Ramanujan's Arithmetic-Geometric Mean

Continued Fractions and Dynamics

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About Verbal Presentations

"I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate. ⋯

The spoken word and the written word are quite different arts. ⋯

I feel that to collect an audience and then read one's material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car."[*]

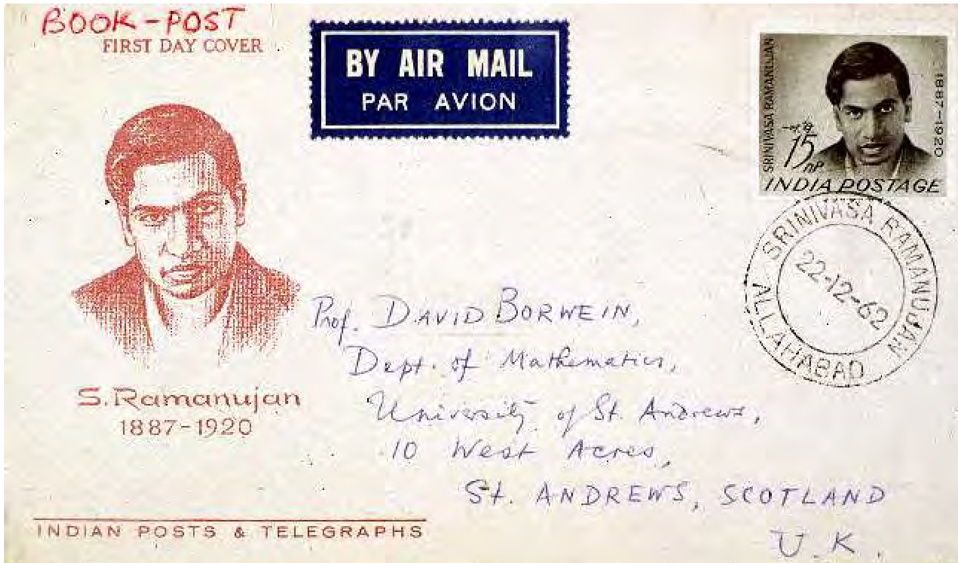
(Sir Lawrence Bragg)

- What would he say about reading overheads?

- *These overheads and a companion set by Russell Luke are on my web page.*

[*]From page 76 of *Science*, July 5, 1996.

Srinivasa Ramanujan (1887–1920)



- G. N. Watson (1886–1965), on reading Ramanujan's work, describes:

  a thrill which is indistinguishable from the thrill I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing 'Day,' 'Night,' 'Evening,' and 'Dawn' which Michelangelo has set over the tomb of Guiliano de'Medici and Lorenzo de'Medici.

1. Abstract

The *Ramanujan AGM continued fraction*

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}$$

enjoys attractive algebraic properties such as a striking **arithmetic-geometric mean** relation & elegant links with **elliptic-function theory**.

- The fraction **presents a computational challenge**, which we could not resist.

♣ Much of this work is to appear in *Experimental Mathematics* [CoLab Preprints #27, #29] and *The Ramanujan Journal* [#253.]

In **Part I**: we show how to rapidly evaluate $\mathcal{R}$ for any positive reals a, b, η. The problematic case being $a \approx b$—then subtle transformations allow rapid evaluation.

- On route we find, e.g., that for rational $a = b$, $\mathcal{R}_\eta$ is an L-series with a 'closed-form.'

- We ultimately exhibit an **algorithm yielding D digits of $\mathcal{R}$ in $O(D)$ iterations**.*

In **Part II** of this talk, we address the harder theoretical and computational dilemmas arising when **(i)** parameters are allowed to be complex, or **(ii)** more general fractions are used.

*The big-O constant is independent of the positive-real triple a, b, η.

2. Preliminaries

PART I. Entry 12 of Chapter 18 of *Ramanujan's Second Notebook* [BeIII] gives the beautiful:

$$\mathcal{R}_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}} \tag{1.1}$$

which we interpret—in most of the present treatment—for **real, positive** $a, b, \eta > 0$.

Remarkably, for such parameters, $\mathcal{R}$ satisfies an AGM relation

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2} \tag{1.2}$$

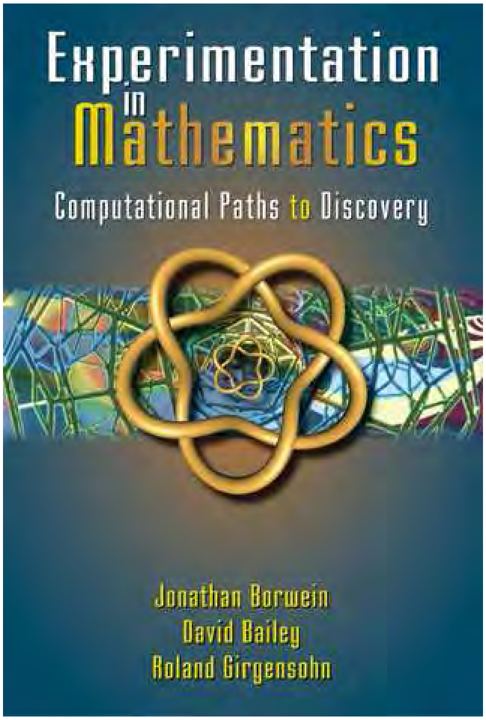
1. (1.2) is one of many relations we develop for computation of $\mathcal{R}_\eta$.

2. "**The hard cases occur when b is near to a,**" including the case $a = b$.

3. We eventually exhibit an algorithm uniformly of geometric/linear convergence across the positive quadrant $a, b > 0$.

4. Along the way, we find attractive identities, such as that for $\mathcal{R}_\eta(r, r)$, with r rational.

5. Finally, we consider complex a, b—obtaining theorems and conjectures on the domain of validity for the AGM relation (1.2).



Research started in earnest when we noted $\mathcal{R}_1(1,1)$ **'seemed close to'** $\log 2$.

Such is the value of experiment: one can be led into deep waters.

Discussed in Ch. 1 of *Experimentation in Mathematics*.

- A useful simplification is

$$\mathcal{R}_\eta(a,b) = \mathcal{R}_1(a/\eta, b/\eta),$$

as can be seen by 'cancellation' of the η elements down the fraction.

Such manipulations are valid because the continued converges.

To prove convergence we put $a/\mathcal{R}_1$ in **RCF** (**reduced continued fraction**) form:

$$\mathcal{R}_1(a,b) = \frac{a}{[A_0; A_1, A_2, A_3, \ldots]} \tag{2.1}$$

$$:= \cfrac{a}{A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \ddots}}}}$$

where the A_i are all positive real.

It is here [Ramanujan's work on elliptic and modular functions] that both the profundity and limitations of Ramanujan's knowledge stand out most sharply.
(G.H. Hardy)

Inspection of $\mathcal{R}$ yields the RCF elements explicitly and gives the asymptotics of A_n: For even n

$$A_n = \frac{n!^2}{(n/2)!^4}\, 4^{-n}\, \frac{b^n}{a^n} \;\sim\; \frac{2}{\pi\,\mathbf{n}}\, \frac{b^n}{a^n}.$$

For odd n

$$A_n = \frac{((n-1)/2!)^4}{n!^2}\, 4^{n-1}\, \frac{a^{n-1}}{b^{n+1}} \;\sim\; \frac{\pi}{2\,ab\,\mathbf{n}}\, \frac{a^n}{b^n}.$$

- This representation leads immediately to:

Theorem 2.1: For any positive real pair a, b the fraction $\mathcal{R}_1(a, b)$ converges.

Proof: An RCF converges **iff** $\sum A_i$ diverges. (This is the *Seidel–Stern theorem* [Kh,LW].)

In our case, such divergence is evident for every choice of real $a, b > 0$. ©

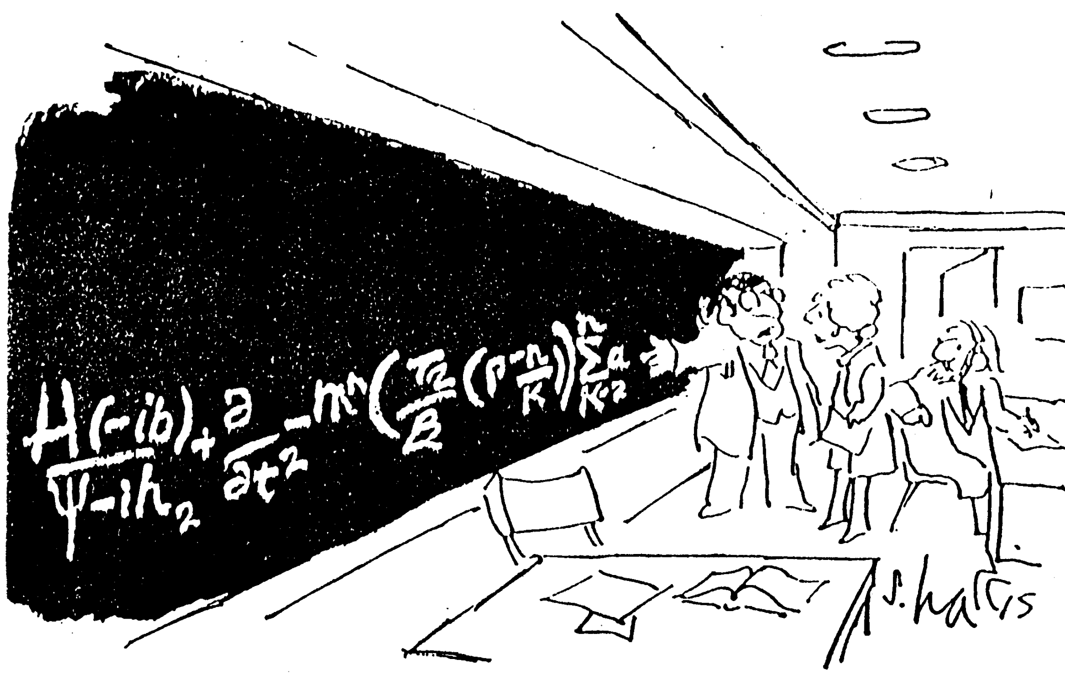
We later show a different fraction for $\mathcal{R}(a)$, and other computationally efficient constructs.

- Note for $a = b$, divergence of $\sum A_i$ is only *logarithmic* —a true indication of slow convergence (we wax more quantitatively later).

- **Our interest started** with asking how, for $a > 0$, to (rapidly) evaluate

$$\mathcal{R}(a) := \mathcal{R}_1(a, a)$$

and thence to prove suspected identities.



"But this *is* the simplified version for the general public."

3. Hyperbolic-elliptic Forms

Links between standard **Jacobi theta functions**

$$\theta_2(q) := \sum q^{(n+1/2)^2}, \qquad \theta_3(q) := \sum q^{n^2}$$

and **elliptic integrals** let us establish various results. We start with:

Theorem 3.1: For real $y, \eta > 0$ and $q := e^{-\pi y}$

$$\eta \sum_{k \in D} \frac{\operatorname{sech}(k\pi y/2)}{\eta^2 + k^2} = \mathcal{R}_\eta(\theta_2^2(q), \theta_3^2(q)),$$

$$\eta \sum_{k \in E} \frac{\operatorname{sech}(k\pi y/2)}{\eta^2 + k^2} = \mathcal{R}_\eta(\theta_3^2(q), \theta_2^2(q)),$$

where D, E denote respectively the odd, even integers.

Consequently, **the Ramanujan AGM identity (1.2) holds for positive triples** η, a, b.

Proof: The sech relations are proved—in equivalent form—in Berndt's treatment (Vol II, Ch. 18) of **Ramanujan's Notebooks** [BeIII].

For the AGM, assume $0 < b < a$. The assignments

$$\theta_2^2(q)/\theta_3^2(q) := b/a$$

$$\eta := \theta_2^2(q)/b$$

are possible (since $b/a \in [0,1)$, see [BB]) and implicitly define q, η, and together with

$$\theta_2^2(q) + \theta_3^2(q) = \theta_3^2(\sqrt{q}),$$

$$2\theta_2(q)\theta_3(q) = \theta_2^2(\sqrt{q})$$

and repeated use of the sech sums above yield

$$\mathcal{R}_1\left(\theta_3^2(q)/\eta, \theta_2^2(q)/\eta\right) + \mathcal{R}_1\left(\theta_2^2(q)/\eta, \theta_3^2(q)/\eta\right)$$

$$= 2\mathcal{R}_1\left(\theta_3^2(\sqrt{q})/(2\eta), \theta_2^2(\sqrt{q})/(2\eta)\right).$$

Since

$$\theta_2^2(q) = \eta\, b, \qquad \theta_3^2(q) = \eta\, a$$

the AGM identity (1.2) holds for all pairs with $a > b > 0$.

The case $0 < a < b$ is handled by symmetry, or on starting by setting

$$\theta_2^2(q)/\theta_3^2(q) := a/b.$$

©

- The wonderful sech identities above stem from classical work of Rogers, Stieltjes, Preece, and of course Ramanujan [BeIII] in which one finds the earlier work detailed.

- In the literature, the proof given for the AGM identity has been claimed for various complex a, b sometimes over ambitiously.*

Mea culpa indirectly.

- These prior sech series can be used in turn to establish two numerical series involving the **complete elliptic integral**

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2(\theta)}}\, d\theta.$$

Below we write $\mathsf{K} := \mathsf{K}(k)$, $\mathsf{K}' := \mathsf{K}(k')$ with $k' := \sqrt{1 - k^2}$.*

Theorem 3.2: For $0 < b < a$ and $k := b/a$ we have

$$\mathcal{R}_1(a, b) = \frac{\pi a\, \mathsf{K}}{2} \sum_{n \in Z} \frac{\operatorname{sech}\left(n\pi \frac{\mathsf{K}'}{\mathsf{K}}\right)}{\mathsf{K}^2 + \pi^2 a^2 n^2}. \qquad (3.1)$$

Correspondingly, for $0 < a < b$ and $k := a/b$ we have

$$\mathcal{R}_1(a, b) = 2\pi b\, \mathsf{K} \sum_{n \in D} \frac{\operatorname{sech}\left(n\pi \frac{\mathsf{K}'}{2\mathsf{K}}\right)}{4\mathsf{K}^2 + \pi^2 b^2 n^2}. \qquad (3.2)$$

*$K(k)$ is fast computable via the AGM iteration.

Proof: The series follow from the assignments $\theta_3^2(q)/\eta := \max(a,b)$, $\theta_2^2(q)/\eta := \min(a,b)$ and Jacobi's *nome* relations

$$e^{-\pi K'/K} = q, \qquad K(k) = \frac{\pi}{2}\theta_3^2(q)$$

inserted appropriately into Theorem 3.1. ©

- The sech-elliptic series (3.1-2) allow fast computation of $\mathcal{R}_1$ for b **not too near** a.

- To get D digits for $\mathcal{R}_1(a,b)$ one requires $O(D\,\mathsf{K}/\mathsf{K}')$ summands.

- So, another motive for the following analysis was **slow convergence of the sech-elliptic forms** for $b \approx a$.

- We also have attractive evaluations such as

$$\mathcal{R}_1\left(1, \frac{1}{\sqrt{2}}\right) = \frac{\pi}{2} \mathrm{K}\left(\frac{1}{\sqrt{2}}\right) \sum_{n \in Z} \frac{\operatorname{sech}(n\pi)}{\mathrm{K}^2(1/\sqrt{2}) + n^2\pi^2},$$

where

$$\mathrm{K}\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}},$$

see [BB], with similar series for $\mathcal{R}_1(1, k_N)$ at the N*-th singular value*, [BBa2].

- A similar relation for $\mathcal{R}_1\left(1/\sqrt{2}, 1\right)$ obtains via (3.2), and via the AGM relation (1.2) yields the oddity

$$\mathcal{R}_1\left(\frac{1+\sqrt{2}}{2\sqrt{2}}, \frac{1}{2^{1/4}}\right) = \pi \mathrm{K}\left(\frac{1}{\sqrt{2}}\right) \sum_{n \in Z} \frac{\operatorname{sech}(n\pi/2)}{4\mathrm{K}^2(1/\sqrt{2}) + n^2\pi^2}.$$

4. Relations for $\mathcal{R}(a)$

- Recalling that $\mathcal{R}(a) := \mathcal{R}_1(a, a)$, we next derive relations for the hard case $b = a$.

Interpreting (3.1) as a Riemann-integral in the limit as $b \to a^-$ (for $a > 0$), gives a slew of relations involving the **digamma function** [St,AS]

$$\psi := \frac{\Gamma'}{\Gamma}$$

and the **Gaussian hypergeometric function**

$$F = {}_2F_1(a, b; c; \cdot).$$

- The following identities are presented in an order that can be serially derived:

For all $a > 0$:

$$\begin{aligned}
\mathcal{R}(a) &= \int_0^\infty \frac{\operatorname{sech}\left(\frac{\pi\, x}{2\, a}\right)}{1+x^2}\, dx \\
&= 2\, a \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{1+(2\, k-1)\, a} \\
&= \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{4a}\right)-\psi\left(\frac{1}{4}+\frac{1}{4a}\right)\right) \\
&= \frac{2a}{1+a}\, F\left(\frac{1}{2a}+\frac{1}{2}, 1; \frac{1}{2a}+\frac{3}{2}; -1\right) \\
&= 2\int_0^1 \frac{t^{1/a}}{1+t^2}\, dt
\end{aligned}$$

and

$$\mathcal{R}(a) = \int_0^\infty e^{-x/a} \operatorname{sech}(x)\, dx.$$

The first series or t-integral yield a recurrence

$$\mathcal{R}(a) = \frac{2a}{1+a} - R\left(\frac{a}{1+2a}\right),$$

while known relations for the digamma [AS,St]—with some symbolic care—lead to

$$\mathcal{R}(a) = C(a) + \frac{\pi}{2}\sec\left(\frac{\pi}{2a}\right) - \tag{4.1}$$

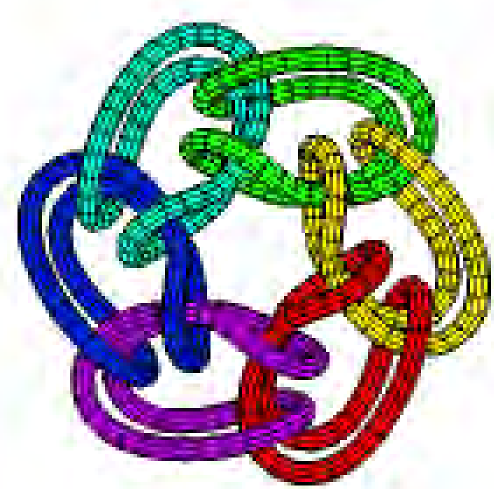
$$\frac{2a^2(1+8a-106a^2+280a^3+9a^4)}{1-12a+25a^2+120a^3-341a^4-108a^5+315a^6}$$

where

$$C(a) = \frac{1}{2}\sum_{n\geq 1}\{\zeta(2n+1)-1\}\,\frac{(3a-1)^{2n}-(a-1)^{2n}}{(4a)^{2n}}$$

is a "rational-zeta" series [BBC].

- Note that (4.1), while rapidly convergent for some a, has sec poles, some being cancelled by the rational function.
  - We require $a > 1/9$ for convergence of the rational-zeta sum.
  - However, the recurrence relation above can be used to force convergence of such a rational-zeta series.



- The hypergeometric form for $\mathcal{R}(a)$ is of special interest because [BBa1] of:

The Gauss continued fraction

$$F(\gamma, 1; 1+\gamma; -1) = [\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots] \quad (4.2)$$

$$= \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\alpha_4 + \ddots}}}}$$

where $\alpha_1 = 1$ and

$$\alpha_n = \gamma\left(((n-1)/2)!\right)^{-2}(n-1+\gamma)\prod_{j=1}^{(n-3)/2}(j+\gamma)^2$$

$$n = 3, 5, 7, \ldots$$

$$\alpha_n = \gamma^{-1}(n/2-1)!^2(n-1+\gamma)\prod_{j=1}^{n/2-1}(j+\gamma)^{-2}$$

$$n = 2, 4, 6, \ldots$$

- An interesting aspect of formal analysis is based upon the first sech-integral for $\mathcal{R}(a)$.

Expanding formally, and using a representation of the **Euler number**

$$E_{2n} = (-1)^n \int_0^\infty \mathrm{sech}(\pi x/2) x^{2n} \, dx$$

one obtains

$$\mathcal{R}(a) \sim \sum_{n \geq 0} E_{2n} \, a^{2n+1},$$

yielding an **asymptotic series of** *zero* **radius** of convergence. Moreover, for the asymptotic error, we have [BBa2,BCP]:

$$\left| \mathcal{R}(a) - \sum_{n=1}^{N-1} E_{2n} \, a^{2n+1} \right| \leq |E_{2N}| \; a^{2N+1},$$

- It is a classic theorem of Borel [St,BBa2] that for every real sequence (a_n) there is a C^∞ function f on R with $f^{(n)}(0) = a_n$.

- The oft-stated success of **Padé approximation** is well exemplified in our case.

Indeed, if one takes the unique $(3,3)$ Padé form* we obtain

$$\mathcal{R}(a) \approx a \frac{1 + 90\,a^2 + 1433\,a^4 + 2304\,a^6}{1 + 91\,a^2 + 1519\,a^4 + 3429\,a^6}.$$

- This simple approximant is remarkably good for small a; e.g., yielding

$$\mathcal{R}(1/10) \approx 0.09904494$$

correct to the implied precision.

For $\mathcal{R}(1/2)$ and the $(30,30)$ Padé approximant† one obtains 4 good digits.

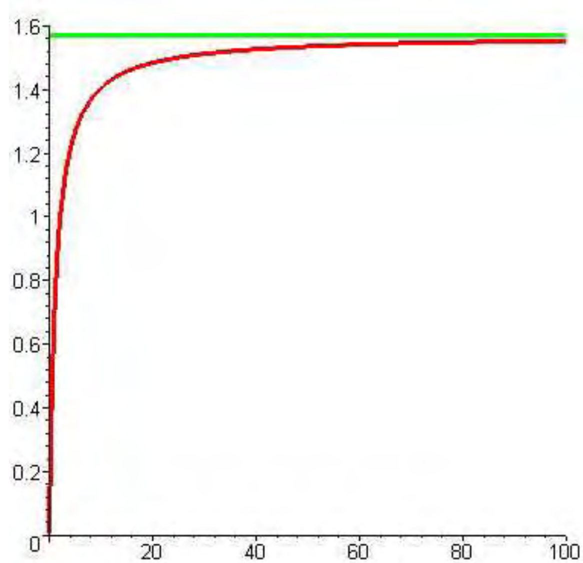
*Thus, numerator and denominator of $\mathcal{R}(a)/a$ each have degree 3 in the variable a^2.

†Numerator and denominator have degree 30 in a^2.

- Though the convergence rate is slower for larger a, the method allows, say, *graphing* $\mathcal{R}$ to reasonable precision.

Having noted a **formal expansion** at $a = 0$, we naturally asked for: **An asymptotic form valid for *large* a**?

- Through a typical asymptotic development, we are rewarded with *more*, namely **a convergent expansion** for all $a > 1$.



A plot of $\mathcal{R}$

Starting with our second sech integral

$$\mathcal{R}(a) = \int_0^\infty e^{-x/a}\,\mathrm{sech}\,x\,dx$$

we again use the Euler numbers and known **Hurwitz-zeta** evaluations of sech-power integrals for *odd* powers.

We obtain a convergent series valid at least for real $a > 1$:

$$\mathcal{R}(a) = \frac{\pi}{2}\sec\left(\frac{\pi}{2a}\right) - 2\sum_{m \in D^+} \frac{\eta(m+1)}{a^m}$$

or, equivalently,

$$\mathcal{R}(a) = 2\sum_{k \geq 0} \eta(k+1)\left(-\frac{1}{a}\right)^k \tag{4.3}$$

- Here D^+ denotes positive odd integers and

  $$\eta(s) := 1/1^s - 1/3^s + 1/5^s - \cdots$$

  is the *primitive L-series mod* 4, not to be confused with Ramanujan's η parameter.

- Remarkably, we find the leading terms for large a involve **Catalan's constant** $G := \eta(2)$ via

$$\mathcal{R}(a) = \frac{\pi}{2} - \frac{2G}{a} + \frac{\pi^3}{16a^2} - \cdots,$$

a development difficult to infer from casual inspection of Ramanujan's fraction.

- Even the asymptote $\mathcal{R}(\infty) = \pi/2$ is hard to so infer, though it follows from various of the previous representations for $\mathcal{R}(a)$.

- Using recurrence relations and various expansions we also obtain results pertaining to the derivatives of $\mathcal{R}$, notably

$$\mathcal{R}'(1) = 8(1 - G), \qquad \mathcal{R}'\left(\frac{1}{2}\right) = \frac{\pi^2}{24}.$$

- A peculiar property of ψ leads to an exact evaluation of the imaginary part of the digamma representation of $\mathcal{R}(a)$ when a lies on the circle

$$C_{1/2} := \{z : |z - 1/2| = 1/2\}$$

in the complex plane.

Imaginary parts of the needed digamma values have a *closed form* [AS,St], and we obtain

$$\mathrm{Im}(\mathcal{R}(a)) = \frac{\pi}{2}\,\mathrm{cosech}\left(\frac{\pi y}{2}\right) - \frac{1}{y}$$

for

$$y := i\left(1 - \frac{1}{a}\right) \quad \text{and} \quad a \in C_{1/2}.$$

- Note that y is always real, and we have an elementary form for $\mathrm{Im}(\mathcal{R})$ on the given continuum set.

- Admittedly we have not yet discussed complex parameters; we do that later.

- The Ramanujan fraction converges at least for $a = b$, $\mathrm{Re}(a) \neq 0$, and it is instructive to compare numerical evaluations of imaginary parts via the above cosech identity.

Firmament—made for Coxeter at ninety

5. The $\mathcal{R}$ Function at Rational Arguments

- For positive integers p, q, we have from the above

  $$\mathcal{R}\left(\frac{p}{q}\right) = 2p\left(\frac{1}{q+p} - \frac{1}{q+3p} + \frac{1}{q+5p} - \dots\right),$$

  which is in the form of a **Dirichlet L-function**.

  One way to evaluate L-functions in finite form is via Fourier-transforms to pick out terms from a general logarithmic series.

We note that an equivalent, elementary form for the digamma at rational arguments is a celebrated result of Gauss. In our case

$$\mathcal{R}\left(\frac{p}{q}\right) = \sum_{\text{odd } k>0}^{4p} e^{-2\pi i\, k(q+p)/(4p)} \times$$

$$\left\{-\log\left(1 - e^{2\pi i\, k/(4p)}\right) - \frac{1}{n}\sum_{n=1}^{q+p-1} e^{2\pi i\, kn/(4p)})\right\}.$$

- After various simplifications, especially forcing everything to be real-valued, we arrive at a finite series in fundamental numbers.

  Namely

$$\mathcal{R}\left(\frac{p}{q}\right) =$$

$$-2p\sum_{n=1}^{p+q-1}\frac{1}{n}\left(\delta_{n\equiv p+q \mod 4p} - \delta_{n\equiv 3p+q \mod 4p}\right)$$

$$-2\sum_{0\ <\text{odd}\ k<2p}\left\{\cos\left(\frac{(p+q)k\pi}{2p}\right)\log\left(2\sin\left(\frac{\pi k}{4p}\right)\right)\right.$$

$$\left. -\pi\left(\frac{1}{2}-\frac{k}{4p}\right)\sin\left(\frac{(p+q)k\pi}{2p}\right)\right\}. \quad (5.1)$$

- When $q = 1$, so that we seek $\mathcal{R}(p)$ for integer p, the first, rational sum vanishes.

  The finite series (5.1) (of $O(p+q)$ total terms) leads quickly to exact evaluations such as:

$$\begin{aligned}
\mathcal{R}(1/4) &= \frac{\pi}{2} - \frac{4}{3}, \quad \mathcal{R}(1/3) = 1 - \log 2, \\
\mathcal{R}(\mathbf{1}) &= \log \mathbf{2}, \quad \mathcal{R}(1/2) = 2 - \pi/2, \\
\mathcal{R}(2/3) &= 4 - \frac{\pi}{\sqrt{2}} - \sqrt{2}\log(1+\sqrt{2}), \\
\mathcal{R}(3/2) &= \pi + \sqrt{3}\log(2-\sqrt{3}), \\
\mathcal{R}(2) &= \sqrt{2}\left\{\frac{\pi}{2} - \log(1+\sqrt{2})\right\}, \\
\mathcal{R}(3) &= \frac{\pi}{\sqrt{3}} - \log 2,
\end{aligned}$$

and, of course, many other attractive forms.

- From (5.1) for positive integer Q one has

$$\mathcal{R}(1/q) = \text{rational} + (-1)^{(q-1)/2}\log 2 \qquad (q \text{ odd})$$

$$\mathcal{R}(1/q) = \text{rational} + (-1)^{q/2}\pi/2 \qquad (q \text{ even})$$

- These can also be derived from $\mathcal{R}(1) = \log 2$, $\mathcal{R}(1/2) = 2 - \pi/2$ and the recurrence

$$\mathcal{R}\left(\frac{1}{q}\right) = \frac{2}{q-1} - \mathcal{R}\left(\frac{1}{q-2}\right).$$

- On the other hand, an alluring integer-argument evaluation involves the *golden mean*:

$$\mathcal{R}(5) = \frac{\pi}{\sqrt{\tau\sqrt{5}}} + \log 2 - \sqrt{5}\,\log \tau,$$

  where $\tau := (1+\sqrt{5})/2$.

  Such evaluations—based on (5.1)—can involve quite delicate symbolic work.

- We have not analyzed evaluating $\mathcal{R}(a)$ for *irrational* a by approximating a first via high-resolution rationals, and then using (5.1).

  Such a development would be of both computational and theoretical interest.

- Armed with exact knowledge of $\mathcal{R}(p/q)$ we find some interesting Gauss-fraction results, in the form of rational multiples of

$$F(\gamma, 1; 1+\gamma; -1) = [\alpha_1, \alpha_2, \ldots].$$

For example, (4.2) yields

$$\mathcal{R}(1) = \log 2 = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \ddots}}}},$$

but alas the beginnings of this fraction are misleading; subsequent elements a_n run

$$\log 2 = [1, 2, 3, 1, 5, \frac{2}{3}, 7, \frac{1}{2}, 9, \frac{2}{5}, \ldots],$$

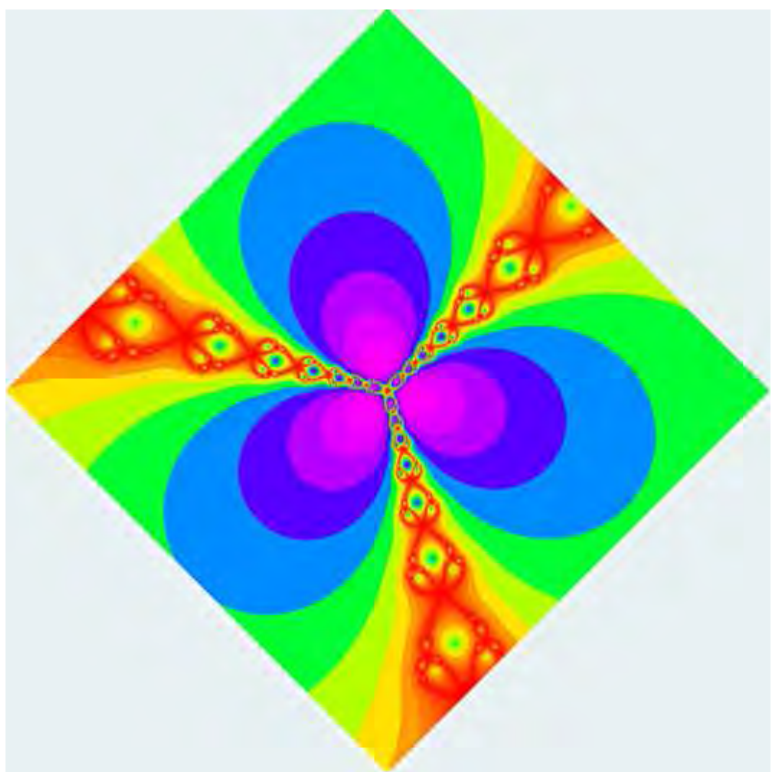
being as $\alpha_n = n, 4/n$ resp. for n odd, even.

Similarly, one can derive

$$2 - \log 4 = [1^3, r_2, 2^3, r_4, 3^3, r_6, 4^3, \ldots],$$

where the even-indexed fraction elements r_{2n} are computable rationals.

- Though these RCFs do not have integer elements, the growths of the α_n provide a clue to the convergence rate, which we study in a subsequent section.



6. Transformation of $\mathcal{R}_1(a,b)$

The big step. We noted that the sech-elliptic series (3.1) (also (3.2)) will converge slowly when $b \approx a$, yet in Sections 4, 5 we successfully addressed the case $b = a$.

We now establish a series representation when $b < a$ but b is very near to a.

We employ the wonderful fact that sech is its own Fourier transform, in that

$$\int_{-\infty}^{\infty} e^{i\,\gamma x} \operatorname{sech}(\lambda x)\, dx = \frac{\pi}{\lambda} \operatorname{sech}\left(\frac{\pi\gamma}{2\lambda}\right).$$

Using this relation, one can perform a *Poisson transform* of the sech-elliptic series (3.1).

- The success of the transform depends on

$$I(\lambda,\gamma) := \int_{-\infty}^{\infty} \frac{\operatorname{sech}\lambda x}{1+x^2}\, e^{i\,\gamma x}\, dx.$$

One may obtain the differential equation:

$$-\frac{\partial^2 I}{\partial \gamma^2} + I = \frac{\pi}{\lambda}\,\mathrm{sech}\left(\frac{\pi\gamma}{2\lambda}\right)$$

and solve it—after some *machinations*—to yield

$$\begin{aligned} I(\lambda,\gamma) &= \frac{\pi}{\cos\lambda}\, e^{-\gamma} \\ &+ \frac{2\pi}{\lambda}\sum_{d\in D^+} \frac{(-1)^{(d-1)/2} e^{-\pi d\,\gamma/(2\lambda)}}{1-\pi^2 d^2/(4\lambda^2)}. \end{aligned}$$

where D^+ denotes the positive odd integers.

- When $\lambda = \pi D/2$ for some odd D, the $1/\cos$ pole conveniently cancels a corresponding pole in the summation, and the result can be inferred either by avoiding $d = D$ in the sum and inserting a precise residual term

$$\Delta I = \pi(-1)^{(D-1)/2} e^{-\gamma}(\gamma + 1/2)/\lambda,$$

  or more simply by taking a numerical limit as $\lambda \to \pi D/2$.

- When $\gamma \to 0$ we can recover from the sum, via analytic relations for $\psi(z)$, the ψ-function form of the integral of $(\mathrm{sech}\lambda x)/(1+x^2)$.

Via **Poisson transformation** of (3.1) we thus obtain, for $0 < b < a$,

$$\mathcal{R}_1(a,b) = \mathcal{R}\left(\frac{\pi a}{2\mathrm{K}'}\right) + \frac{\pi}{\cos\frac{\mathrm{K}'}{a}}\frac{1}{e^{2\mathrm{K}/a}-1} + 8\pi a\,\mathrm{K}'\sum_{d\in D^+}\frac{(-1)^{(d-1)/2}}{4\mathrm{K}'^2-\pi^2 d^2 a^2}\frac{1}{e^{\pi d\,\mathrm{K}/\mathrm{K}'}-1} \tag{6.1}$$

where $k := b/a$, $\mathrm{K} := \mathrm{K}(k)$, $\mathrm{K}' := \mathrm{K}(k')$, and D^+ again denotes the positive odd integers.

- A similar Poisson transform obtains from (3.2) in the case $b > a > 0$.

- Such transforms appear recondite, but we have what we desired: *convergence is rapid for $b \approx a$*: because $K/K' \sim \infty$.

7. Convergence Results

For an RCF $x = [a_0, a_1, \ldots]$ (so each a_i is non-negative but need not be integer) one has the usual recurrence relations* for **convergents**

$$p_n = a_n \cdot p_{n-1} + 1 \cdot p_{n-2},$$

$$q_n = a_n \cdot q_{n-1} + 1 \cdot q_{n-2},$$

with $(p_0, p_{-1}, q_0, q_{-1}) := (a_0, 1, 1, 0)$.

We also have the approximation rule

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}},$$

so that convergence rates can be bounded by virtue of the growth of the q_n.

- One may iterate the recurrence in various ways, obtaining for example

$$q_n = \left(1 + a_n a_{n-1} + \frac{a_n}{a_{n-2}} \right) q_{n-2} - \frac{a_n}{a_{n-2}} q_{n-4}.$$

*The corresponding matrix scheme with b_n inserted for '1' applies generally to CF's.

An immediate application is

Theorem 7.1: For the **RCF form of the Gauss fraction**, $F(\gamma, 1; 1+\gamma; -1) = [\alpha_1, \alpha_2, \ldots]$, and for $\gamma > 1/2$ we have

$$\left| F - \frac{p_n}{q_n} \right| < \frac{c}{8^{n/2}},$$

where c is an absolute constant.

Remark: One can obtain sharper γ-dependent bounds. We intend here just to show geometric convergence; i.e. that the number of good digits grows at least linearly in the number of iterates.

Also note that for the $\mathcal{R}(a)$ evaluation of interest, $\gamma = 1/2 + 1/(2a)$ so that the condition on γ is natural.

Proof: From the element assignments following (4.2) we have

$$\alpha_n \alpha_{n-1} = \frac{4}{(n-1)^2}(n-1+\gamma)(n-2+\gamma);$$

$$1 < n \text{ odd },$$

$$\alpha_n \alpha_{n-1} = \frac{1}{(n/2-1+\gamma)^2}(n-1+\gamma)(n-2+\gamma);$$

$$n \text{ even }.$$

We also have $q_1 = 1$, $q_2 = 1 + 1/\gamma > 2$ so that for sufficiently large n we have

$$\alpha_n \alpha_{n-1} + 1 > 4, 2$$

respectively as n is odd, even.

From the estimate $q_n > (\alpha_n \alpha_{n-1} + 1) q_{n-2}$ the desired bound follows. ©

- A clever computational acceleration for Gauss fractions is described in [BBa1,AAR,LW]. Consider the previously displayed fraction

$$\log 2 = [1, 2, 3, 1, 5, 2/3, \ldots].$$

Generally a "tail" t_N of this construct, meaning a subfraction starting from the N-th element, runs like so:

$$t_N := \cfrac{1}{\cfrac{4}{N} + \cfrac{1}{N + 1 + \cfrac{1}{\cfrac{4}{N+2} + \cfrac{1}{N + 3 + \ddots}}}}.$$

The clever idea in the literature is that this tail t_N should be near the *periodic* fraction

$$[4/N, N, 4/N, N, \ldots] = N(\sqrt{2} - 1)/2.$$

- This suggests that if we evaluate the Gauss fraction and stop at element $4/N$, this one element should be replaced by $2(1+\sqrt{2})/N$.

  - In our own numerical experiments, this trick always adds a few digits precision.

- As suggested in [LW], there are higher-order takes of this idea; e.g., the use of longer periods for the tail sub-fraction.

  - As the reference shows, experimentally, the acceleration can be significant.

We now attack convergence of **the Ramanujan RCF**, viz

$$\frac{a}{\mathcal{R}_1(a,b)} = [A_0; A_1, A_2, A_3, \ldots].$$

- The q_n convergents are linear combinations of $a^i b^j$'s for i,j even integers, and the explicit coefficients can be isolated.

This leads to

$$q_n \geq 1 + \frac{b^{n-2}}{a^n} \prod_{m \text{ even}}^{n} (1 - 1/m)^2$$

$$> 1 + \frac{1}{2n} \frac{b^{n-2}}{a^n}$$

$$n \text{ even},$$

$$q_n \geq \frac{1}{b^2} + \frac{a^{n-1}}{b^{n+1}} \prod_{m \text{ even}}^{n-1} (m/(m+1))^2$$

$$> \frac{1}{b^2} + \frac{1}{n} \frac{a^{n-1}}{b^{n+1}}$$

$$n \text{ odd}.$$

- We are ready for a convergence result for the *original Ramanujan construct:*

Theorem 7.2: For the **Ramanujan RCF**

$$\frac{a}{\mathcal{R}_1(a,b)} = [A_0; A_1, A_2, A_3, \ldots]$$

we have for $b > a > 0$

$$\left|\frac{a}{\mathcal{R}_1(a,b)} - \frac{p_n}{q_n}\right| < \frac{2nb^4}{(b/a)^n},$$

while for $a > b > 0$ we have

$$\left|\frac{a}{\mathcal{R}_1(a,b)} - \frac{p_n}{q_n}\right| < \frac{nb/a}{(a/b)^n}.$$

Proof: The given bounds follow directly upon inspection of the products $q_n\, q_{n+1}$. ©

Remark: Again sharper bounds should be possible. Our aim was merely to show geometric convergence *when* a, b *are not near each other.*

- As previously intimated, convergence for $a = b$ is slow. What we can prove is:

Theorem 7.3: For real $a > 0$, we have

$$\left|\frac{a}{\mathcal{R}(a)} - \frac{p_n}{q_n}\right| < \frac{c(a)}{n^{h(a)}},$$

where $c(a), h(a)$ are n-independent constants.

The exponent $h(a)$ can be taken to be

$$c_0 \min(1, 4\pi^2/a^2)$$

where the constant c_0 is absolute.

Remark: While the bound is computationally poor, as noted, convergence does occur.

- With effort, the exponent $h(a)$ can be sharpene and made more explicit.

- Indeed, for $a = b$ or even $a \approx b$ we now have many other, rapidly convergent options.

Proof: With a view to induction, assume that for some constants (n-independent) $d(a), g(a)$ and for $n \in [1, N-1]$ we have $q_n < dn^g$.

Note that the asymptotics following (2.1) mean that $A_n > f(a)/n$ for an n-independent f.

Then we have a bound for the next q_N:

$$q_N > \frac{f}{N} d(N-1)^g + d\ (N-2)^g\,.$$

Since for $g < 1$, $0 < x \leq 1/2$ we have

$$(1-x)^g > 1 - gx - gx^2,$$

and the constants d, g can be arranged so that $q_N > dN^g$ and the induction goes through. ©

We reprise the import of these three theorems:

- (Theorem 7.1) The *Gauss fraction* for $\mathcal{R}(a)$ exhibits (*at least*) geometric/linear convergence.

- (Theorem 7.2) So does the *original Ramanujan form* $\mathcal{R}_1(a,b)$ when a/b or b/a is (significantly) greater than unity .

- (Theorem 7.3) When $a=b$ we still have convergence in the original form.

  As suggested by Theorem 7.3 convergence is far below geometric/linear.

8. A Uniformly Convergent Algorithm

- We may now give a complete algorithm to evaluate the original Ramanujan fraction $\mathcal{R}_\eta(a,b)$ for positive real parameters.

- Convergence is uniform—for any positive real triple η, a, b we obtain D good digits in no more than $c\,D$ computational iterations, where c is independent of the size of η, a, b.*

0. Observe that $\mathcal{R}_\eta(a,b) = \mathcal{R}_1(a/\eta, b/\eta)$ so that with impunity we may assume $\eta = 1$ and subsequently evaluate only $\mathcal{R}_1$.

*By iterations here we mean either continued-fraction recurrence steps, or series-summand additions.

Algorithm for $\mathcal{R}_\eta(a,b)$ with real $\eta, a, b > 0$:

1. If $(a/b > 2$ or $b/a > 2)$ return the original fraction (1.1), or equivalently (2.1);

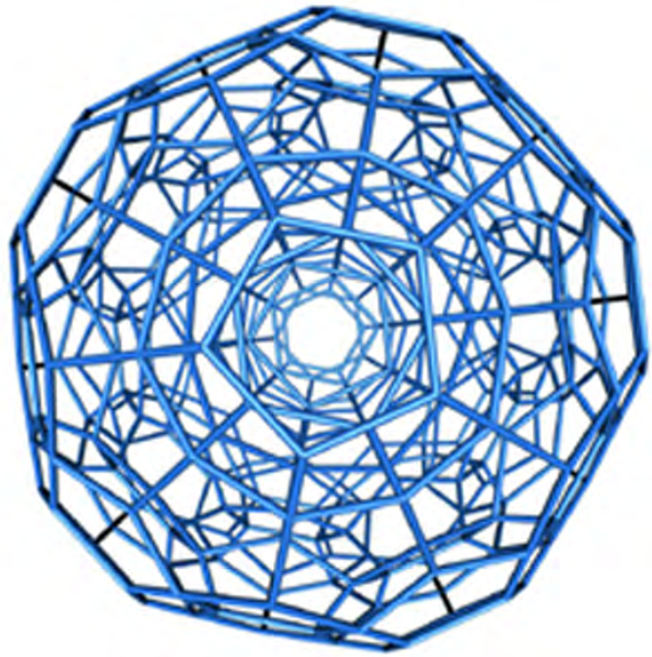
2. If $(a = b)$ $\{$
   if ($a = p/q$ rational) return finite form (5.1); else return the Gauss RCF (4.2) or rational-zeta form (4.1) or (4.3) or some other scheme such as rapid ψ computations; $\}$

3. If $(b < a)$ $\{$
   if (b is *not too close* to a)*, return sech-elliptic result (3.1); else return Poisson-transform result (6.1); $\}$

4. (We have $b > a$) Return, as in (1.2),
$$2\mathcal{R}_1\left((a+b)/2, \sqrt{ab}\right) - \mathcal{R}_1(b,a).$$

©

*Say, $|1 - b/a| > \varepsilon > 0$ for any fixed $\varepsilon > 0$.



- It is an implicit tribute to Ramanujan's ingenuity that the final step (4) of the algorithm allows entire procedure to go through for *all* positive real parameters.

- One may avoid step (4) by invoking a Poisson transformation of (3.2), but Ramanujan's AGM identity is finer!

9. About Complex Parameters

PART II. *Complex* parameters a, b, η are **complex**, as we found via extensive experimentation.

- We attack this stultifying scenario by assuming $\eta = 1$ and defining

$$\mathcal{D} = \{(a, b) \in \mathrm{C}^2 : \mathcal{R}_1(a, b) \text{ converges}\},$$

meaning the convergents of the original fraction (1.1) have a well-defined limit.

- There are literature claims [BeIII] that

$$\{(a, b) \in \mathrm{C}^2 : \mathrm{Re}(a), \mathrm{Re}(b) > 0\} \subseteq \mathcal{D},$$

i.e., that convergence occurs whenever both parameters have positive real part.

- **This is false**—as we shall show.

- The *exact identification* of $\mathcal{D}$ is very delicate.

- $\mathcal{R}_1(a,b)$ *typically diverges* for $|a| = |b|$: we observed numerically* that

$$\mathcal{R}_1(1/2 + \sqrt{-3}/2, 1/2 - \sqrt{-3}/2)$$

and $\mathcal{R}_1(1,i)$ have '**period two**'—as is generic–while $\mathcal{R}_1(t\,i, t\,i)$ is '**chaotic**' for $t > 0$.

*After a *caution* on checking only even terms!

- An observation that led to the results below is that we have implicitly used, for positive reals $a \neq b$ and perforce for the Jacobian parameter

$$q := \frac{\min(a,b)}{\max(a,b)} \in [0,1),$$

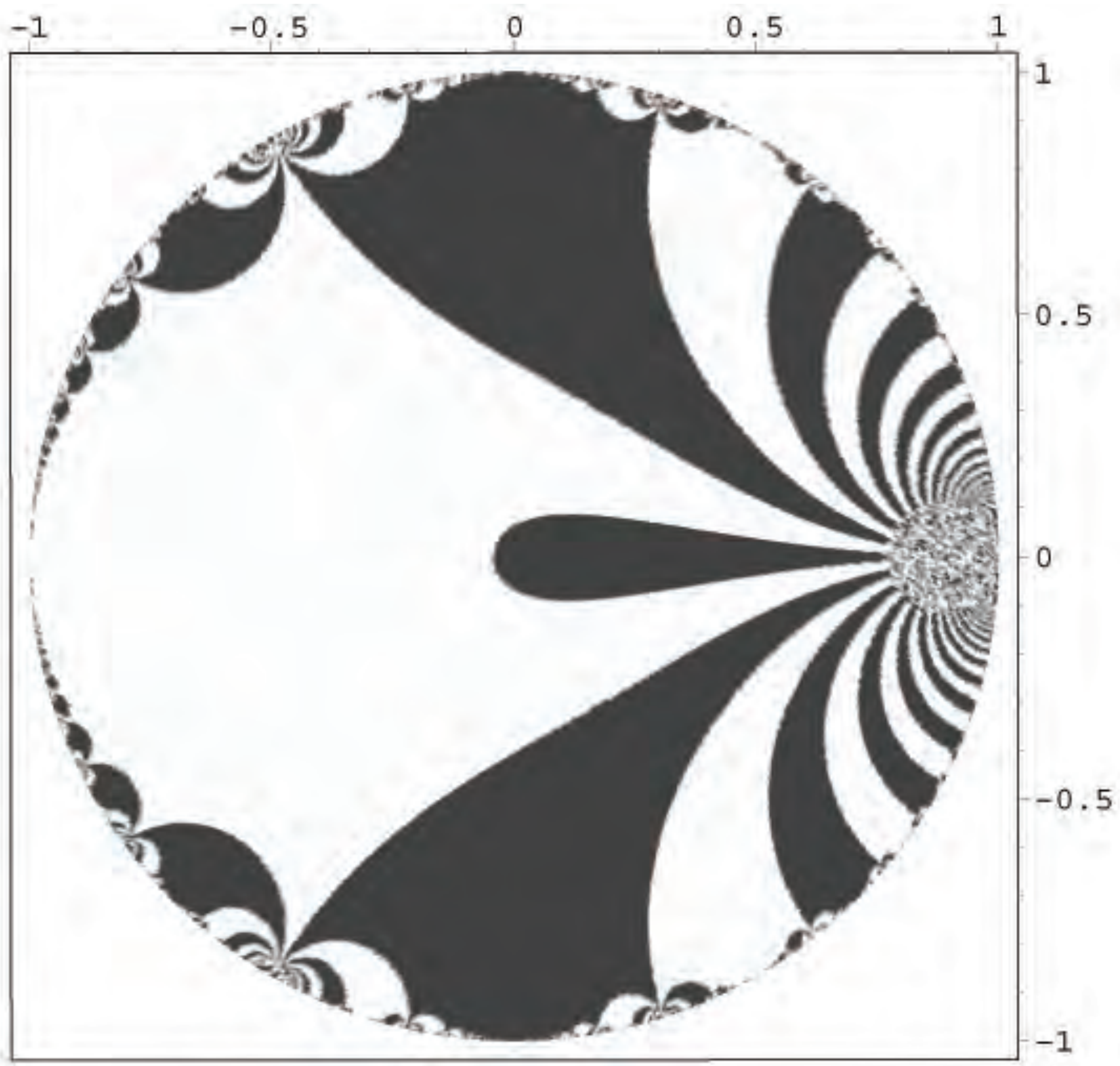
the fact that

$$0 \leq \frac{\theta_2(q)}{\theta_3(q)} < 1.$$

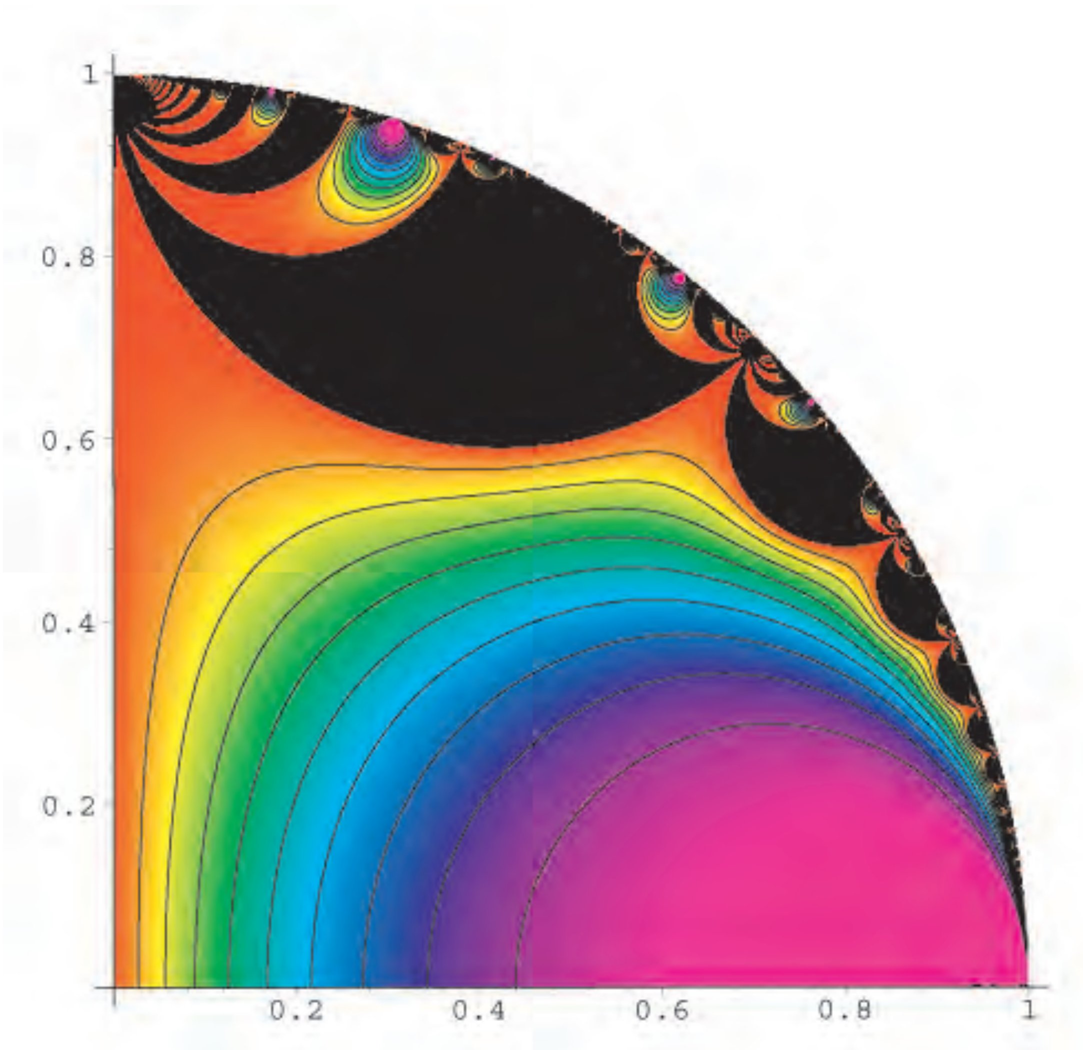
- If, however, one plots *complex* q with this ratio of **absolute value less than one**, a complicated **fractal structure emerges**, as shown in the Figures below—this leads to the *theory of modular forms* [BB].

- Thence the sech relations of Theorem 2.1 are suspect for complex q.

- Numerically, the identities appear to fail when $|\theta_2(q)/\theta_3(q)|$ exceeds unity as graphed in white for $|q| < 1$:

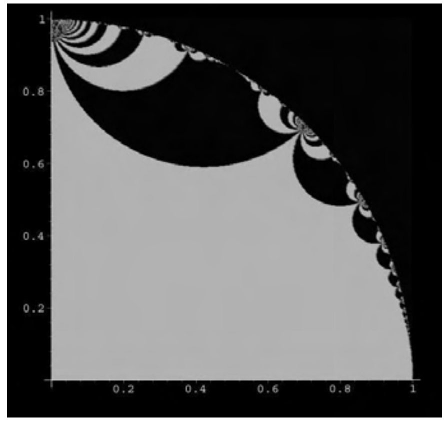


- Such *fractal behaviour* is ubiquitous.



- Where $|\theta_4(q)/\theta_3(q)| > 1$ in first quadrant.*

*Colours show gradations between zero and one.



- Though the original fraction $\mathcal{R}_1(a,b)$ converges widely, the celebrated AGM relation (1.2) does *not* hold across $\mathcal{D}$.

Using **a:=1, b:=-3/2+i/4** the computationalist will find that the AGM relation fails:

$$\mathcal{R}_1\left(-\frac{1}{4}+\frac{i}{8}, \sqrt{-\frac{3}{2}+\frac{i}{4}}\right) \neq \frac{\mathcal{R}_1\left(1,b\right)+\mathcal{R}_1\left(b,1\right)}{2}.$$

- The *key to determining the domain for the AGM relation* seems to be the ordering of the *moduli* of the relevant parameters.

- We take the elliptic integral $\mathsf{K}(k)$ for complex k to be defined by

$$\mathsf{K}(k) = \frac{\pi}{2}\,\mathsf{F}\left(\frac{1}{2},\frac{1}{2};1;k^2\right).$$

- The hypergeometric function F converges absolutely for k in the disk $(|k| < 1)$ and is continued analytically.

- For any numbers $z = re^{i\phi}$ under discussion, $r \geq 0$, $\arg(z) \in (-\pi, +\pi]$ and so

$$\sqrt{z} := \sqrt{r}e^{i\phi/2}.$$

- We start with some numerically based *Conjectures* now *largely* **proven**:

Conjecture 9.0 (Analytic continuation): Consider complex pairs (a, b). Then

i. If $|a| > |b|$ the original fraction $\mathcal{R}_1(a, b)$ exists and agrees with the sech series (3.1).

ii. If $|a| < |b|$ the original fraction $\mathcal{R}_1(a, b)$ exists and agrees with the sech series (3.2).

Theorem 9.1: $\mathcal{R}(a) := \mathcal{R}_1(a, a)$ converges **iff** $a \notin \mathcal{I}$. That is, the fraction diverges if and only if a is pure imaginary. Moreover, for $a \in \mathcal{C} \backslash \mathcal{I}$ the fraction converges to a holomorphic function of a in the appropriate open half-plane.

Theorem 9.2: $\mathcal{R}_1(a, b)$ converges for all real pairs; that is whenever $\text{Im(a)} = \text{Im(b)} = 0$.

Theorem 9.3: (i) The even/odd parts of $\mathcal{R}_1(1, i)$ (e.g.) converge to **distinct limits**. (ii) There are $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ such that $\mathcal{R}_1(a, b)$ diverges.

★ Define

- $\mathcal{H} := \{z \in \mathcal{C} : \left|\frac{2\sqrt{z}}{1+z}\right| < 1\}$,
- $\mathcal{K} := \{z \in \mathcal{C} : \left|\frac{2z}{1+z^2}\right| < 1\}$.

Theorem 9.4: If $a/b \in \mathcal{K}$ then both $\mathcal{R}_1(a, b)$ and $\mathcal{R}_1(b, a)$ converge.

Theorem 9.5: $\mathcal{H} \subset \mathcal{K}$ (properly).

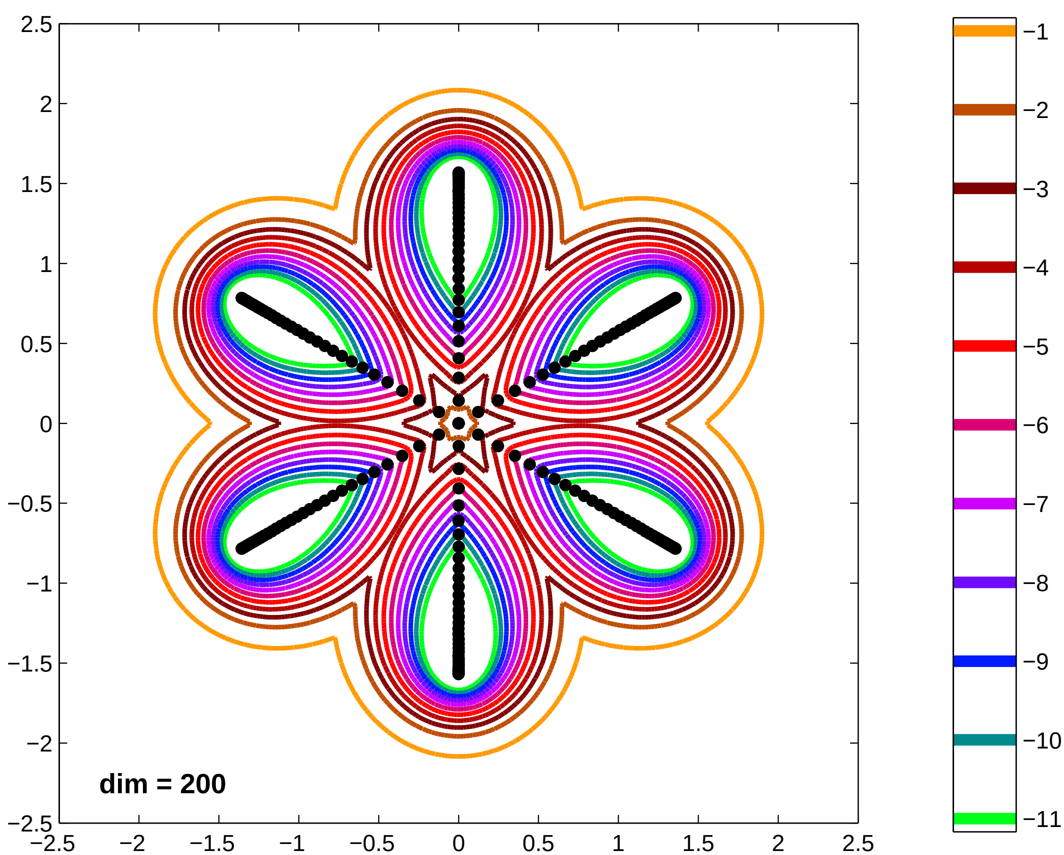
The results so far combine to give a region of validity for the AGM relation, in the form of:

Theorem 9.6: If $a/b \in \mathcal{H}$ then $\mathcal{R}_1(a, b)$ & $\mathcal{R}_1(b, a)$ converge, and the arithmetic mean $(a + b)/2$ dominates the geometric mean $\sqrt{ab}$ in modulus.

- Now to the problematic remainders of the AGM relation (1.2) $\cdots$.

- We performed "**scatter diagram**" analysis to find computationally where the AGM relation holds in the parameter space.

  ★ The results were quite spectacular! And lead to the Theorems above.



- Our resulting **tough conjecture** with $\mathcal{C}' := \{z \in \mathcal{C} : |z| = 1, z^2 \neq 1\}$, led to:

Theorem 9.11: The precise domain of convergence for $\mathcal{R}_1(a,b)$ is

$$\mathcal{D}_0 = \{(a,b) \in \mathcal{C}\times\mathcal{C} : (a/b \notin \mathcal{C}') \text{ or } (a^2 = b^2,\ b \notin \mathcal{I})\}$$

In particular, for $a/b \in \mathcal{C}'$ we have divergence.

Moreover, **provably**, $\mathcal{R}_1(a,b)$ converges to an analytic function of both a or b on the domain

$$\mathcal{D}_2 := \{(a,b) \in \mathcal{C} \times \mathcal{C} : \ |a/b| \neq 1\} \subset \mathcal{D}_0.$$

- Note, we are not harming Theorems 9.4–9.6 because neither $\mathcal{H}$ nor $\mathcal{K}$ intersects $\mathcal{C}'$.

- The "**bifurcation**" of Theorem 9.11 is very subtle.

Theorem 9.12: The restriction $a/b \in \mathcal{H}$ implies the truth of the AGM relation (1.2) with all three fractions converging.

Proof: For $a/b \in \mathcal{H}$, the ratio $(a+b)/(2\sqrt{ab}) \notin \mathcal{C}'$ and via 9.11 we have sufficient analyticity to apply Berndt's technique. ©

- A picturesque take on Theorems 9.4–9.6 and 9.12 follows:

  Equivalently, a/b belongs to the closed exterior of $\partial\mathcal{H}$, which in polar-coordinates is given by the **cardioid-knot**

  $$r^2 + (2\cos\phi - 4)r + 1 = 0$$

  drawn in the complex plane ($r = |a/b|$).

Proof: A pair from Theorem 9.11 meets

$$1 \le \left|\frac{a+b}{2\sqrt{ab}}\right|^2 = \frac{1}{4}\left|\sqrt{z}+\frac{1}{\sqrt{z}}\right|^2,$$

with $z := a/b$. We obtain thus, for $z := re^{i\phi}$ that $4 \le r + 2\cos\phi + 1/r$, which defines the closed exterior* of the cardioid-knot curve. ©

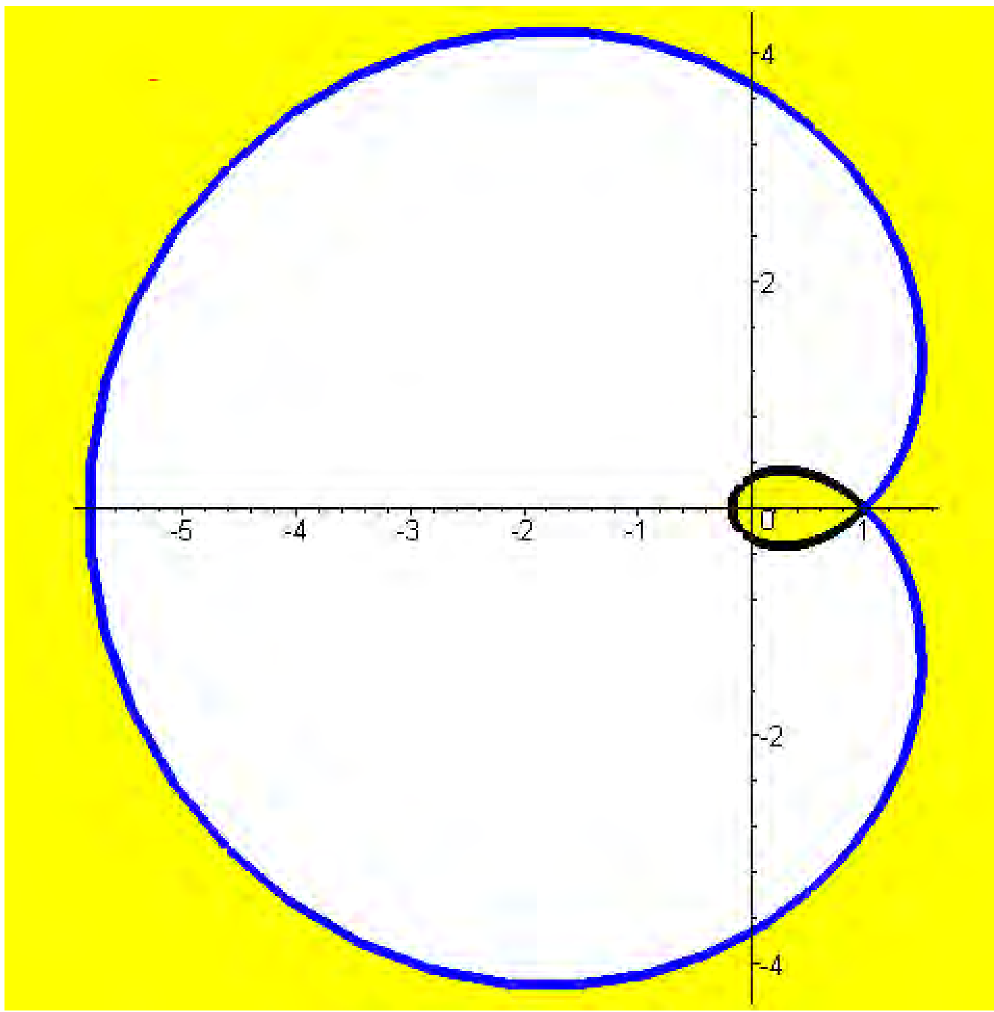
- To clarify, consider the two lobes of $\partial\mathcal{H}$:
  - We fuse the orbits of the $\pm$ instances, for $\theta \in [0, 2\pi]$ yielding:

    $$r = 2 - \cos\theta \pm \sqrt{(1-\cos\theta)(3-\cos\theta)}.$$

  - Thus, $\mathcal{H}$ has a small loop around the origin, with left-intercept $\sqrt{8} - 3 + 0i$, and a wider contour whose left-intercept is $-3 - \sqrt{8} + 0i$.

*As determined by Jordan crossings.

- $a/b \in \mathcal{H}$: the *arithmetic mean dominating the geometric mean in modulus*.



A cardioid-knot, on the (yellow) exterior of which we can prove the truth of the Ramanujan AGM relation (1.2), (9.1).

- The condition $a/b \in \mathcal{H}(\mathcal{K})$ is symmetric: if a/b is in $\mathcal{H}(\mathcal{K})$ then so is b/a, since $r \to 1/r$ leaves the polar formula invariant.

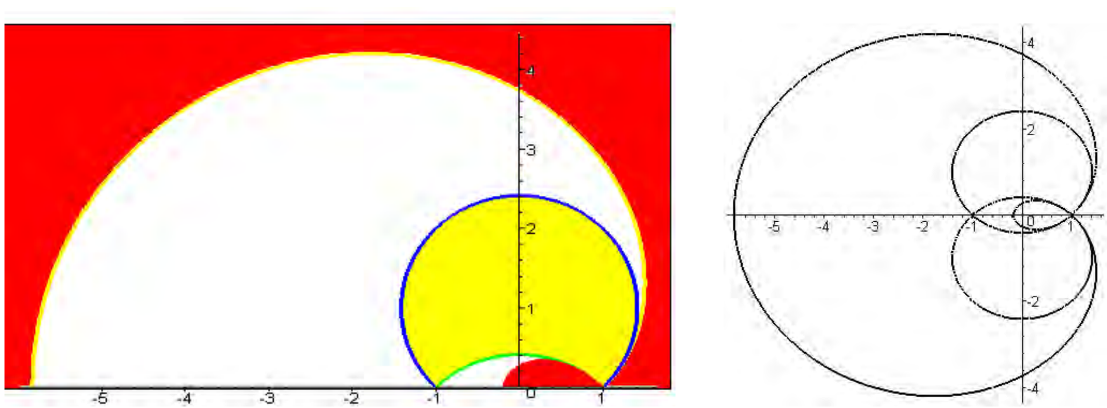
- In particular, the AGM relation holds whenever $(a, b) \in \mathcal{D}$ and a/b lies on the exterior rays:
$$\frac{a}{b} \text{ or } \frac{b}{a} \in [\sqrt{8} - 3, \infty) \cup (\infty, -3 - \sqrt{8}],$$
thus including all positive real pairs (a, b) as well as a somewhat wider class.

- Similarly, the AGM relation holds for pairs $(a, b) = (1, i\beta)$ with
$$\pm\beta \in [0, 2 - \sqrt{3}] \cup [2 + \sqrt{3}, \infty).$$

Remark. We performed extensive numerical experiments *without faulting* Theorems 9.11 and 9.12.

- Even with $a = b$ we need $(a, a) \in \mathcal{D}$; recall (i, i) (also $(1, i)$) **provably** is not in $\mathcal{D}$.

- The unit circle only intersects $\mathcal{H}$ at $z = 1$.



Where provably $\mathcal{R}$ exists (not yellow) and where the AGM holds (red).

$$\left|2\frac{\sqrt{z}}{1+z}\right| \Rightarrow \left|2\frac{z}{1+z^2}\right| < 1.$$

$\diamond$ A key component of our proofs, actually valid in any B^* algebra, is:

Theorem 9.13. Let (a_n), (b_n) be sequences of $k \times k$ complex matrices.

Suppose that $\prod_{j=1}^{n} a_j$ converges as $n \to \infty$ to an **invertible limit** while $\sum_{j=1}^{\infty} \|b_j\| < \infty$. Then

$$\prod_{j=1}^{n} (a_j + b_j)$$

also converges to a finite complex matrix.

- Theorem 9.13 appears new even in $\mathbf{C}^1$!

★ It allows one to linearize nonlinear recursions—ignoring $O\left(1/n^2\right)$ terms for convergence purposes.

- This is how the issue of the dynamics of (t_n) arose.

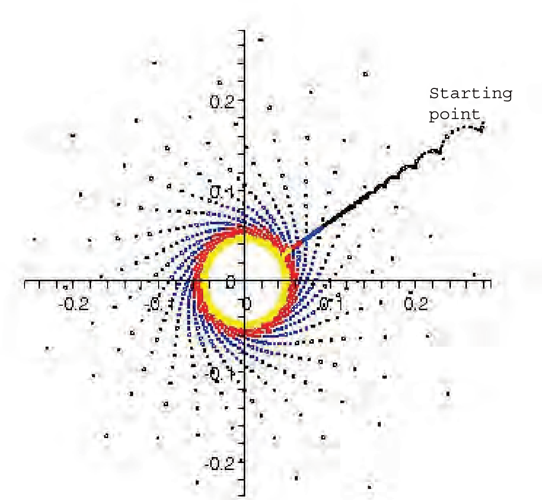
10. Visual Dynamics from a 'Black Box'

• Six months later we had a beautiful proof using genuinely new dynamical results. Starting from the *dynamical system* $t_0 := t_1 := 1$:

$$t_n \hookleftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

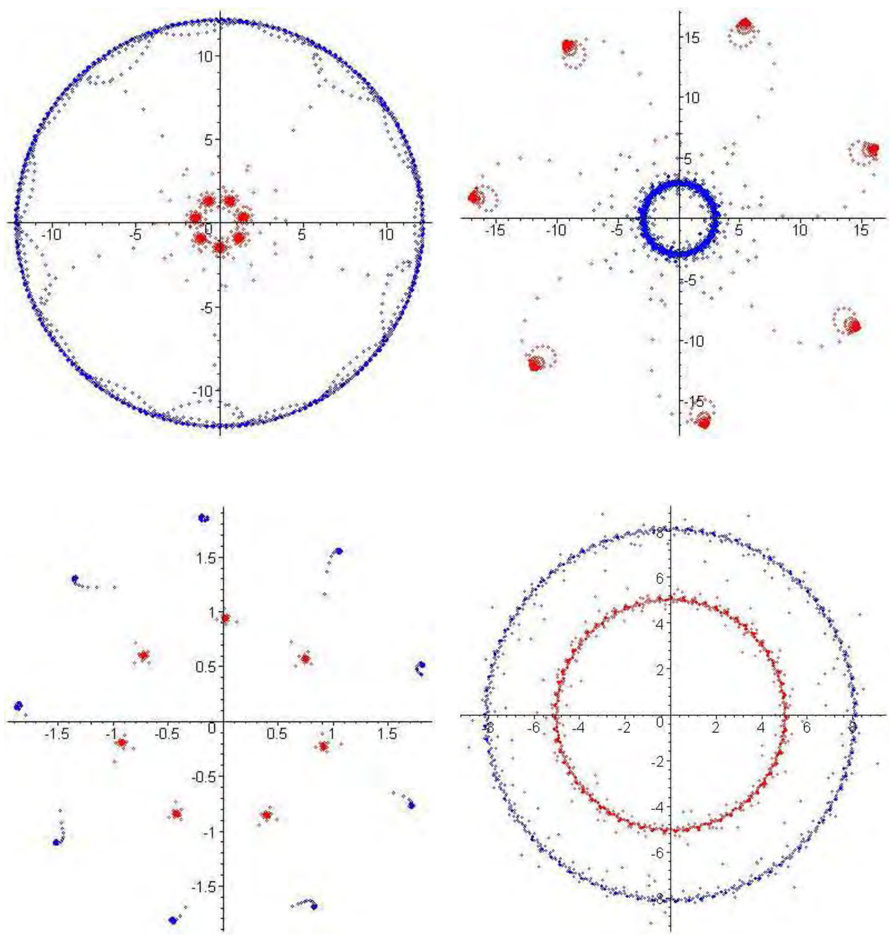
where $\omega_n = a^2, b^2$ for n even, odd respectively—or is much more general.*

- Numerically all one learns is that is tending to zero slowly. Pictorially we see significantly more:



$^*\sqrt{n}\, t_n$ is bounded iff $\mathcal{R}_1(a,b)$ *diverges*.

- Scaling by $\sqrt{n}$, and coloring odd and even iterates, fine structure appears.



The attractors for various $|a| = |b| = 1$.

★ This is *now* fully explained, especially the original rate of convergence, which follows by a fine *singular-value* argument.

11. The 'Chaotic' Case

Jacobsen-Masson theory used in Theorem 9.1 shows, unlike $\mathcal{R}_1(1,i)$, even/odd fractions for $\mathcal{R}_1(i,i)$ behave "chaotically," neither converge.

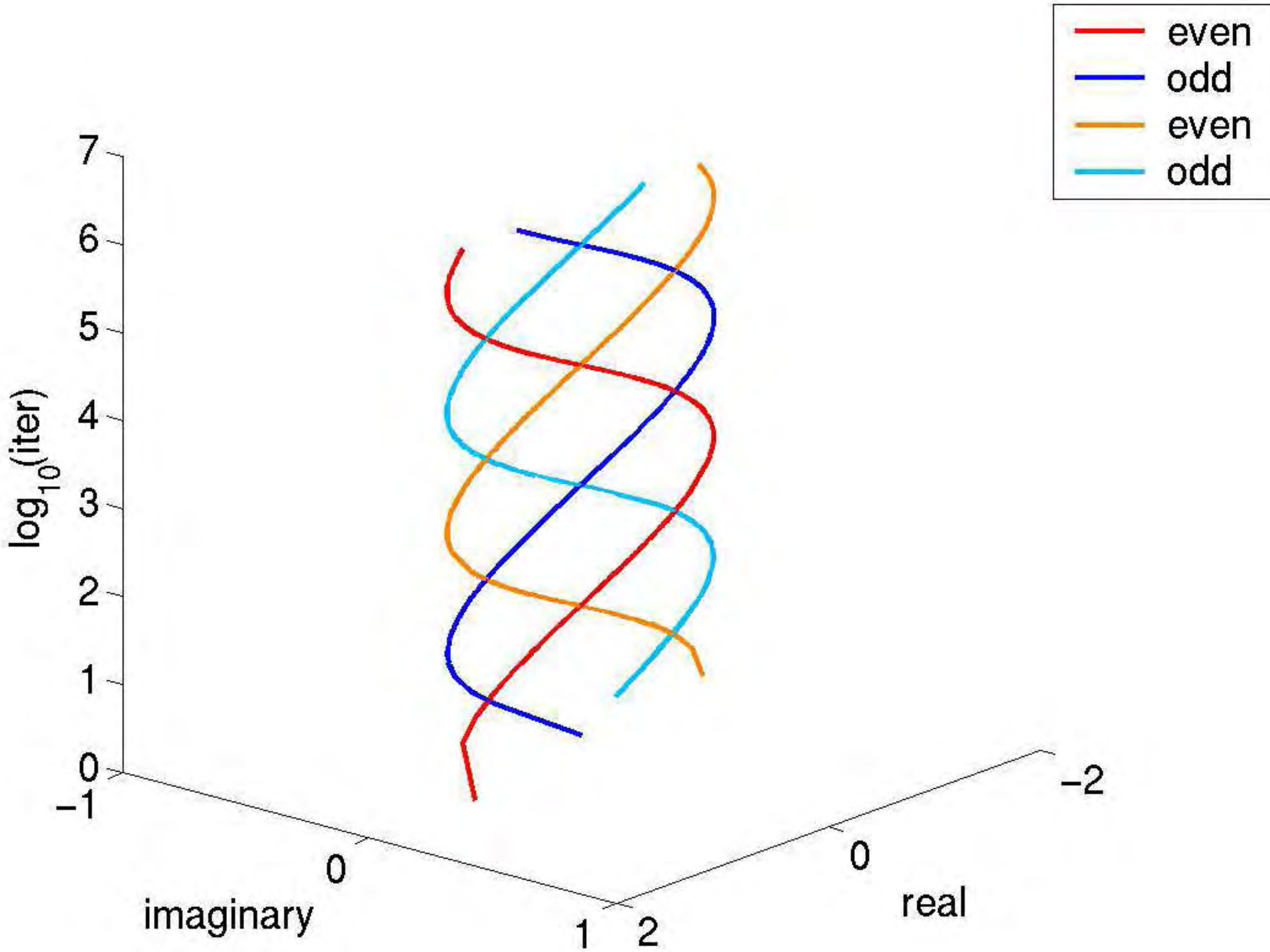
When $a = b = i$, (t_n) exhibit a fourfold quasi-oscillation, as n runs through values mod 4. Plotted versus n, the (real) sequence $t_n(\mathbf{1,1})$ exhibits the "serpentine oscillation" of four separate "necklaces."

For $a = i$, the detailed asymptotic is

$$t_n(1,1) = \sqrt{\frac{2}{\pi}\cosh\frac{\pi}{2}}\,\frac{1}{\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right) \times$$

$$\begin{cases} (-1)^{n/2}\cos(\theta - \log(2n)/2) & n \text{ is even} \\ (-1)^{(n+1)/2}\sin(\theta - \log(2n)/2) & n \text{ odd} \end{cases}$$

where $\theta := \arg\Gamma((1+i)/2)$.



The subtle four fold serpent.

♠ This behavior seems very difficult to infer directly from the recurrence.

♡ Analysis is based on a striking hypergeometric parametrization which was both **experimentally discovered and computer proved**!

It is

$$t_n(1,1) = \frac{1}{2}F_n(a) + \frac{1}{2}F_n(-a),$$

where

$$F_n(a) := -\frac{a^n 2^{1-\omega}}{\omega\,\beta(n+\omega,-\omega)}\,{}_2\mathrm{F}_1\left(\omega,\omega;n+1+\omega;\frac{1}{2}\right)$$

while

$$\beta(n+1+\omega,-\omega) = \frac{\Gamma\,(n+1)}{\Gamma\,(n+1+\omega)\,\Gamma\,(-\omega)},$$

and

$$\omega = \frac{1-1/a}{2}.$$

12. More General Fractions

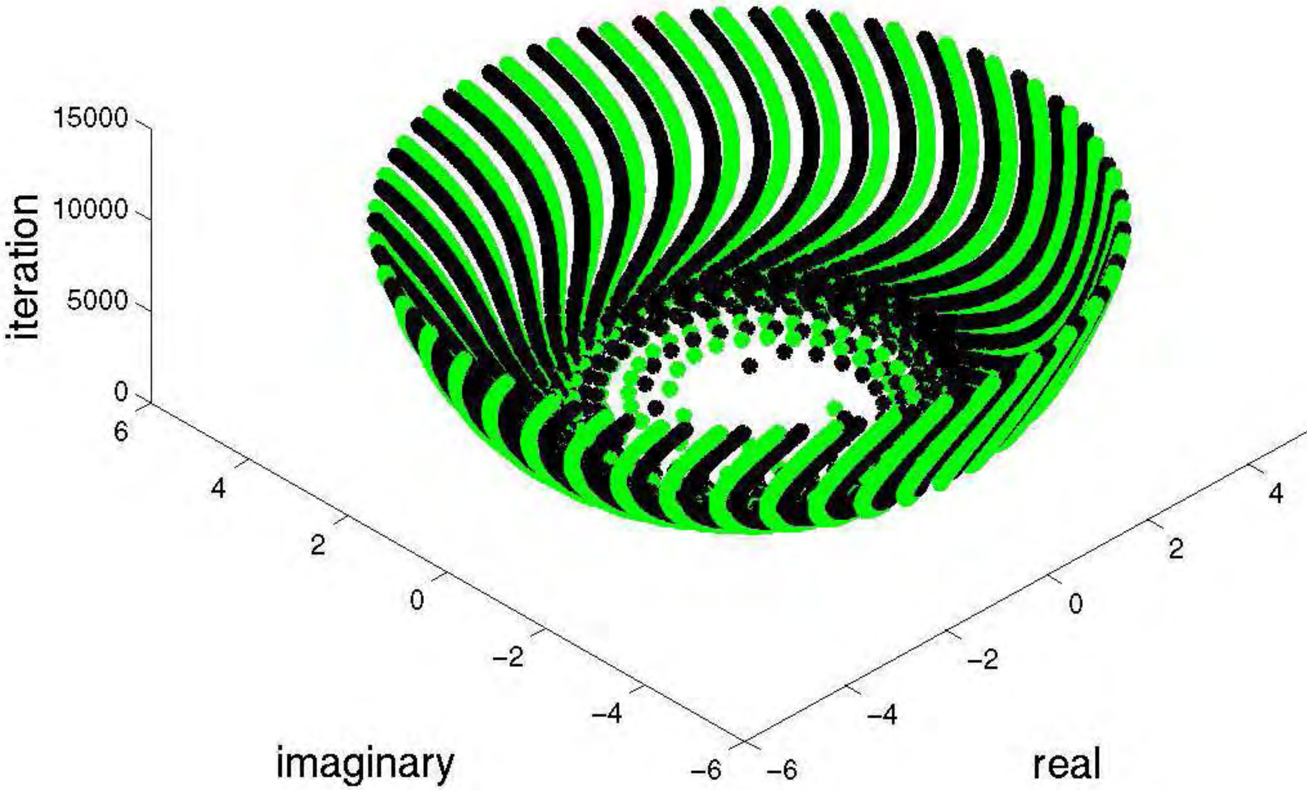
Study of $\mathcal{R}$ devolved to *hard but compelling* conjectures on complex dynamics, with many interesting *proven* and *unproven* generalizations (e.g., Borwein-Luke, 2004).

For *any* sequence $a \equiv (a_n)_{n=1}^{\infty}$, we consider continued fractions like

$$\mathcal{S}_1(a) = \cfrac{1^2 a_1^2}{1 + \cfrac{2^2 a_2^2}{1 + \cfrac{3^2 a_3^2}{1 + \ddots}}}$$

- We studied convergence properties for deterministic and random sequences (a_n).

- For the deterministic case the best results are for periodic sequences, satisfying $a_j = a_{j+c}$ for all j and some finite c.



A period three dynamical system
(odd and even iterates)

- The cases (i) $a_n = Const \in \mathbb{C}$, (ii) $a_n = -a_{n+1} \in \mathbb{C}$, (iii) $|a_{2n}| = 1,\ a_{2n+1} = i$, and (iv) $a_{2n} = a_{2m},\ a_{2n+1} = a_{2m+1}$ with $|a_n| = |a_m|\ \forall\ m, n \in \mathbb{N}$, were already covered.

13. Final Open Problems

- Again on the basis of numerical experiments, we acknowledge that some "deeper" AGM identity might hold.

  - There are pairs $\{a, b\}$ so

    $$\frac{\mathcal{R}_1(a,b) + \mathcal{R}_1(b,a)}{2} \neq \mathcal{R}_1\left(\frac{a+b}{2}, \sqrt{ab}\right)$$

    but *the LHS agrees* numerically with some variant, call it $\mathcal{S}_1((a+b)/2, \sqrt{ab})$, naively chosen as one of (3.1) or (3.2).

  - Such coincidences are remarkable, and difficult so far to predict.

- *We maintain hope that* there should ultimately be a comprehensive theory under which *the lovely AGM relation*— suitably modified—*holds for all complex values*.

1. What precisely is the domain of pairs for which $\mathcal{R}_1(a,b)$ converges, *and* some AGM holds?

2. Relatedly, when does the fraction depart from its various analytic representations?

3. What is the precise domain of validity of the sech formulae (3.1) and (3.2)? Is Conjecture 9.0 right?

4. While $\mathcal{R}(i) := \mathcal{R}_1(i,i)$ does not converge, the ψ-function representation of Section 4 has a definite value at $a = i$. Does some limit such as $\lim_{\epsilon \to 0} \mathcal{R}_1(i+\epsilon, i)$ exist and coincide?

5. Despite a host of closed forms for $\mathcal{R}(a) := \mathcal{R}_1(a,a)$, we know no *nontrivial* closed form for $\mathcal{R}_1(a,b)$ with $a \neq b$.

G.H. Hardy

All physicists and a good many quite respectable mathematicians are contemptuous about proof.

...

Beauty is the first test. There is no permanent place in the world for ugly mathematics.

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