehat{\eta}) = \beta \frac{e^{ik|x|}}{|x|^{1/2}} u^\infty(\widehat{x},\widehat{\eta}) + o\left(\frac{1}{|x|^{1/2}}\right), \qquad \widehat{x} = \frac{x}{|x|} \quad |x| \to \infty,$$

where $u^\infty : \mathbb{S} \to \mathbb{C}$ is the far field pattern,
$\mathbb{S} := \left\{ x \in \mathbb{R}^2 \mid |x| = 1 \right\}$ and $\widehat{x} := \frac{x}{|x|}$.

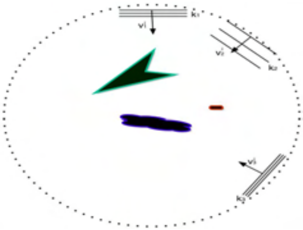
The parameter $\widehat{\eta}$ in the argument of the fields above keeps track of the direction of the incident field.

Mathematical Model



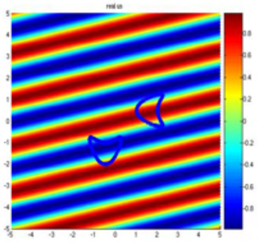
Problem Statement

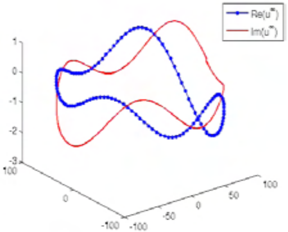
Given one or more triplets $(k, \widehat{\eta}, u^{\infty})$, determine $\partial\mathbb{D}$ and as much information about $n(x)$, the **index of refraction**, as possible.



Problem Statement

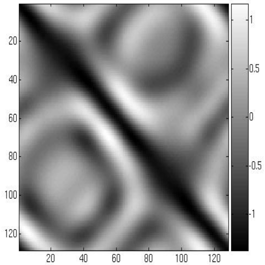
For a single incident plane wave, the data would consist of



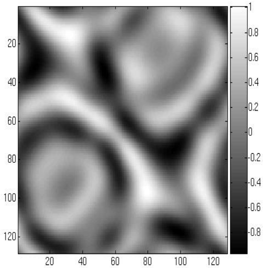


Problem Statement

For N incident plane waves, the data is a 2-D array of numbers u_{ij}^{∞} indexing the measurement point $\widehat{x}_i$ in the far field, and the incident field direction $\widehat{\eta}_j$.



(a)



(b)

Outline

Where Is the Scatterer? How Big is It?

Define the Herglotz Wave function

$$v_g^i(x,k) := \int_{\mathbb{S}} e^{-\mathrm{i}kx\cdot\widehat{y}} g(-\widehat{y})\, ds(\widehat{y})$$

(superpositon of plane waves). Denote the corresponding scattered and far fields by v_g^s and v_g^∞.

Where Is the Scatterer? How Big is It?

By linearity and boundedness of the scattering operator, we have

$$v_g^s(x,k)) := \int_{\mathbb{S}} u^s(x,-\widehat{y},k)g(-\widehat{y})\,ds(\widehat{y})$$

and

$$\begin{aligned} v_g^\infty(\widehat{x},k)) &:= \int_{\mathbb{S}} u^\infty(\widehat{x},-\widehat{y},k)g(-\widehat{y})\,ds(\widehat{y}) \\ &= \int_{\mathbb{S}} u^\infty(\widehat{y},-\widehat{x},k)g(-\widehat{y})\,ds(\widehat{y}) \qquad \text{(reciprocity)} \end{aligned}$$

Where Is the Scatterer? How Big is It?

Given the far field pattern $u^\infty(\widehat{\eta}, -\widehat{x})$ for $\widehat{\eta} \in \Lambda \subset \mathbb{S}$ due to an incident plane wave $u^i(\cdot, -\widehat{x})$ with fixed direction $-\widehat{x}$, let v_g^i denote the incident Herglotz wave field defined above and $v_g^\infty(\widehat{x})$ the corresponding far field pattern. The **scattering test response** for the test domain $\mathbb{D}_t$ is defined by

$$\mu(\mathbb{D}_t, u^\infty(\Lambda, -\widehat{x})) := \sup\left\{ \left|v_g^\infty(\widehat{x})\right| \;\middle|\; g \in L^2(-\Lambda) \text{ with } \|v_g^i\|_{\partial\mathbb{D}_t} = 1 \right\}.$$

Where Is the Scatterer? How Big is It?

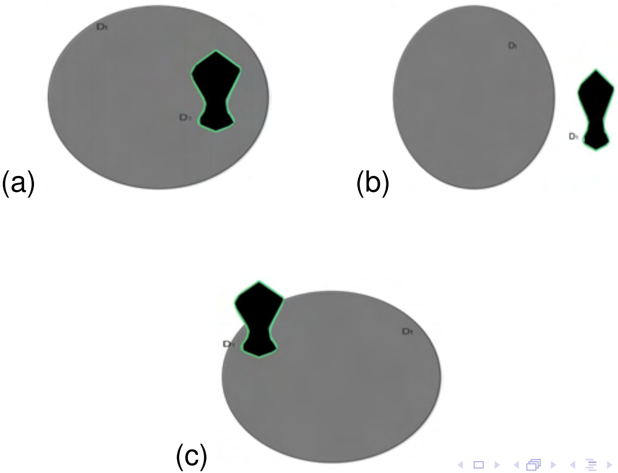
Theorem [L.&Potthast, 2003]

(a) If $\mathbb{D} \subset \mathbb{D}_t$, then $\mu(\mathbb{D}_t, u^\infty(\Lambda, -\widehat{x})) < \infty$.

(b) If, on the other hand, $\mathbb{D}_t \cap \mathbb{D} = \emptyset$, and $\mathbb{R}^n \setminus \overline{\mathbb{D}_t} \cup \overline{\mathbb{D}}$ is connected, then $\mu(\mathbb{D}_t, u^\infty(\Lambda, -\widehat{x})) = \infty$.

(c) If $\mathbb{R}^m \neq (\mathbb{D}_t \cap \mathbb{D})^c \neq \mathbb{D}^c$, and if the scattered field u cannot be analytically continued throughout $(\mathbb{D}_t \cap \mathbb{D})^c$, then $\mu(\mathbb{D}_t, u^\infty(\Lambda, -\widehat{x})) = \infty$ [Potthast, 2005] .

Where Is the Scatterer? How Big is It?



Where Is the Scatterer? How Big is It?

Let $\mathbb{D}_0$ denote a fixed, bounded smooth test domain. Denote translations of $\mathbb{D}_0$ by $\mathbb{D}_0(z) := \mathbb{D}_0 + z$ for $z \in \mathbb{R}^2$. Define the **corona** of the scatterer $\mathbb{D}$, relative to the scattering test response μ by

$$\mathbb{M}_\mu \quad := \bigcup_{\substack{z\in\mathbb{R}^2 \\ s.t.\ \mu\left(\mathbb{D}_t(z), u^\infty(\Lambda, -\widehat{x})\right) < \infty}} \mathbb{D}_t(z).$$

Approximate size and location of scatterers (L.& Potthast, 2003):

The scatterer $\mathbb{D}$ is a subset of its corona, $\mathbb{M}_\mu$

Where Is the Scatterer? How Big is It?

Advantages:

- Requires only one incident field

  and

- the boundary condition is irrelevant

Disadvantages:

- involves solving an infinite dimensional optimization problem at each translation point z

Where Is the Scatterer? How Big is It?

Let $\mathbb{D}_0(0)$ be a circle of radius r centered at the origin where r is large enough that $\mathbb{D} \subset \mathbb{D}_0(z)$ for some $z \in \mathbb{R}^2$. For each $\widehat{y} \in \mathbb{S}$ and all $x \in \partial\mathbb{D}_t(0)$, let g_0 solve

$$\| (\mathcal{H}g)(\cdot, r\widehat{y}) - \Phi(\cdot, r\widehat{y})\|_{L^2(\partial\mathbb{D})} < \epsilon$$

where

$$(\mathcal{H}g_0)(x, r\widehat{y}) := \int_{\mathbb{S}} e^{-\mathrm{i}kx\cdot\widehat{\eta}} g(-\widehat{\eta}, r\widehat{y}, 0)\; ds(\widehat{\eta}).$$

Define the *partial scattering test response* , $\delta : \mathbb{R}^2 \to \mathbb{R}_+$, by

$$\delta(z) := \int_{\mathbb{S}} \left| \int_{\Lambda} e^{-\mathrm{i}k\widehat{\eta}\cdot z} u^{\infty}(\widehat{\eta}, -\widehat{x}) g(-\widehat{\eta}, r\widehat{y}, 0)\; d\widehat{\eta} \right| ds(\widehat{y}), \qquad \Lambda \subset \mathbb{S}$$

Where Is the Scatterer? How Big is It?

Conjecture. For any $\widehat{x} \in \mathbb{S}$ there exist constants $0 < M' < M$ such that

$$\delta(z) \begin{cases} > M & \forall\, z \in \mathbb{R}^2 \quad \text{where} \quad \mathbb{D} \cap \mathbb{D}_t(0) + z = \emptyset \\ < M' & \forall\, z \in \mathbb{R}^2 \quad \text{where} \quad \mathbb{D} \subset \operatorname{int}\left(\mathbb{D}_t(0) + z\right). \end{cases}$$

Where Is the Scatterer? How Big is It?

Details. The equation

$$\left(\mathcal{H}g(\cdot, r\widehat{y}, 0)\right)(x, r\widehat{y}) = \Phi(x, r\widehat{y}), \qquad x \in \partial\mathbb{D}_t(0).$$

is ill-posed, albeit linear, with respect to g.
We regularize the problem by solving the regularized least squares problem

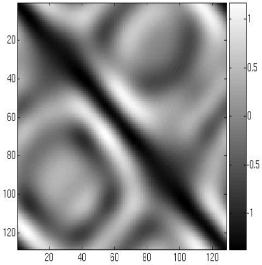
$$\underset{g \in L^2(\mathbb{S})}{\text{minimize}} \; \|\mathcal{H}g - \Phi^\infty(\cdot, r\widehat{y})\|^2 + \alpha\|g\|^2.$$

which yields

$$g(\cdot; r\widehat{y}, 0) \approx (\alpha I + \mathcal{H}^*\mathcal{H})^{-1}\mathcal{H}^*\Phi^\infty(\cdot, r\widehat{y}).$$

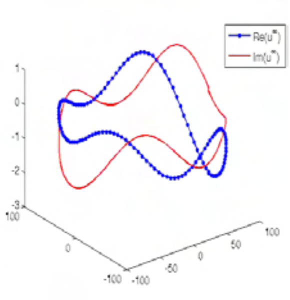
Where Is the Scatterer? How Big is It? I

For our data set, we take one column of



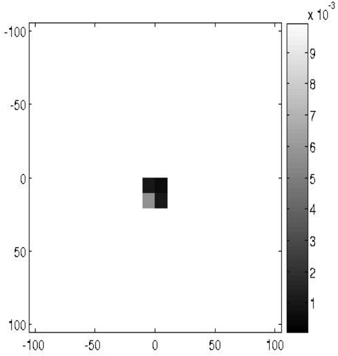
corresponding to

Where Is the Scatterer? How Big is It? II



and preform the above test to yield

Where Is the Scatterer? How Big is It? III



Is the Scatterer Absorbing? I

To answer this question, we study the spectral properties of the far field operator:

$$\mathcal{F}f(\widehat{x}) := \int_{\mathbb{S}} u^{\infty}(\widehat{x}, -\widehat{\eta}) f(-\widehat{\eta})\, ds(\widehat{\eta}).$$

Fact:
There exists some $g_z \in L^2(\mathbb{S})$ such that $u^s(z, \widehat{\eta}) = (\mathcal{F}g_z)(\widehat{\eta})$, i.e. the scattered field lies in the range of the far field operator.

$\Longrightarrow$ the spectrum of the far field operator should say **something** about the types of scattered fields that can be generated, and hence something about the nature of the scatterer(s).

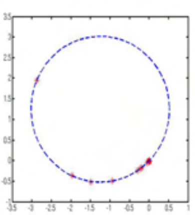
Is the Scatterer Absorbing?

Theorem (Colton&Kress))

Let the scattering inhomogeneity have index of refraction n mapping $\mathbb{R}^2$ to the upper half of the complex plane. The scattering inhomogeneity is nonabsorbing, that is, $\mathrm{Im}\, n(x) = 0$ for all x, if and only if the eigenvalues of $\mathcal{F}$ lie on the circle centered at $\frac{1}{2k}\left(\mathrm{Im}(\beta^{-1}), \mathrm{Re}(\beta^{-1})\right)$ and passing through the origin. Otherwise, the eigenvalues of $\mathcal{F}$ lie on the interior of this disk.

Requires all incident directions on $\mathbb{S}$ and all far field measurements on $\mathbb{S}$.

Is the Scatterer Absorbing?



The eigenvalues (asterisks) of the far field matrix are shown to line up on the circle passing through the origin with center $1/2k(\mathrm{Im}\beta, \mathrm{Re}\beta)$ for $k = 2$ and β a known constant. This implies that the inhomogeneity is nonabsorbing.

What Is the Shape of the Scatterer?

We will construct an indicator function to determine the feasibility of an auxilliary problem that allows us to tell whether a point is inside or outside the scatterer.

What Is the Shape of the Scatterer?

$$(*) \quad \triangle w(x)+k^2 n(x) w(x) = 0, \quad \triangle v(x)+k^2 v(x) = 0 \text{ for } x \in \operatorname{int}(\mathbb{D})$$

$$(**) \quad w - v = f(\cdot, z), \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial f}{\partial \nu} \quad \text{on } \partial\mathbb{D}.$$

(1a) Uniqueness.

If the medium is absorbing, that is $\mathrm{Im}(n(x)) > 0$, then there are no nontrivial solutions to the homogeneous problem (*)-(**) with $f = 0$, hence the inhomogeneous problem will have a unique solution when a solution exists.

What Is the Shape of the Scatterer?

From the previous experiment, however, our medium is nonabsorbing, i.e. $\mathrm{Im}(n(x)) = 0$ for all x, so there is still the threat of nonuniqueness.

(1b) Uniqueness, (Colton and Päivärinta (2000)).

The set of values of k for which the solution to (*)-(**) with $f = 0$ has a nontrivial solution – called **transmission eigenvalues** – is a discrete set. Hence, for almost all k (*)-(**) has a unique solution, if it exists.

What Is the Shape of the Scatterer?

(2) Existence (Kirsch).

Let

$$f(y) = h_p^{(1)}(k|y|)\,Y_p(\widehat{y}), \tag{1}$$

a spherical wave function of order p. Then...

The integral equation

$$\int_{\mathbb{S}} u^{\infty}(\widehat{x};\widehat{y})g(-\widehat{x})\, ds(\widehat{x}) = \frac{\mathrm{i}^{p-1}}{\beta k} Y_p(\widehat{y}), \qquad \widehat{y} \in \mathbb{S} \qquad (2)$$

has a solution $g \in L^2(\mathbb{S})$ if and only if there exists $w \in C^2(\mathrm{int}\,(\mathbb{D})) \cap C^1(\mathbb{D})$ and a function v given by

$$v(x) = \int_{\mathbb{S}} e^{\mathrm{i}kx\cdot(-\widehat{y})} g(-\widehat{y})\, ds(\widehat{y})$$

such that the pair (w, v) is a solution to (*)-(**)

What Is the Shape of the Scatterer?

(3) Theorem (Colton&Kress).

There exists a unique weak solution to (*)-(**) with $f(x;z) := \Phi(x,z)$ for every $z \in \mathrm{int}(\mathbb{D})$ with Φ the free-space fundamental solution, that is, the pair (w, v) satisfies

$$w + k^2 \int_{\mathbb{R}^m} \Phi(x,y)(1-n(y))w(y)\, dy = v \qquad \text{on } \mathrm{int}(\mathbb{D}) \quad (3)$$

and

$$-k^2 \int_{\mathbb{R}^m} \Phi(x,y)(1-n(y))w(y)\, dy = \Phi(x,z) \qquad \text{for } x \in \partial\mathbb{B} \quad (4)$$

where $\mathbb{B} \subset \mathbb{R}^2$ is a ball with $\mathrm{int}(\mathbb{D}) \subset \mathbb{B}$.

What Is the Shape of the Scatterer?

Recap.

Equation (2) has a solution if and only if there is a corresponding solution to (*)-(**) with f given by (1); moreover, (*)-(**) with $f = \Phi(x, z)$ is solvable for every $z \in \operatorname{int}(\mathbb{D})$.

Question:

For $z \in \mathbb{R}^2 \setminus \mathbb{D}$, or, just as $z \to \partial\mathbb{D}$ from $\operatorname{int}(\mathbb{D})$, what happens to solutions to

$$\int_{\mathbb{S}} u^\infty(\widehat{x}; \widehat{y}) g(-\widehat{x})\, ds(\widehat{x}) = \Phi^\infty(\widehat{y}, z), \qquad \widehat{y} \in \mathbb{S} \tag{5}$$

where $\Phi^\infty(\widehat{y}, z)$ is the far field pattern of the fundamental solution $\Phi(y, z)$?

What Is the Shape of the Scatterer?

Linear Sampling. For every $\epsilon > 0$ and $z \in \mathbb{D}$ there exists a $g(\cdot; z) \in L^2(\mathbb{S})$ satisfying

$$\|(\mathcal{F}g)(\cdot, z) - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S})} \leq \epsilon \tag{6}$$

such that

$$\lim_{z \to \partial\mathbb{D}} \|g(\cdot, z)\|_{L^2(\mathbb{S})} = \infty. \tag{7}$$

What Is the Shape of the Scatterer?

Details. The equation

$$(\mathcal{F}g)(\cdot) := \int_{\mathbb{S}} u^\infty(\cdot, -\widehat{\eta})g(-\widehat{\eta})\, ds(\widehat{\eta}) = \Phi^\infty(\cdot, z)$$

is ill-posed, albeit linear, with respect to g.
We regularize the problem by solving the regularized least squares problem

$$\underset{g \in L^2(\mathbb{S})}{\text{minimize}} \ \|\mathcal{F}g - \Phi^\infty(\cdot, z)\|^2 + \alpha\|g\|^2.$$

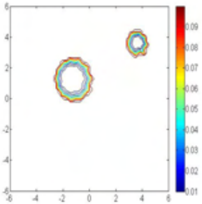
which yields

$$g(\cdot; z, \alpha) := (\alpha I + \mathcal{F}^*\mathcal{F})^{-1}\mathcal{F}^*\Phi^\infty(\cdot, z).$$

What Is the Shape of the Scatterer?

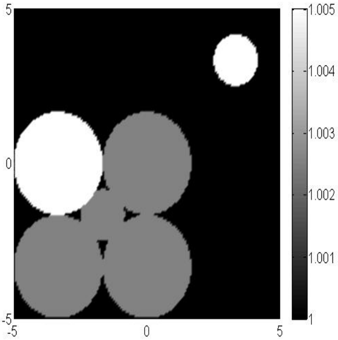
Since we already know from the scattering test response approximately where and how big the scatterer is, we needn't calculate $g(\cdot; z, \alpha)$ at all points $z \in \mathbb{D}$, but rather just on the corona $\mathbb{M}_\mu$, or, if we are confident of the earlier Conjecture, then on the corona of the partial scattering response $\mathbb{M}_\delta$ for a given δ. We identify the boundary of the scatterer by those points z_j on a grid where the norm of the density $g(\cdot; z_j, \alpha)$ becomes large relative to the norm of the density at neighboring points.

What Is the Shape of the Scatterer?



Shown is $\|g(\cdot; z_j, \alpha)\|_{L^2(\mathbb{S})}$ for $g(\cdot; z_j, \alpha)$ the regularized density with $\alpha = 10^{-8}$ for all grid points z_j on the domain $[-6, 6] \times [-6, 6]$ sampled at a rate of 40 points in each direction. The cutoff is 2.

What Is the Answer?

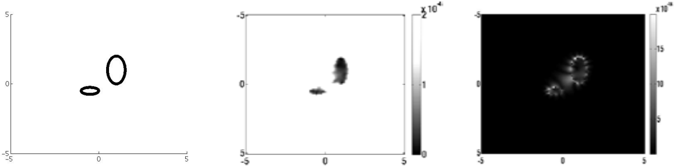


The true scatterer consisting of 6 circles of different sizes and indices of refraction indicated by the color.

Outline

Recall: Mathematical Experiment

The numerical far field patterns are sampled at 128 points for 128 incident plane wave directions evenly distributed on $\mathbb{S}$. We use 12 singular vectors corresponding to the 12 smallest singular values of $\mathcal{F}$ for our constructed incident field.



Obstacle $|v_g^i|$ $|v_g^s|$

Recall

Linear Sampling. For every $\epsilon > 0$ and $z \in \mathbb{D}$ there exists a $g(\cdot; z) \in L^2(\mathbb{S})$ satisfying

$$\|(\mathcal{F}g)(\cdot, z) - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S})} \leq \epsilon$$

such that

$$\lim_{z \to \partial\mathbb{D}} \|g(\cdot, z)\|_{L^2(\mathbb{S})} = \infty.$$

Normalized Linear Sampling

Theorem (L. Devaney). Let $\mathbb{D}$ be a domain with smooth boundary and assume that k^2 is not a transmission eigenvalue for $-\triangle$ on $\mathbb{D}$. If $z \in \mathbb{D}$ then for every $\epsilon > 0$ there exists a solution g_z to

$$\|\mathcal{F}g_z(\cdot) + \Phi^\infty(\cdot; z)\|_{L^2(\mathbb{S})} < \epsilon$$

such that

$$\lim_{z\to\partial\mathbb{D}} \|\mathcal{F}\widehat{g}_z\|_{L^2(\mathbb{S})} = 0 \quad \text{and} \quad \lim_{z\to\partial\mathbb{D}} \left\| \mathcal{H}\widehat{g}_z - \frac{f_z}{\|g_z\|_{L^2(\mathbb{S})}} \right\|_{H^{1/2}(\partial\mathbb{D})} = 0.$$

where

$$\widehat{g}_z := \frac{g_z}{\|g_z\|_{L^2(\mathbb{S})}} \quad \text{and} \quad f_z \quad \text{solves} \quad \mathcal{B}f_z(\cdot) = -\Phi^\infty(\cdot; z).$$

Nonscattering fields

Corollary. Let $\widehat{g}_z$ be the density in the Normalized Linear Sampling Theorem. Then the scattered field, $v^s_{\widehat{g}_z}$, corresponding to the incident Herglotz wave function $v^i_{\widehat{g}_z} = \mathcal{H}\widehat{g}_z$ has the behavior

$$\lim_{z \xrightarrow{\mathbb{D}} \partial\mathbb{D}} v^s_{\widehat{g}_z}(x) = 0 \quad \text{for} \quad x \in \mathbb{D}^o \quad \text{while} \quad \lim_{x \xrightarrow{\mathbb{D}^o} \partial\mathbb{D}} \lim_{z \xrightarrow{\mathbb{D}} \partial\mathbb{D}} v^i_{\widehat{g}_z}(x) = 0.$$

MUSIC

Denote the singular system of $\mathcal{F}$ by $(\sigma_n, \xi_n, \psi_n)$ where

$$\mathcal{F}\xi_n = \sigma_n \psi_n, \quad \text{and} \quad \mathcal{F}^*\psi_n = \sigma_n \xi_n$$

with singular values $|\sigma_n| > |\sigma_m|$ for $m > n$, left and right singular functions ψ_n and ξ_n respectively.

MUSIC

Theorem. Let $\mathbb{D}$ be a domain with smooth boundary and assume that k^2 is not a Dirichlet eigenvalue for the negative Laplacian on $\mathbb{D}$. Let $(\sigma_n, \xi_n, \psi_n)$, $n \in \mathbb{N}$, be the singular system for the far field operator $\mathcal{F}$ with $|\sigma_n| \leq |\sigma_m|$ for $n > m$. Given any $\gamma > 0$ there is a vector $a \in l^2$ with $\|a\|_2 = 1$ and $\rho > 0$ such that for any $x \in \mathbb{D}^o$ satisfying dist $(x, \mathbb{D}) < \rho$ we have

$$\sum_{n=1}^{\infty} \left| a_n \langle \xi_n, \Phi^\infty(\cdot; x) \rangle_{L^2(\mathbb{S})} \right| < \gamma.$$

MUSIC

Remark. The Normalized Linear Sampling Theorem does not tell us how to calculate the desired density $\widehat{g}$ but the MUSIC observation suggests the following algorithm:

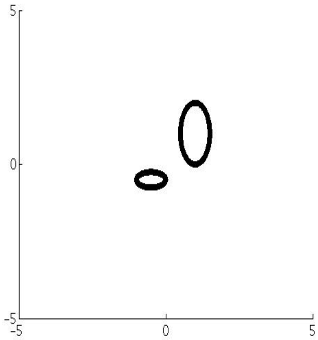
MUSIC

Algorithm.

- Denote the noise subspace of $\mathcal{F}$ by $\mathcal{N}_\epsilon$ corresponding to the span of the singular functions ψ_n with singular values $|\sigma_n| \leq \epsilon$ for $n > N_\epsilon$.
- Construct a $\widehat{g} \in \mathcal{N}_\epsilon$ for ϵ sufficiently small, that is let $\widehat{g}$ be a linear combination of the elements $\xi_n \in \mathcal{N}_\epsilon$ for a large enough cutoff.
- At each sample point $z \in \mathbb{G}$, on some computational grid $\mathbb{G}$, plot

$$\sum_{n=N_\epsilon}^{\overline{N}} \left| a_n \langle \xi_n(\cdot), \Phi^\infty(z,\cdot) \rangle_{L^2(\mathbb{S})} \right|$$

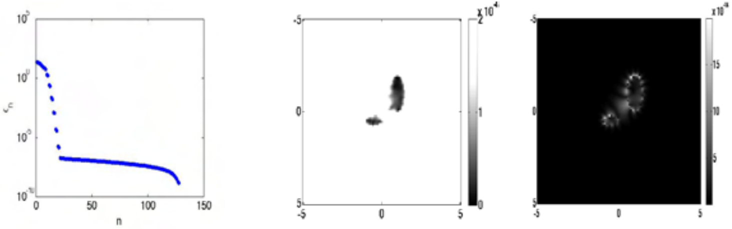
Numerical Example



Sound-soft obstacles to be recovered.

Numerical Example

128 incident fields, 128 far field samples, 12 singular functions



Outline

Final observations

- We can use this technique to **image** scatterers (essentially the same as linear sampling)
- We can construct an incident field from the singular vectors of the far field operator that has **very low scattering** for the obstacle, i.e. we can generate fields for which fixed scatterers are **invisible**.
- How can we modify the scatterer so that for a fixed class of irradiating incident fields, the scattered field is small? In other words, can we use this technique to **protect** certain objects from radiation while targeting others?
- What constitutes an image?
- Object discrimination?

References I

D. R. Luke and A. J. Devaney,
Identifying scattering obstacles by the construction of nonscattering waves
SIAM J. Appl. Math. (to appear)

S. Hou, K. Solna, and H. Zhao.
A direct imaging algorithm for extended targets.
Inverse Problems, 22(4):1151–1178, 2006.

H. Ammari, E. Iakovleva, and D. Lesselier.
Two numerical methods for recovering small inclusions from the scattering amplitude at a fixed frequency.
SIAM J. Sci. Comp., 27(1):130–158, 2005.

References II

A. J. Devaney.
Time reversal imaging of obscured targets from multistatic data.
IEEE Trans. on Anten. and Prop., 53(5):1600–1610, 2005.

C. Hazard and K. Ramdani.
Selective acoustic focusing using time-harmonic reversal mirrors.
SIAM J. App. Math., 64(3):1057–1076, 2004.

M. Cheney.
The linear sampling method and the MUSIC algorithm.
Inverse Problems, 17:591–595, 2001.

References III

A. Kirsch.
The MUSIC algorithm and the factorization method in inverse scattering theory for inhomogeneous media.
Inverse Problems, 18:1025–1040, 2001.

T. Arens.
Why linear sampling works.
Inverse Problems, 20:163–173, 2004.

F. Cakoni and D. Colton.
Qualitative Methods in Inverse Scattering Theory.
Springer Verlag, Berlin, 2006.

References IV

A. Kirsch.
Characterization of the shape of a scattering obstacle using the spectral data of the far field operator.
Inverse Problems, 14:1489–1512, 1998.