Integer Powers of Arcsin

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Abstract: New simple nested sum representations for powers of the arcsin function are given. This generalization of Ramanujan's work makes connections to finite binomial sums and polylogarithms.

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1 Introduction

This note discovers, derives, and then studies, *simple* closed-form Taylor series expressions for integer powers of $\arcsin(x)$. Specifically, we show that for $|x| \le 2$ and $N = 1, 2, \ldots$

$$\frac{\arcsin^{2N}\left(\frac{x}{2}\right)}{(2N)!} = \sum_{k=1}^{\infty} \frac{H_N(k)}{\binom{2k}{k} k^2}\, x^{2\,k}, \tag{1}$$

where $H_1(k) = 1/4$ and

$$H_{N+1}(k) := \frac{1}{4} \sum_{n_1=1}^{k-1} \frac{1}{(2n_1)^2} \sum_{n_2=1}^{n_1-1} \frac{1}{(2n_2)^2} \cdots \sum_{n_N=1}^{n_{N-1}-1} \frac{1}{(2n_N)^2},$$

and also that for $|x| \le 2$ and $N = 0, 1, 2, \ldots$

$$\frac{\arcsin^{2N+1}\left(\frac{x}{2}\right)}{(2N+1)!} = \sum_{k=0}^{\infty} \frac{G_N(k)\binom{2\,k}{k}}{2(2k+1)4^{2k}} x^{2k+1}, \tag{2}$$

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where $G_0(k) = 1$ and

$$G_N(k) := \sum_{n_1=0}^{k-1} \frac{1}{(2n_1+1)^2} \sum_{n_2=0}^{n_1-1} \frac{1}{(2n_2+1)^2} \cdots \sum_{n_N=0}^{n_{N-1}-1} \frac{1}{(2n_N+1)^2}.$$

The convention is that the sum is zero if the starting index exceeds the finishing index.

Nested sums are not new. The last decade saw many interesting results concerning *Euler sums* or *multizeta values*, wonderful generalizations of the classical ζ-function, whose discovery can be traced to a letter from Goldbach to Euler [2, pp.99–100] and [3, Chapter 3]—a letter that played a seminal role in the discovery of the ζ-function.

When Gauss was criticized for the lack of motivation in his writings, he remarked that the architects of great cathedrals do not obscure the beauty of their work by leaving the scaffolding in place after the construction has been completed. While we find (1) and (2) worthy of undistracted attention, in truth their discovery was greatly facilitated by the use of *experimental mathematics*—the relatively new approach to doing mathematical research with the intelligent use of computers. This perspective is elucidated throughout this paper. It is also illustrative of the changing speed of mathematical communication that the special cases (3), (4), (5), and (6) given below are already online at [11].

2 Experiments and Proofs

The first identity below is very well known:

$$\arcsin^2\left(\frac{x}{2}\right) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{x^{2k}}{\binom{2k}{k}k^2}. \tag{3}$$

It is explored at some length in [4, pp.384–386]. While it is seen in various calculus books (see [6, pp.88–90], where the series for $\arcsin^3(x)$ is also proven), it dates back at least two centuries and was given by Ramanujan among many others; see [10, pp.262–63]. As often in Mathematics, history is complicated. Equation (3) has been rediscovered repeatedly. For example, an equivalent form is elegantly solved as a 1962 MAA Monthly problem ("A Well-Known Constant", Problem E 1509, p.232). We quote in extenso, the editors' attempts to trace the history of the formula:

> The series was located in the *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, 6.42 No. 5, p. 122; Chrystal, *Algebra*, vol.

2. 1906 ed, Ex. xx, No. 7, p. 335, (cites Pfaff as source); Bromwich, *An Introduction to the Theory of rnfinite Series*, 1908 ed, Prob. 2, p. 197 (claims known to Euler); Knopp, *Theory and Application of Infinite Series*, Ex. 123, Chap. VIII, p.271; Schuh, *Leerboek der Differtiaal en Integraalrekening*, vol. 2, pp. 154–6; Hobson, *Treatise on Plane Trigonometry*, eqs. 20, 21, 22, pp. 279–80; M.R. Speigel, this Monthly, 60 (1953) 243–7; Taylor, *Advanced Calculus*, p. 632; Edwards, *Differential Calculus for Beginners* (1899), p.78.

Note the appearance in the Monthly itself in 1953.

The second identity, slightly rewritten (see [10]), is less well known:

$$\arcsin^4\left(\frac{x}{2}\right) = \frac{3}{2}\sum_{k=1}^{\infty}\left\{\sum_{m=1}^{k-1}\frac{1}{m^2}\right\}\frac{x^{2k}}{\binom{2k}{k}k^2}, \tag{4}$$

and when compared, they hint at the third and fourth identities below—subsequently confirmed numerically—from the prior if flimsy pattern:

$$\arcsin^6\left(\frac{x}{2}\right) = \frac{45}{4}\sum_{k=1}^{\infty}\left\{\sum_{m=1}^{k-1}\frac{1}{m^2}\sum_{n=1}^{m-1}\frac{1}{n^2}\right\}\frac{x^{2k}}{\binom{2k}{k}k^2}. \tag{5}$$

$$\arcsin^8\left(\frac{x}{2}\right) = \frac{315}{2}\sum_{k=1}^{\infty}\left\{\sum_{m=1}^{k-1}\frac{1}{m^2}\sum_{n=1}^{m-1}\frac{1}{n^2}\sum_{p=1}^{n-1}\frac{1}{p^2}\right\}\frac{x^{2k}}{\binom{2k}{k}k^2}. \tag{6}$$

Reassured by this confirmation we conjectured that in general

$$\frac{\arcsin^{2N}\left(\frac{x}{2}\right)}{(2N)!} = \sum_{k=1}^{\infty}\frac{H_N(k)}{\binom{2k}{k}k^2}\,x^{2k}, \tag{7}$$

where $H_{N+1}(k)$ is as in (1) above. Subsequently, we were naturally led to discovering the corresponding odd formulas.

We next provide a proof of both (1), equivalently (7), and (2).

Proof. ((1) and (2)). The formulae for $\arcsin^k(x)$ with $2 \le k \le 4$ are given on pages 262–63 of [10], and Berndt comments that [6] is the best source he knows for $k = 2$ and 3. Berndt's proof also implicitly gives our desired result since he establishes, via a differential equation argument, that for all real parameters a one has

$$e^{a\,\arcsin(x)} = \sum_{n=0}^{\infty} c_n(a)\,\frac{x^n}{n!} \tag{8}$$

where

$$c_{2n+1}(a) := a \prod_{k=1}^{n} \left(a^2 + (2k-1)^2\right), \qquad c_{2n}(a) := \prod_{k=1}^{n} \left(a^2 + (2k-2)^2\right).$$

Now expanding the power of a^n on each side of (8) provides the asserted formula. Note that (8) is equivalent to the somewhat-less-elegant if more-concise [7, Formula 10.49.33], specifically,

$$\sum_{k=0}^{\infty} \frac{(ia)_{k/2}}{k!(ia+1)_{-k/2}} (-ix)^k = \exp\left[2a \sin^{-1}\left(\frac{x}{2}\right)\right]. \tag{9}$$

Another proof can be obtained from the hypergeometric identity

$$\frac{\sin(ax)}{a\sin(x)} = {}_2\mathrm{F}_1\left(\frac{1+a}{2}, \frac{1-a}{2}; \frac{3}{2}; \sin^2(x)\right)$$

given in [4, Exercise 16, p.189]. □

Maple can also prove identities such as (8) as the following code shows.

```
> ce:=n->product(a^2+(2*k)^2,k=0..n-1):
> co:=n->a*product(a^2+(2*k+1)^2,k=0..n-1):
> sum(ce(n)*x^(2*n)/(2*n)!,n=0..infinity) assuming x>0;
                          cosh(a arcsin(x))

> simplify(expand(sum(co(n)*x^(2*n+1)/(2*n+1)!,n=0..infinity)))
  assuming x>0;
                          sinh(a arcsin(x))
```

A — necessarily equivalent — formula for powers of arcsin is listed by Hansen in [7, 88.2.2],

$$\sum_{n=0}^{\infty} x^{2n} \prod_{k=1}^{m-1} \left\{ \sum_{n_k=0}^{n_{k-1}} \frac{(2n_{k-1}-2n_k)!}{[(n_{k-1}-n_k)!]^2(2n_{k-1}-2n_k+1)} 2^{2n_k-2n_{k-1}} \right\} \frac{(2n_{m-1})!}{(n_{m-1}!)^2(2n_{m-1}+1)} 2^{-2n_{m-1}}$$

$$= \left(\frac{\sin^{-1} x}{x}\right)^m \tag{10}$$

but its relation to (1) and (2) is not obvious; our nested-sum representations are definitely more elegant. While Ramanujan listed only the first two cases in his notebooks [10, pp.262–63], his previous entries suggest an approach for any power was being assembled. One can only imagine how much farther his intuition would have taken him if he had today's computational power!

Powers of arcsin play an important role in analytical calculations of massive Feynman diagrams, see [8], and in the construction of Laurent expansions of different types of hypergeometric functions with respect to small parameters. In [8], series expansions for small powers of arcsin are given, but they do not have the compact nested-sum form seen in equations (1) and (2). Likewise, formula (8) has proven central to recent work on effective asymptotic expansions for Laguerre polynomials and Bessel functions, [1]. These results are again motivated largely by applications in mathematical physics and in prime computation.

3 Properties of Coefficient Functions

It is of some independent interest, at least to the present authors, to determine a few properties of $H_N(k)$ and $G_N(k)$.

Corollary 1 *The following properties obtain:*

(a) $H_N(k) = G_N(k) = 0$ *if* $N > k$.

(b) $H_N(k) = \sum_{j=N-1}^{k-1} \frac{H_{N-1}(j)}{(2j)^2}, \quad G_N(k) = \sum_{j=N-1}^{k-1} \frac{G_{N-1}(j)}{(2j+1)^2}$

(c) $H_k(k) = \frac{1}{4^k (k-1)!^2}, \quad G_k(k) = \frac{4^k k!^2}{(2k)!^2}$

(d) $\sum_{N=1}^{k} (-4)^N H_N(k) = 0$ *for* $k \geq 2$, $\sum_{N=0}^{k} (-1)^N G_N(k) = 0$ *for* $k \geq 1$.

Proof. Parts (a)–(c) follow from the definition of $H_N(k)$ and $G_N(k)$. To prove part (d), first note that equation (1) may be rewritten as

$$x^{2N} = \frac{(2N)!}{4^N} \sum_{k=1}^{\infty} \frac{4^{k+N} H_N(k)}{\binom{2k}{k} k^2} (\sin x)^{2k}.$$

Use this to obtain

$$\begin{aligned} 1 - 2\sin^2 x = \cos(2x) &= 1 + \sum_{N=1}^{\infty} \frac{(-1)^N 4^N x^{2N}}{(2N)!} \\ &= 1 + \sum_{N=1}^{\infty} (-1)^N \sum_{k=1}^{\infty} \frac{4^{k+N} H_N(k)}{\binom{2k}{k} k^2} (\sin x)^{2k} \\ &= 1 + \sum_{k=1}^{\infty} \frac{4^k (\sin x)^{2k}}{\binom{2k}{k} k^2} \sum_{N=1}^{k} (-4)^N H_N(k). \end{aligned}$$

Now, matching powers of $(\sin x)^2$ implies the first part of (d). Equation (2) may be similarly manipulated to give the second part of (d). $\square$

Correspondingly,

Corollary 2 *For non-negative integers N and k one has*

$$\frac{4^{2k} H_N(k)}{\binom{2k}{k} k^2} = -\sum_{j=0}^{k-1} \frac{\binom{2j}{j}\binom{2k-2j}{k-j}}{2j-1} G_N(k-j) \tag{11}$$

and

$$\frac{\binom{2k-2}{k-1} G_N(k-1)}{2^{2k-1}(2k-1)} = -\sum_{j=0}^{k-1} \frac{\binom{2j}{j} 4^{k-2j}}{(k-j)(2j-1)\binom{2k-2j}{k-j}} H_{N+1}(k-j). \tag{12}$$

Proof. Differentiate equation (2) to obtain

$$\arcsin^{2N}\left(\frac{x}{2}\right) = \sqrt{1-\left(\frac{x}{2}\right)^2}\,(2N)! \sum_{k=0}^{\infty} \frac{G_N(k)\binom{2\,k}{k}}{16^k} x^{2k}. \tag{13}$$

Using the binomial theorem and comparing to equation (1) gives equation (11). Similarly, differentiating equation (1) leads to (12). $\square$

Even the simplest case, with $N = 1$, yields the non-obvious identity:

$$\frac{4^{2k-1}}{\binom{2k}{k} k^2} = -\sum_{j=0}^{k-1} \frac{\binom{2j}{j}\binom{2k-2j}{k-j}}{2j-1} \sum_{n=0}^{k-j-1} \frac{1}{(2n+1)^2}. \tag{14}$$

Asking a student to prove this identity directly is instructive, particularly from an experimental mathematics perspective. While *Maple* stares dumbly at the right side, it immediately redeems itself after one interchanges the order of summation, producing

$$\sum_{n=0}^{k-1} \frac{1}{(2n+1)^2} \sum_{j=0}^{k-n-1} \frac{\binom{2j}{j}\binom{2k-2j}{k-j}}{2j-1} = -\frac{\sqrt{\pi}\, 4^k\, \Gamma(k)}{4k\, \Gamma(k+1/2)},$$

which may be easily rewritten as the desired expression (14).

A host of partition identities tumble directly from (2) and (7) on comparing various ways of combining powers of arcsin.

Corollary 3 *Given m, n, $\{M_i\}_{i=1}^m \geq 1$ and $\{N_i\}_{i=1}^n \geq 0$, let*

$$T := 2M_1 + \cdots + 2M_m + (2N_1+1) + \cdots + (2N_n+1)$$

and

$$W := 2^{-n}\,(2M_1)!\cdots(2M_m)!(2N_1+1)!\cdots(2N_n+1)!\,.$$

Then for any $s \geq n$ where s and n have the same parity,

$$W\sum \frac{\binom{2k_1}{k_1}\cdots\binom{2k_n}{k_n}H_{M_1}(j_1)\cdots H_{M_m}(j_m)G_{N_1}(k_1)\cdots G_{N_n}(k_n)\,4^{2n}}{\binom{2j_1}{j_1}\cdots\binom{2j_m}{j_m}(j_1\cdots j_m)^2(2k_1-1)\cdots(2k_n-1)4^{2k_1+\cdots+2k_n}}$$

$$= \begin{cases} 4T!\,\dfrac{H_{T/2}(s/2)}{\binom{s}{s/2}s^2}, & n \text{ even} \\[2ex] \frac{T!}{2}\binom{s-1}{(s-1)/2}\dfrac{G_{(T-1)/2}((s-1)/2)}{s\,4^{s-1}}, & n \text{ odd} \end{cases}$$

where the sum is taken over all partitions *of*

$$\frac{s+n}{2} = j_1 + \cdots + j_m + k_1 + \cdots + k_n$$

where $j_1, j_2, \ldots, j_m \geq 1$ and $k_1, k_2, \ldots, k_n \geq 1$.

An interesting special case occurs when we specify $n = 0$, $m = 2$, $M_1 = M_2 = 1$ and $s = 2k$:

$$\sum_{j=1}^{k-1} \frac{1}{\binom{2j}{j}\binom{2k-2j}{k-j}j^2(k-j)^2} = \frac{6}{\binom{2k}{k}k^2}\sum_{j=1}^{k-1}\frac{1}{j^2}, \tag{15}$$

hence

$$\lim_{k\to\infty}\sum_{j=1}^{k-1}\frac{\binom{2k}{k}}{\binom{2j}{j}\binom{2k-2j}{k-j}}\frac{k^2}{j^2(k-j)^2} = \pi^2.$$

4 Powers of Arcsin via Iterated Integrals

Since $\arcsin'(x) = 1/\sqrt{1-x^2}$, we may also express any power of $\arcsin(x)$ as an iterated integral. Specifically,

$$\frac{\arcsin^n x}{n!} = \int_0^x \frac{dy_1}{\sqrt{1-y_1^2}}\int_0^{y_1}\frac{dy_2}{\sqrt{1-y_2^2}}\int_0^{y_2}\frac{dy_3}{\sqrt{1-y_3^2}}\cdots\int_0^{y_{n-1}}\frac{dy_n}{\sqrt{1-y_n^2}}. \tag{16}$$

To convert the multiple integral into a multiple sum, note that the binomial theorem gives

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \frac{1}{4^k}\binom{2k}{k} x^{2k},$$

which may be repeatedly used in (16) to yield

$$\begin{aligned}
\frac{\arcsin^n x}{n!} &= \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} \frac{\binom{2k_1}{k_1}\binom{2k_2}{k_2}\cdots\binom{2k_n}{k_n} x^{2w_n+n}}{4^{w_n}(2k_1+1)(2k_1+2k_2+2)\cdots(2w_n+n)} \\
&= x^n \sum_{k=0}^{\infty} \left\{ \sum_{k_1=0}^{k} \frac{\binom{2k_1}{k_1}}{4^{k_1}(2k_1+1)} \sum_{k_2=0}^{k-k_1} \frac{\binom{2k_2}{k_2}}{4^{k_2}(2k_1+2k_2+2)} \cdots \sum_{k_n=0}^{k+k_n-w_n} \frac{\binom{2k_n}{k_n}}{4^{k_n}(2w_n+n)} \right\} x^k \\
&= x^n \sum_{k=0}^{\infty} \left\{ \sum_{k_1=0}^{k} \frac{\binom{2k_1}{k_1}}{(2k_1+1)} \sum_{k_2=k_1}^{k} \frac{\binom{2k_2-2k_1}{k_2-k_1}}{(2k_2+2)} \cdots \sum_{k_n=k_{n-1}}^{k} \frac{\binom{2k_n-2k_{n-1}}{k_n-k_{n-1}}}{(2k_n+n)} \frac{1}{4^{k_n}} \right\} x^k
\end{aligned}$$

where $w_n := k_1 + k_2 + \cdots + k_n$.

Though this process also writes the coefficients in terms of nested-sums, these terms are not nearly as simple as H_N and G_N.

5 Related Series Manipulations

After having discovered formulas such as (1) and (2), the analyst's natural inclination is to "mine" them for striking examples. We rescale (1) and (2) to obtain

$$\arcsin^{2N}(x) = (2N)! \sum_{k=1}^{\infty} \frac{H_N(k) 4^k}{\binom{2k}{k} k^2} x^{2k},$$

and

$$\arcsin^{2N+1}(x) = (2N+1)! \sum_{k=0}^{\infty} \frac{G_N(k)\binom{2k}{k}}{(2k+1)4^k} x^{2k+1},$$

These series, or their derivatives, may be evaluated at values such as $x = \pi/2, \pi/3, \pi/4, \pi/6, i/2$, and i to obtain many formulae along the lines of those found by Lehmer and others as described in [4, pp.384–86]. A few examples are:

$$\sum_{k=1}^{\infty} \frac{H_N(k) 4^k}{\binom{2k}{k} k^2} = \frac{1}{(2N)!}\left(\frac{\pi}{2}\right)^{2N}, \quad \sum_{k=1}^{\infty} \frac{H_N(k) 3^k}{\binom{2k}{k} k^2} = \frac{1}{(2N)!}\left(\frac{\pi}{3}\right)^{2N},$$

$$\sum_{k=1}^{\infty} \frac{H_N(k)2^k}{\binom{2k}{k}k^2} = \frac{1}{(2N)!}\left(\frac{\pi}{4}\right)^{2N}, \quad \sum_{k=1}^{\infty} \frac{H_N(k)}{\binom{2k}{k}k^2} = \frac{1}{(2N)!}\left(\frac{\pi}{6}\right)^{2N},$$

$$\sum_{k=1}^{\infty} \frac{H_N(k)(-1)^k}{\binom{2k}{k}k^2} = \frac{(-1)^N}{(2N)!}\left(\log\frac{\sqrt{5}-1}{2}\right)^{2N},$$

and

$$\sum_{k=1}^{\infty} \frac{H_N(k)(-4)^k}{\binom{2k}{k}k^2} = \frac{(-1)^N}{(2N)!}\left(\log(1+\sqrt{2})\right)^{2N}.$$

Integrating (1) and (2) is, naturally, much more challenging. Replacing x by $i\,x$ in (1) and (2) provides the Maclaurin series for positive integer powers of the form

$$\log^N(x+\sqrt{x^2+1}).$$

These expressions may then be integrated (or differentiated). In particular, see [4, Exercise 17, p.189], we have

$$\frac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3\binom{2n}{n}} = \int_0^{1/2} \frac{\log^2(x+\sqrt{x^2+1})}{x}\,dx = \frac{\zeta(3)}{10}. \tag{17}$$

Likewise, with more work, see [5], we obtain

$$\begin{aligned}\int_0^{1/2} \frac{\ln^4(x+\sqrt{x^2+1})}{x}dx &= -\frac{3}{2}\,\mathrm{Li}_5\left(g^2\right) + 3\,\mathrm{Li}_4\left(g^2\right)\ln(g) + \frac{3}{2}\,\zeta(5) \\ &\quad -\frac{12}{5}\,\zeta(3)\ln^2(g) - \frac{4}{15}\,\pi^2\ln^3(g) + \frac{4}{5}\,\ln^5(g),\end{aligned} \tag{18}$$

where $g = (\sqrt{5}-1)/2$ is the golden ratio and where $\mathrm{Li_n}(z) = \sum_{k=1}^{\infty} z^k/k^n$ is the *polylogarithm* of order n.

This process may be generalized as follows. Define

$$\mathcal{L}_n := \frac{2^n}{n!}\int_0^{1/2} \frac{\ln^n(x+\sqrt{x^2+1})}{x}dx.$$

We leave it to the reader to determine the explicit series expression for $\mathcal{L}_n$, for n even and for n odd.

Extensive experimentation with *Maple* coupled with pattern lookups with Sloane's *Online Encyclopedia of Integer Sequences*[1] produced

$$\mathcal{L}_n = \zeta(n+1) - \frac{n(-\log g^2)^{n+1}}{2(n+1)!} - \sum_{j=2}^{n+1} \frac{(-\log g^2)^{n+1-j}}{(n+1-j)!}\,\mathrm{Li_j}(g^2).$$

[1]URL: `www.research.att.com/personal/njas/sequences/eisonline.html`

Multiplying by x^n, and summing over $n \geq 1$, shows this is equivalent to the next generating function, written in terms of ψ, the *digamma* function:

$$\begin{aligned} -x \int_0^{-\log g} (e^{2xy}-1)\coth(y)\,dy &= \frac{1}{2} + x\,(\gamma + \psi(1-x)) - \frac{1/2 + x\log g}{e^{2x\log g}} \\ &+ \ e^{-2x\log g} \sum_{k=2}^{\infty} \mathrm{Li_k}(g^2)\,x^k. \end{aligned}$$

These observations can be made rigorous with the next result.

Theorem 4 *For $n \geq 1$ and $|x| \leq 1$,*

$$\begin{aligned} \frac{1}{n!}\int_0^x \frac{\arcsin^n(y)}{y}dy &= -\sum_{k=2}^{n+1} \mathrm{Li_k}((\sqrt{1-x^2}+ix)^2)\frac{(-2i)^{1-k}\ \arcsin(x)^{n+1-k}}{(n+1-k)!} \\ &-\frac{i\arcsin^{n+1}(x)}{(n+1)!} + \frac{\arcsin^n(x)}{n!}\log\left(2x^2 - 2ix\sqrt{1-x^2}\right) + \left(\frac{i}{2}\right)^n \zeta(n+1). \end{aligned}$$

Proof. The derivatives of each side match, and the equation holds for $x = 0$. Alternatively, this is a reworking of formula (7.48) in [9, p.199]. $\square$

In hindsight, this equation should not come as much of a surprise since Ramanujan's entries immediately preceding (3) give similar formulae using the Clausen functions. One can substitute (1) or (2) to obtain further formulae. This allows for extensions to the complex plane. For example:

Corollary 5 *For $n \geq 1$ and $|x| \leq 1$, and $\mathrm{Re}(x) \geq 0$*

$$\begin{aligned} \frac{(-1)^n}{2}\sum_{k=1}^{\infty}\frac{H_n(k)(-4)^k}{\binom{2\,k}{k}k^3}\,x^{2\,k} &= -\sum_{k=2}^{2n+1}\mathrm{Li_k}((x-\sqrt{1+x^2})^2)\frac{2^{1-k}\mathrm{arcsinh}(x)^{2n+1-k}}{(2n+1-k)!} \\ &+\frac{\mathrm{arcsinh}^{2n+1}(x)}{(2n+1)!} + \frac{\mathrm{arcsinh}^{2n}(x)}{2n!}\log\left(2x\sqrt{1+x^2}-2x^2\right) + \frac{\zeta(2n+1)}{4^n}. \end{aligned}$$

Example. Substituting $x = i$ gives

$$\begin{aligned} \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_n(k)4^k}{\binom{2k}{k}k^3} &= \sum_{k=1}^{n}(1-2^{-2k})\zeta(2k+1)\left(-\frac{1}{4}\right)^k\frac{(\pi/2)^{2n-2k}}{(2n-2k)!} \\ &+\frac{(\pi/2)^{2n}}{(2n)!}\log 2 + \left(-\frac{1}{4}\right)^n\zeta(2n+1) \end{aligned}$$

with the special case $n = 1$ yielding

$$\sum_{k=1}^{\infty}\frac{4^k}{\binom{2k}{k}k^3} = \pi^2\log 2 - \frac{7}{2}\zeta(3). \tag{19}$$

6 Conclusion

We hope this examination of (1) and (2) encourages readers to similarly explore what transpires for arctan and other functions. Armed with a good computer algebra system, and an internet connection, one can quite fearlessly undertake this task.

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