Joint Mathematics Meetings
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New Orleans

Dalhousie Distributed Research Institute and Virtual Environment

Computer-assisted Discovery and Proof

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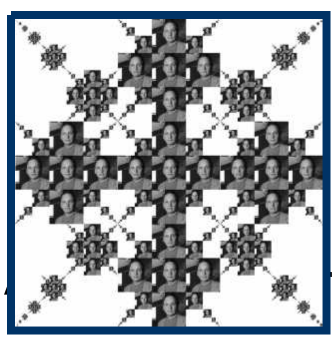
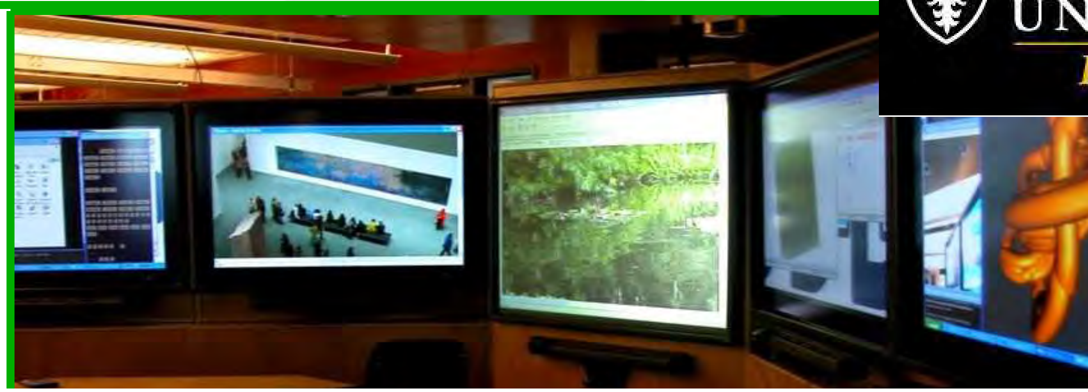
"Elsewhere Kronecker said ``In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas." ... I would rather say ``computations" than ``formulas", but my view is essentially the same."

Harold Edwards, *Essays in Constructive Mathematics*, 2004

DALHOUSIE UNIVERSITY
Inspiring Minds





AMS Special Session

Jan 5 2007

Revised 29/12/2006

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PART I. Numerical Experimentation

Computer-assisted Discovery and Proof of Generating Functions for Riemann's Zeta

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240 core Glooscap

Details in **Experimental Mathematics in Action**
Borwein, Bailey, Calkin, Girgensohn, Luke and Moll, A.K. Peters, 2007
- based on eponymous 2006 MAA short Course

"All truths are easy to understand once they are discovered; the point is to discover them." – Galileo Galilei

Algorithms Used in Experimental Mathematics

- Symbolic computation for algebraic and calculus manipulations.
- Integer-relation methods, especially the “PSLQ” algorithm.
- High-precision integer and floating-point arithmetic.
- High-precision evaluation of integrals and infinite series summations.
- The Wilf-Zeilberger algorithm for proving summation identities.
- Iterative approximations to continuous functions.
- Identification of functions based on graph characteristics.
- Graphics and visualization methods targeted to mathematical objects.

“High-Precision” or “Arbitrary Precision” Arithmetic

- High-precision integer arithmetic is required in symbolic computing packages.
- High-precision floating-point arithmetic is required to permit identification of mathematical constants using PSLQ or online constant recognition facilities.
- Most common requirement is for 200-500 digits, although more than 1,000-digit precision is sometimes required.
- One problem required 50,000-digit arithmetic.

"Equations are more important to me, because politics is for the present, but an equation is something for eternity." - Albert Einstein

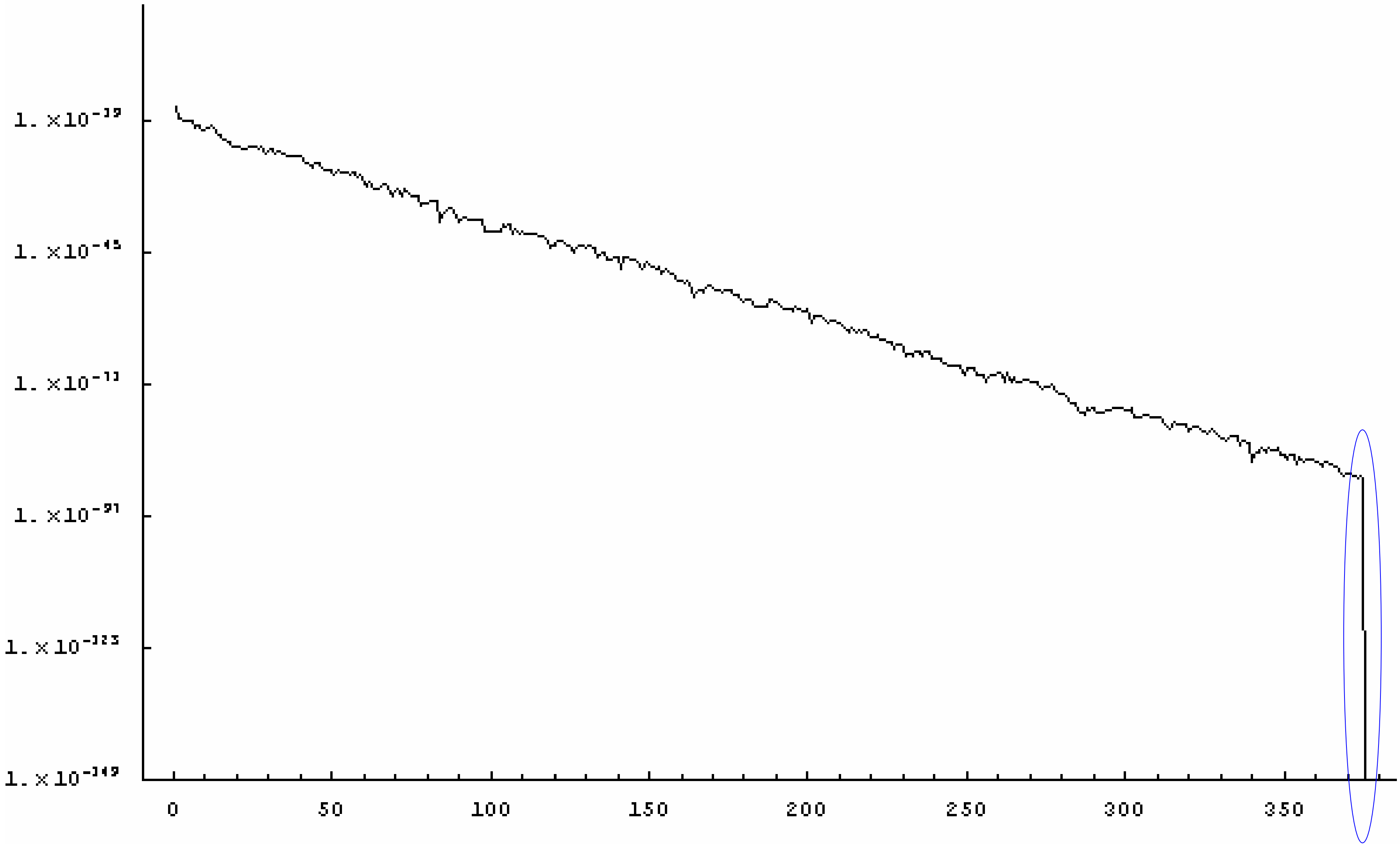
The PSLQ Integer Relation Algorithm

Let (x_n) be a vector of real numbers. An integer relation algorithm finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \;=\; 0$$

- At present the PSLQ algorithm of mathematician-sculptor *Helaman Ferguson* (featured in *Science* in October 2006) is the best-known integer relation algorithm
- PSLQ was named one of ten “algorithms of the century” by *Computing in Science and Engineering*.
- High precision arithmetic software is required: at least d £ n digits, where d is the size (in digits) of the largest of the integers a_k.

Decrease of $\min_j |A_j x|$ in PSLQ
(error versus iterations)



Peter Borwein
in front of
Helaman Ferguson's
work

CMS Meeting
December 2003
SFU Harbour Centre

Ferguson uses high tech tools and micro engineering at NIST to build monumental math sculptures

The David Borwein CMS Career Award

$$= {\sum_{n,m,p}}' \frac{(-1)^{n+m+p}}{\sqrt{n^2+m^2+p^2}}$$

This polished solid silicon bronze sculpture is inspired by the work of David Borwein, his sons and colleagues, on the conditional series above for salt, Madelung's constant. This series can be summed to give uncountably many constants; one is Madelung's constant for sodium chloride.

This constant is a period of an elliptic curve, a real surface in four dimensions. There are uncountably many ways to imagine that surface in three dimensions; one has negative gaussian curvature and is the tangible form of this sculpture. (As described by the artist.)

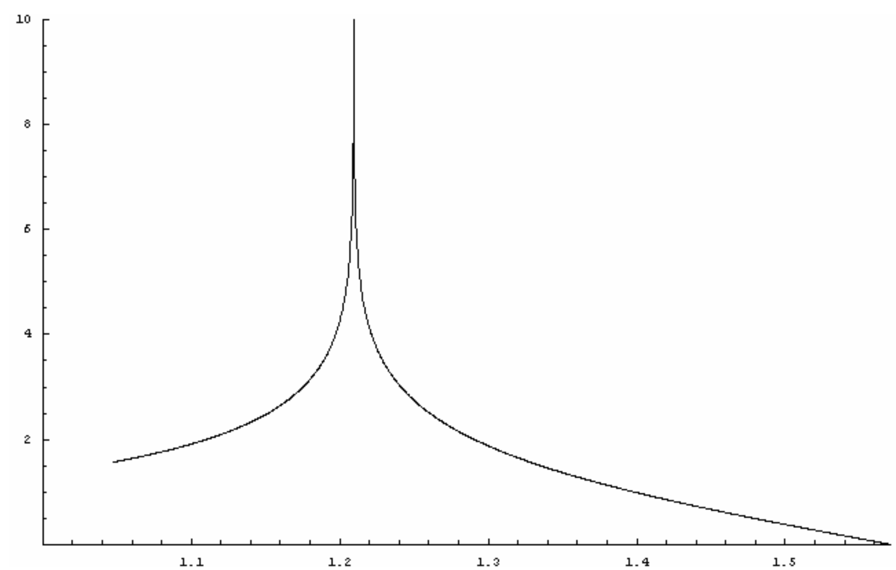
I. Extreme Quadrature (EQ)

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt$$
$$\stackrel{?}{=} \sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2} \right]$$

This arises in mathematical physics, from analysis of the volumes of *ideal tetrahedra* in hyperbolic space.

This "identity" has now been verified numerically to **20,000** digits, but no proof is known.

Note that the integrand function has a nasty singularity.



Extreme Quadrature … 20,000 Digits (50 Certified) on 1024 CPUs

- The integral was split at the nasty interior singularity
- The sum was `easy'.
- All fast arithmetic & function evaluation ideas used

Run-times and speedup ratios on the Virginia Tech G5 Cluster

CPUs	Init	Integral #1	Integral #2	Total	Speedup
1	*190013	*1534652	*1026692	*2751357	1.00
16	12266	101647	64720	178633	15.40
64	3022	24771	16586	44379	62.00
256	770	6333	4194	11297	243.55
1024	199	1536	1034	2769	993.63

Parallel run times (in seconds) and speedup ratios for the 20,000-digit problem

Expected and unexpected scientific spinoffs

- **1986-1996**. Cray used quartic-Pi to check machines in factory
- **1986**. Complex FFT sped up by factor of two
- **2002**. Kanada used hex-pi (20hrs not 300hrs to check computation)
- **2005**. Virginia Tech (this integral pushed the limits)
- **1995-** Math Resources (another lecture)

Further Equations

Define

$$J_n = \int_{n\pi/60}^{(n+1)\pi/60} \log\left|\frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}}\right| dt$$

Then

$$\begin{aligned} 0 \stackrel{?}{=} & -2J_2 - 2J_3 - 2J_4 + 2J_{10} + 2J_{11} + 3J_{12} \\ & +3J_{13} + J_{14} - J_{15} - J_{16} - J_{17} - J_{18} \\ & -J_{19} + J_{20} + J_{21} - J_{22} - J_{23} + 2J_{25} \end{aligned}$$

This has been verified to over **1000** digits. The interval in J_{23} includes the singularity.

"We [Kaplansky and Halmos] share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury."

(Irving Kaplansky, 1917-2006)

II. New Ramanujan-Like Identities

Guillera has recently found Ramanujan-like identities, including:

$$\begin{aligned}
\frac{128}{\pi^2} &= \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32}\right)^{2n} \\
\frac{32}{\pi^2} &= \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2}\right)^{2n} \\
\frac{32}{\pi^3} &= \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{32}\right)^{2n}.
\end{aligned}$$

where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$$

Guillera proved the first two of these using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods.

Are there any higher-order analogues?

Not as far as we can tell

Searches for Additional Formulas

We searched for additional formulas of either the following forms:

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} r(n)^{2m+1}(p_0 + p_1 n + \cdots + p_m n^m)\alpha^{2n}$$

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} (-1)^n r(n)^{2m+1}(p_0 + p_1 n + \cdots + p_m n^m)\alpha^{2n}.$$

where c is some linear combination of

$1, 2^{1/2}, 2^{1/3}, 2^{1/4}, 2^{1/6}, 4^{1/3}, 8^{1/4}, 32^{1/6}, 3^{1/2}, 3^{1/3}, 3^{1/4}, 3^{1/6}, 9^{1/3}, 27^{1/4}, 243^{1/6}, 5^{1/2}, 5^{1/4}, 125^{1/4}, 7^{1/2}, 13^{1/2}, 6^{1/2}, 6^{1/3}, 6^{1/4}, 6^{1/6}, 7, 36^{1/3}, 216^{1/4}, 7776^{1/6}, 12^{1/4}, 108^{1/4}, 10^{1/2}, 10^{1/4}, 15^{1/2}$

where each of the coefficients p_i is a linear combination of

$1, 2^{1/2}, 3^{1/2}, 5^{1/2}, 6^{1/2}, 7^{1/2}, 10^{1/2}, 13^{1/2}, 14^{1/2}, 15^{1/2}, 30^{1/2}$

and where α is chosen as one of the following:

$1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, \sqrt{5}-2, (2-\sqrt{3})^2, 5\sqrt{13}-18, (\sqrt{5}-1)^4/128, (\sqrt{5}-2)^4, (2^{1/3}-1)^4/2, 1/(2\sqrt{2}), (\sqrt{2}-1)^2, (\sqrt{5}-2)^2, (\sqrt{3}-\sqrt{2})^4$

Relations Found by PSLQ
(in addition to Guillera's three relations)

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} r(n)^3(1+6n)\left(\frac{1}{2}\right)^{2n}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} r(n)^3(5+42n)\left(\frac{1}{8}\right)^{2n}$$

$$\frac{12^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-15+9\sqrt{3}-36n+24\sqrt{3}n)\left(2-\sqrt{3}\right)^{4n}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-1+5\sqrt{5}+30n+42\sqrt{5}n)\left(\frac{(\sqrt{5}-1)^4}{128}\right)^{2n}$$

$$\frac{5^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3(-525+235\sqrt{5}-1200n+540\sqrt{5}n)\left(\sqrt{5}-2\right)^{8n}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(1+6n)\left(\frac{1}{2\sqrt{2}}\right)^{2n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(-5+4\sqrt{2}-12n+12\sqrt{2}n)\left(\sqrt{2}-1\right)^{4n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(23-10\sqrt{5}+60n-24\sqrt{5}n)\left(\sqrt{5}-2\right)^{4n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3(177-72\sqrt{6}+420n-168\sqrt{6}n)\left(\sqrt{3}-\sqrt{2}\right)^{8n}$$

✓all are in Ramanujan (Pi and the AGM)

Proofs?

Echoes of the elliptic theory in

Pi and the AGM

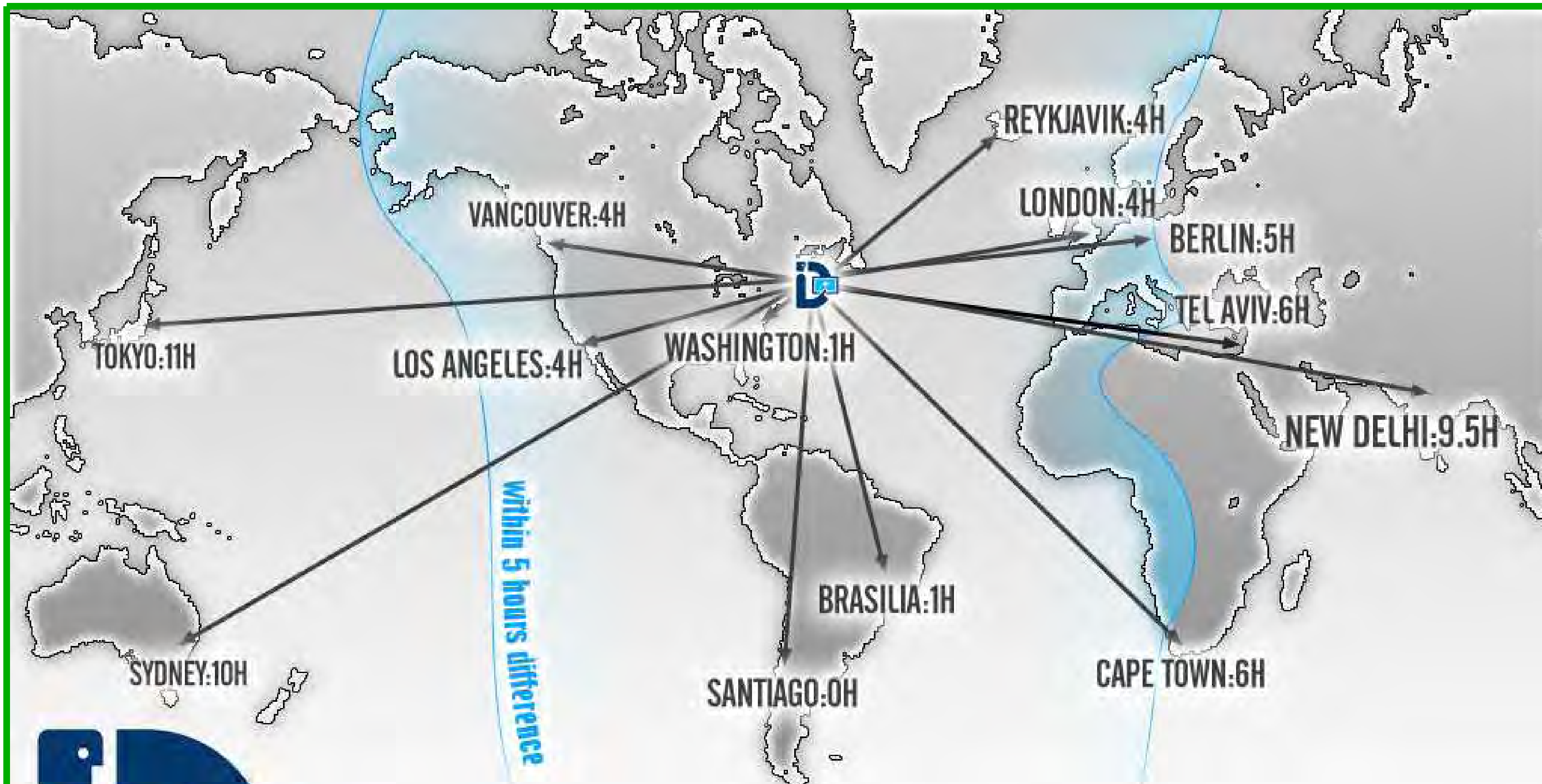
explain the various series for $1/\pi$.

Details are in given

Experimental Mathematics in Action.

"No. I have been teaching it all my life, and I do not want to have my ideas upset."

Isaac Todhunter (1820-1884) recording *Maxwell's response to being asked whether he would like to see an experimental demonstration of conical refraction.*



D-Drive's Nova Scotia location lends us unusual freedom when interacting globally. Many cities around the world are close enough in a chronological sense to comfortably accommodate real-time collaboration.

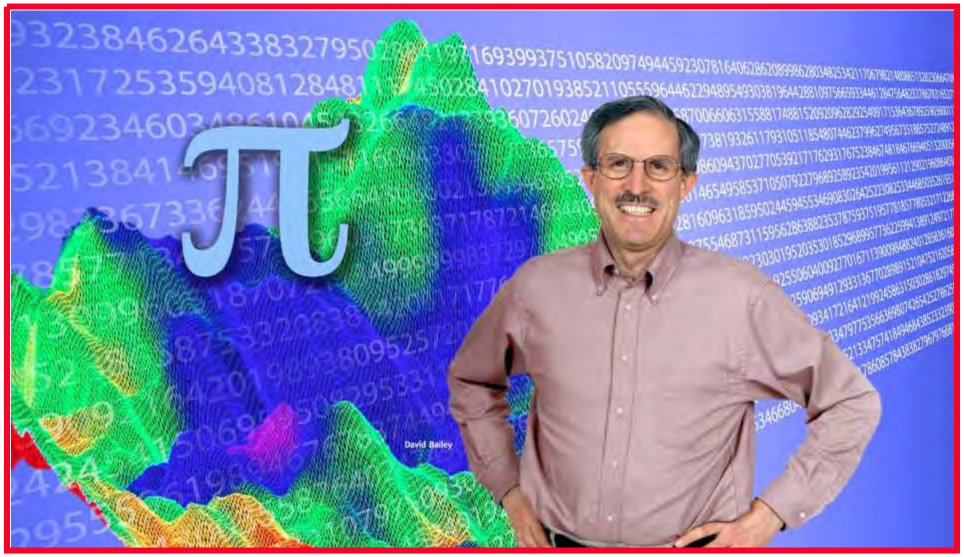
Drive

Dalhousie Distributed Research Institute and Virtual Environment

Part II.
Experiment and Proof

JM Borwein and DH Bailey

with DA Bradley



"Anyone who is not shocked by quantum theory has not understood a single word." - Niels Bohr

The Wilf-Zeilberger Algorithm for Proving Identities

- A slick, computer-assisted proof scheme to prove certain types of identities
- Provides a nice complement to PSLQ

  - **PSLQ and the like permit one to discover new identities but do not constitute rigorous proof**

  - **W-Z methods permit one to prove certain types of identities but do not suggest any means to discover the identity**

"The formulas move in advance of thought, while the intuition often lags behind; in the oft-quoted words of d'Alembert, ***"L'algebre est genereuse, elle donne souvent plus qu'on lui demande."*** " (Edward Kasner, 1905)

Example Usage of W-Z

Consider these experimentally-discovered identities (the later from Part I):

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^4}{2^{16n}} \left(120n^2 + 34n + 3\right) = \frac{32}{\pi^2}$$

$$\boxed{B = 4A}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} \left(820n^2 + 180n + 13\right) = \frac{128}{\pi^2}$$

Guillera ***cunningly*** started by defining

$$G(n,k) = \frac{(-1)^k}{2^{16n}2^{4k}} \left(120n^2 + 84nk + 34n + 10k + 3\right) \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}$$

He then used the **EKHAD** software package to obtain the companion

$$F(n,k) = \frac{(-1)^k 512}{2^{16n}2^{4k}} \frac{n^3}{4n-2k-1} \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}$$

Example Usage of W-Z, II

When we define

$$H(n,k) \;=\; F(n+1,n+k)+G(n,n+k)$$

Zeilberger's theorem gives the identity

$$\sum_{n=0}^{\infty} G(n,0) \;=\; \sum_{n=0}^{\infty} H(n,0)$$

which when written out is

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4\binom{4n}{2n}}{2^{16n}}\left(120n^2+34n+3\right) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n+7}}\frac{(n+1)^3}{2n+3}\frac{\binom{2n+2}{n+1}^4\binom{2n}{n}^3\binom{2n+4}{n+2}}{\binom{2n+2}{n}\binom{2n+1}{n+1}^2}$$

$$+\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n}}\left(204n^2+44n+3\right)\binom{2n}{n}^5 \;=\; \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n\binom{2n}{n}^5}{2^{20n}}\left(820n^2+180n+13\right)$$

Now for integer k

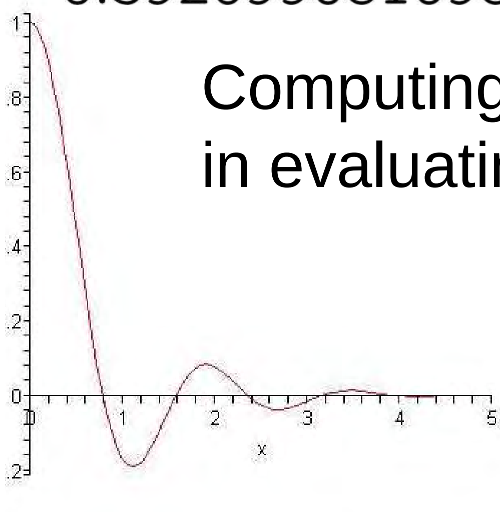
$$\sum_{n=0}^{\infty} G(n,k) \;=\; \sum_{n=0}^{\infty} G(n,k+1)$$

and so for all real k: taking the limit at t=1/2 completes the proof.

IIIa. A Cautionary Example

These **constants agree to 42 decimal digits** accuracy,
but are **NOT** equal:

$$\int_0^\infty \cos(2x) \prod_{n=0}^{\infty} \cos(x/n)\, dx =$$
$$0.39269908169872415480783042290993786052464\mathbf{543418723}\ldots$$
$$\frac{\pi}{8} =$$
$$0.39269908169872415480783042290993786052464\mathbf{617492189}\ldots$$



Computing this integral is nontrivial, due largely to difficulty in evaluating the integrand function to high precision.

Fourier analysis explains this happens when a hyperplane meets a hypercube (LP) …

$$\sum_k^n 1/k > 2$$

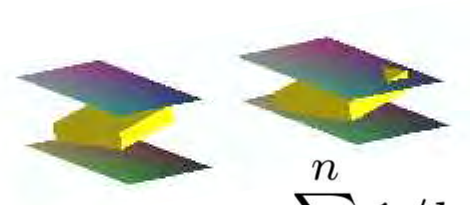
IIIb. A Cautionary Example

Relatedly, very recently, Baillie, D. Borwein and J. Borwein discovered and showed why

$$\int_0^\infty \operatorname{sinc}^N(x)\,dx - \sum_{n=1}^{\infty} \operatorname{sinc}^N(n) = \frac{1}{2}$$

exactly for N=1, 2,.., 6. For N>6 they differ by a polynomial of degree N in π.

•The integral is always a rational multiple of π.



$$\sum_{k}^{n} 1/k > 2$$

IV. Apery-Like Summations

The following formulas for $\zeta(n)$ have been known for many decades or more:

$$
\begin{aligned}
\zeta(2) &= 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}, \\
\zeta(3) &= \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}}, \\
\zeta(4) &= \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}.
\end{aligned}
$$

These results have led many to speculate that

$$
Q_5 := \zeta(5) / \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}}
$$

might be some nice rational or algebraic value.

Sadly, PSLQ calculations have established that if Q_5 satisfies a polynomial with **degree** at most **25**, then at least **one coefficient** has **380** digits.

Nothing New under the Sun

Margo Kondratieva found a formula of Markov in 1890:

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)^3} \;=\; \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \, (n!)^6}{(\,2n+1)!} \times \frac{\left(5\,(n+1)^2 + 6\,(a-1)\,(n+1) + 2\,(a-1)^2\right)}{\prod_{k=0}^{n} (a+k)^4}.$$

Note: *Maple* establishes this identity as

$$-1/2\,\Psi\,(2,a) = -1/2\,\Psi\,(2,a) - \zeta\,(3) + 5/4\,{}_4\mathrm{F}_3\,([1,1,1,1],[3/2,2,2],-1/4)$$

Hence

$$\zeta(4) = -\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\binom{2\,m}{m}\, m^4} + \frac{10}{3} \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \sum_{k=1}^{m} \frac{1}{k}}{\binom{2\,m}{m}\, m^3}.$$

The case a=0 above is Apery's formula for $\zeta(3)$!

Apery-Like Relations Found Using Integer Relation Methods

$$\begin{aligned}
\zeta(5) &= 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2},\\
\zeta(7) &= \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\\
\zeta(9) &= \frac{9}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^9\binom{2k}{k}} - \frac{5}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2} + 5\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^5\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\\
&\quad + \frac{45}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^6} - \frac{25}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\sum_{j=1}^{k-1}\frac{1}{j^2},\\
\zeta(11) &= \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{11}\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\\
&\quad - \frac{75}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^8} + \frac{125}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\sum_{i=1}^{k-1}\frac{1}{i^4}
\end{aligned}$$

Formulas for 7 and 11 were found by JMB and David Bradley; 5 and 9 by Kocher 25 years ago, as part of the general formula:

$$\sum_{k=1}^{\infty}\frac{1}{k(k^2-x^2)} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\frac{5k^2-x^2}{k^2-x^2}\prod_{m=1}^{k-1}\left(1-\frac{x^2}{m^2}\right)$$

Newer (2005) Results

Using **bootstrapping** and the "**Pade/pade**" function JMB and Dave Bradley then found the following remarkable result (1996):

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1-x^4/k^4)} = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}(1-x^4/k^4)} \prod_{m=1}^{k-1}\left(\frac{1+4x^4/m^4}{1-x^4/m^4}\right)$$

Following an analogous – but more deliberate – experimental-based procedure, we have obtained a similar general formula for $\zeta(2n+2)$ that is pleasingly parallel to above:

$$\sum_{k=1}^{\infty} \frac{1}{k^2-x^2} = 3\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}\left(1-x^2/k^2\right)} \prod_{m=1}^{k-1}\left(\frac{1-4\,x^2/m^2}{1-x^2/m^2}\right)$$

Note that this gives an *Apery-like formula* for $\zeta(2n)$, since the LHS equals

$$\sum_{n=0}^{\infty} \zeta(2n+2)x^{2n} = \frac{1-\pi x \cot(\pi x)}{2x^2}$$

- We sketch our experimental discovery of this in the new few slides

BBB, Exp Mathematics, 15 (2006), 281-289.

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The Experimental Scheme

1. We first supposed that $\zeta(2n+2)$ is a rational combination of terms of the form:

$$\sigma(2r;[2a_1,\cdots,2a_N]) \;:=\; \sum_{k=1}^{\infty} \frac{1}{k^{2r}\binom{2k}{k}} \prod_{i=1}^{N} \sum_{n_i=1}^{k-1} \frac{1}{n_i^{2a_i}}$$

where $r + a_1 + a_2 + \ldots + a_N = n + 1$ and a_i are listed increasingly.

2. We can then write:

$$\sum_{n=0}^{\infty} \zeta(2n+2)\,x^{2n} \;\stackrel{?}{=}\; \sum_{n=0}^{\infty}\sum_{r=1}^{n+1}\sum_{\pi\in\Pi(n+1-r)} \alpha(\pi)\,\sigma(\mathbf{2r};\mathbf{2}\pi)\,x^{2n}$$

where $\Pi(m)$ denotes the additive partitions of m.

3. We can then deduce that

$$\sum_{n=0}^{\infty} \zeta(2n+2)\,x^{2n} \;=\; \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2-x^2)} P_k(x)$$

where $P_k(x)$ are functions **whose general form we hope to discover:**

The Bootstrap Process

$$\begin{aligned}
\zeta(2) &= 3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^2} = 3\sigma(2,[0]),\\
\zeta(4) &= \mathbf{3}\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^4} - 9\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-2}}{\binom{2k}{k}k^2} = \mathbf{3}\sigma(4,[0]) - 9\sigma(2,[2])\\
\zeta(6) &= \mathbf{3}\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^6} - \mathbf{9}\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-2}}{\binom{2k}{k}k^4} - \frac{45}{2}\sum_{k=1}^{\infty}\frac{\sum_{j=1}^{k-1} j^{-4}}{\binom{2k}{k}k^2}\\
&\quad + \frac{27}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\frac{\sum_{i=1}^{k-1} i^{-2}}{j^2\binom{2k}{k}k^2},\\
&= \mathbf{3}\sigma(6,[]) - \mathbf{9}\sigma(4,[2]) - \frac{45}{2}\sigma(2,[4]) + \frac{27}{2}\sigma(2,[2,2])\\
\zeta(8) &= \mathbf{3}\sigma(8,[]) - \mathbf{9}\sigma(6,[2]) - \frac{45}{2}\sigma(4,[4]) + \frac{27}{2}\sigma(4,[2,2]) - 63\sigma(2,[6])\\
&\quad + \frac{135}{2}\sigma(2,[4,2]) - \frac{27}{2}\sigma(2,[2,2,2])\\
\zeta(10) &= \mathbf{3}\sigma(10,[]) - \mathbf{9}\sigma(8,[2]) - \frac{45}{2}\sigma(6,[4]) + \frac{27}{2}\sigma(6,[2,2]) - \mathbf{63}\sigma(4,[6])\\
&\quad + \frac{135}{2}\sigma(4,[4,2]) - \frac{27}{2}\sigma(4,[2,2,2]) - \frac{765}{4}\sigma(2,[8]) + 189\sigma(2,[6,2])\\
&\quad + \frac{675}{8}\sigma(2,[4,4]) - \frac{405}{4}\sigma(2,[4,2,2]) + \frac{81}{8}\sigma(2,[2,2,2,2])
\end{aligned}$$

Coefficients Obtained

Partition	Alpha	Partition	Alpha	Partition	Alpha
[empty]	3/1	1	-9/1	2	-45/2
1,1	27/2	3	-63/1	2,1	135/2
1,1,1	-27/2	4	-765/4	3,1	189/1
2,2	675/8	2,1,1	-405/4	1,1,1,1	81/8
5	-3069/5	4,1	2295/4	3,2	945/2
3,1,1	-567/2	2,2,1	-2025/8	2,1,1,1	405/4
1,1,1,1,1	-243/40	6	-4095/2	5,1	9207/5
4,2	11475/8	4,1,1	-6885/8	3,3	1323/2
3,2,1	-2835/2	3,1,1,1	567/2	2,2,2	-3375/16
2,2,1,1	6075/16	2,1,1,1,1	-1215/16	1,1,1,1,1,1	243/80
7	-49149/7	6,1	49140/8	5,2	36828/8

Partition	Alpha	Partition	Alpha	Partition	Alpha
5,1,1	-27621/10	4,3	32130/8	4,2,1	-34425/8
4,1,1,1	6885/8	3,3,1	-15876/8	3,2,2	-14175/8
3,2,1,1	17010/8	3,1,1,1,1	-1701/8	2,2,2,1	10125/16
2,2,1,1,1	-6075/16	2,1,1,1,1,1	729/16	1,1,1,1,1,1,1	-729/560
8	-1376235/56	7,1	1179576/56	6,2	859950/56
6,1,1	-515970/56	5,3	902286/70	5,2,1	-773388/56
5,1,1,1	193347/70	4,4	390150/64	4,3,1	-674730/56
4,2,2	-344250/64	4,2,1,1	413100/64	4,1,1,1,1	-41310/64
3,3,2	-277830/56	3,3,1,1	166698/56	3,2,2,1	297675/56
3,2,1,1,1	-119070/56	3,1,1,1,1,1	10206/80	2,2,2,2	50625/128
2,2,2,1,1	-60750/64	2,2,1,1,1,1	18225/64	2,1,1,1,1,1,1	-1458/64
1,1,1,1,1,1,1,1	2187/4480				

Resulting Polynomials

$$P_3(x) \approx 3 - \frac{45}{4}x^2 - \frac{45}{16}x^4 - \frac{45}{64}x^6 - \frac{45}{256}x^8 - \frac{45}{1024}x^{10} - \frac{45}{4096}x^{12} - \frac{45}{16384}x^{14} - \frac{45}{65536}x^{16}$$

$$P_4(x) \approx 3 - \frac{49}{4}x^2 + \frac{119}{144}x^4 + \frac{3311}{5184}x^4 + \frac{38759}{186624}x^6 + \frac{384671}{6718464}x^8 + \frac{3605399}{241864704}x^{10} + \frac{33022031}{8707129344}x^{12} + \frac{299492039}{313456656384}x^{14}$$

$$P_5(x) \approx 3 - \frac{205}{16}x^2 + \frac{7115}{2304}x^4 + \frac{207395}{331776}x^6 + \frac{4160315}{47775744}x^8 + \frac{74142995}{6879707136}x^{10} + \frac{1254489515}{990677827584}x^{12} + \frac{20685646595}{142657607172096}x^{14} + \frac{336494674715}{20542695432781824}x^{16}$$

$$P_6(x) \approx 3 - \frac{5269}{400}x^2 + \frac{6640139}{1440000}x^4 + \frac{1635326891}{5184000000}x^6 - \frac{5944880821}{18662400000000}x^8 - \frac{212874252291349}{67184640000000000}x^{10} - \frac{14143638495690738l}{241864704000000000000}x^{12} - \frac{70524260274859115989}{870712934400000000000000}x^{14} - \frac{315334571688192146555 41}{31345665638400000000000000000}x^{16}$$

$$P_7(x) \approx 3 - \frac{5369}{400}x^2 + \frac{8210839}{1440000}x^4 - \frac{199644809}{5184000000}x^6 - \frac{680040118121}{18662400000000}x^8 - \frac{278500311775049}{67184640000000000}x^{10} - \frac{841367152178726 81}{241864704000000000000}x^{12} - \frac{22363377813883431689}{870712934400000000000000}x^{14} - \frac{556009084026391142884 1}{31345665638400000000000000000}x^{16}$$

After Using "Pade" Function in Mathematica or Maple

$$
\begin{aligned}
P_1(x) &\stackrel{?}{=} 3\\
P_2(x) &\stackrel{?}{=} \frac{3(4x^2-1)}{(x^2-1)}\\
P_3(x) &\stackrel{?}{=} \frac{12(4x^2-1)}{(x^2-4)}\\
P_4(x) &\stackrel{?}{=} \frac{12(4x^2-1)(4x^2-9)}{(x^2-4)(x^2-9)}\\
P_5(x) &\stackrel{?}{=} \frac{48(4x^2-1)(4x^2-9)}{(x^2-9)(x^2-16)}\\
P_6(x) &\stackrel{?}{=} \frac{48(4x^2-1)(4x^2-9)(4x^2-25)}{(x^2-9)(x^2-16)(x^2-25)}\\
P_7(x) &\stackrel{?}{=} \frac{192(4x^2-1)(4x^2-9)(4x^2-25)}{(x^2-16)(x^2-25)(x^2-36)}
\end{aligned}
$$

… and factoring

which immediately suggests the general form:

$$\sum_{n=0}^{\infty} \zeta(2n+2)x^{2n} \stackrel{?}{=} 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2-x^2)} \prod_{m=1}^{k-1} \frac{4x^2-m^2}{x^2-m^2}$$

Several Confirmations of Z(2n+2)=Zeta(2n+2) Formula

- We symbolically computed the power series coefficients of the LHS and the RHS , and verified that they agree up to the term with $\mathbf{x^{100}}$.

- We verified that Z(1/6), Z(1/2), Z(1/3), Z(1/4) give numerically correct values (analytic values are known).

- We then affirmed that the formula gives numerically correct results for 100 pseudorandomly chosen arguments
  – to high precision near radius of convergence

We subsequently proved this formula two different ways, including using the **Wilf-Zeilberger method**….

To SUMMARIZE ….

iDrive

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Euler
(1707-73)

1. via PSLQ to 50,000 digits
(250 terms)

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \cdots$$

Riemann
(1826-66)

$$\mathcal{Z}(x) = 3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2}$$

$$= \sum_{k=0}^{\infty} \zeta(2k+2) x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}$$

$$= \frac{1 - \pi x \cot(\pi x)}{2x^2}$$

1

2. reduced as hoped

2

2005 Bailey, Bradley & JMB *discovered and proved* - in 3Ms - three *equivalent* binomial identities

$$3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2 - m^2}{n^2 - m^2}}{\binom{2k}{k}\left(k^2 - n^2\right)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}$$

3

$$ {}_3F_2\left(\begin{matrix} 3n, n+1, -n \\ 2n+1, n+1/2 \end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}$$

3. was easily **computer proven**
(Wilf-Zeilberger)
?human/MAA?

Automating the Steps?

$$\sigma(2r;[2a_1,\cdots,2a_N]) \;:=\; \sum_{k=1}^{\infty}\frac{1}{k^{2r}\binom{2k}{k}}\prod_{i=1}^{N}\sum_{n_i=1}^{k-1}\frac{1}{n_i^{2a_i}}$$

1. **HUMAN CONJECTURE** "There is a generating function for $\zeta(2n+2)$ in terms of σ"

2. **DATA COLLECTION** via PSLQ and *Maple* or *Mathematica*

3. PATTERN DETECTION

4. **STRUCTURE DETERMINATION** via *Maple/Mathematica*
   - INFINITE IDENTITY

5. **ANALYTIC CONTINUATION** via *Gosper*
   - FINITE IDENTITY I

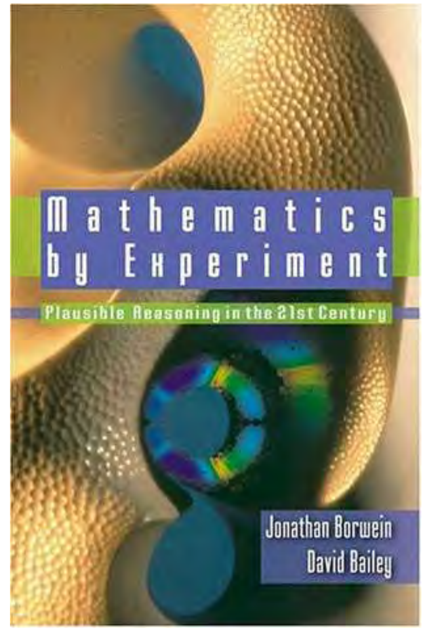
6. **HUMAN PURIFICATION**
   - FINITE IDENTITY II

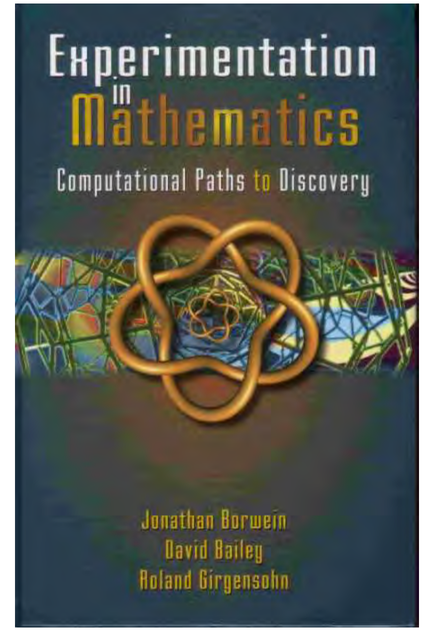
***7. WILF-ZEILBERGER* PROOF**

Summary

New techniques now permit integrals, infinite series sums and other entities to be evaluated to high precision (hundreds or thousands of digits), thus permitting PSLQ-based schemes to discover new identities.

These methods typically do not suggest proofs, but often it is much easier to find a proof when one "knows" the answer is right.





Details are in *Experimental Mathematics in Action*, or in these two slightly older books by Borwein, Bailey and (for vol 2) Girgensohn (also on **CD**). A "Reader's Digest" version of these two books is at **www.experimentalmath.info**.

"The plural of 'anecdote' is not 'evidence'."
- Alan L. Leshner, *Science* publisher