# Calculating Bessel Functions via the Exp-arc Method

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Illinois Number Theory Fest, May 2007

# Outline

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The second order differential equation

$$z^2 y'' + zy' + (z^2 - \nu^2)y = 0$$

is called Bessel's Equation.

The ordinary Bessel function or order $\nu$, or the Bessel function of the first kind of order $\nu$, denoted $J_\nu(z)$, is a solution to this differential equation.

$J_\nu(z)$ can be represented as an ascending series

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

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It is not difficult to show that for $\nu \notin \mathbb{Z}$, $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent. Since Bessel's Equation is second order, for non-integer $\nu$ this pair generates all the solutions.

When $\nu = n$ is an integer, $J_n(z)$ is also given by the generating function

$$e^{\frac{z}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(z)t^n.$$

Replace $t$ by $-t$ and we find that

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So to obtain the second solution to Bessel's Equation at integer order, define the Bessel function of the second kind $Y_n(z)$, $n \in \mathbb{Z}$, by

$$Y_n(z) := \lim_{\nu \to n} \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi}.$$

For general $\nu$, $Y_\nu(z)$ is defined as above without the limit.

We also have the following ascending series representation for $Y_n(z)$, $n \in \mathbb{Z}$.

$$Y_n(z) = \frac{1}{\pi}\left(2(\gamma + \log(z/2))J_n(z) - \sum_{k=0}^{n-1} \frac{(n-k-1)!(z/2)^{2k-n}}{k!} - \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}(H_k + H_{k+n})}{k!(n+k)!}\right).$$

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In addition to the $J$ and $Y$ Bessel functions, there are also the *modified Bessel functions* $I_\nu(z)$ and $K_\nu(z)$, which are solutions to the differential equation

$$z^2 y'' + zy' - (z^2 + \nu^2)y = 0.$$

$I_\nu(z)$ is usually known as the Bessel function of imaginary argument, and is related to $J_\nu$ by

$$I_\nu(z) = e^{\pi i\nu/2} J_\nu(iz) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)},$$

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As $z \to \infty$, we have the asymptotic expansion

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \left(\cos(z - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}) \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}-\nu)_{2k} (\frac{1}{2}+\nu)_{2k}}{k!(2z)^{2k}} - \sin(z - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}) \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}-\nu)_{2k+1} (\frac{1}{2}+\nu)_{2k+1}}{k!(2z)^{2k+1}}\right),$$

as well as similar expressions for $Y$, $I$, and $K$.

Here the notation $(a)_k$ is the Pochhammer symbol given by

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There exists, however, a *Hadamard expansion* that is convergent:

$$I_\nu(z) = \frac{e^z}{\Gamma(\nu+\frac{1}{2})\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}-\nu)_k}{k!(2z)^k} \int_0^{2z} t^{\nu+k-\frac{1}{2}} e^{-t} dt.$$

Note that the error behaves like $N^{-\nu-1/2}$ when the series is truncated after $N$ terms.

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Bessel's Equation arises as a special case of Laplace's Equation with cylindrical symmetry.

Thus, Bessel functions occur often in the study of waves in two dimensions.

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## Applications to Number Theory

### Hardy and the Circle Problem

Let $r_2(n)$ denote the number of representations of the positive integer $n$ as a sum of two squares. The “circle problem” is to determine the precise order of magnitude for the “error term" $P(x)$ defined by

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + P(x),$$

where the prime $'$ on the summation sign on the left side indicates that if $x$ is an integer, only $\frac{1}{2}r_2(x)$ is counted.

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In 1915, Hardy proved that

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n)\left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}).$$

This is equivalent to the following result of Berndt and Zaharescu

$$\sum_{0\le n\le x}{}' r_2(n) = \pi x$$

$$+2\sqrt{x}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\frac{1}{4})x}\right)}{\sqrt{m(n+\frac{1}{4})}} - \frac{J_1\left(4\pi\sqrt{m(n+\frac{3}{4})x}\right)}{\sqrt{m(n+\frac{3}{4})}}\right\}.$$

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The BZ result is a corollary of an entry on page 335 of Ramanujan's Lost Notebook.

That entry is one of a pair of equations involving a doubly-infinite series of Bessel functions. If we let

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases}$$

where, $[x]$ is the greatest integer less than or equal to $x$, and

$$Z_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z);$$

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## Entry

*For* $x > 0$ *and* $0 < \theta < 1$,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\sin(2\pi n\theta) = \pi x\left(\tfrac{1}{2}-\theta\right) - \tfrac{1}{4}\cot(\pi\theta)$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{J_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} - \frac{J_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}.$$

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*For $x > 0$ and $0 < \theta < 1$,*

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I wanted to evaluate a truncation of the right-hand side of the second entry to verify it against the left-hand side. As an example, computing the right-hand side (with one of the sums truncated at 50 terms, the other truncated at 1000 terms, for a total of 50,000 summands) at 28-digit precision in PARI took nearly 4 hours.

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## What is “exp-arc”?

In their recent paper to find asymptotics for Laguerre polynomials with effective (explicit) error bounds, D. Borwein, J. M. Borwein, and R. Crandall were led to consider the following integral

$$\mathcal{I}(p,q) := \int_{-\pi/2}^{\pi/2} e^{-iq\omega} e^{p\cos\omega} \, d\omega.$$

A simple change of variable yields

$$\mathcal{I}(p,q) = 4e^{p} \int_0^{1/\sqrt{2}} \frac{\cosh(-2iq \arcsin x) e^{-2px^2}}{\sqrt{1-x^2}} \, dx.$$

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This integral, therefore, reduces to an integral involving $e^{\tau \arcsin x}$, or what BBC calls an "exp-arc" integral.

They then exploit the fact that the exp-arc function has a very nice series expansion on $(-1, 1)$, namely

$$e^{\tau \arcsin x} = 1 + \sum_{k=1}^{\infty} \frac{c_k(\tau) x^k}{k!},$$

where

$$c_{2k+1}(\tau) = \tau \prod_{j=1}^{k} (\tau^2 + (2j-1)^2), \qquad c_{2k} = \prod_{j=1}^{k} (\tau^2 + (2j-2)^2).$$

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Plugging this into the expression for $\mathcal{I}(p, q)$ and interchanging summation and integration we obtain

$$\mathcal{I}(p,q) = 4e^p \sum_{k=0}^{\infty} \frac{g_k(-2iq)}{(2k)!} B_k(p), \tag{1}$$

where

$$g_0 := 1, \qquad g_k(\nu) := \prod_{j=1}^{k} \left((2j-1)^2 + \nu^2\right) \text{ for } k \geq 1,$$

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$$B_k(p) := \int_0^{1/\sqrt{2}} x^{2k} e^{-2px^2} dx = \frac{1}{2^k \sqrt{2}} \int_0^1 e^{-pu} u^{k-\frac{1}{2}} du.$$

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Note that

- $g_k$ and $B_k$ are rapidly computable via recursion
- The integral $\int_0^1 e^{-pu} u^{k-1/2} du$ is uniformly bounded for all $k > 0$, and so the $B_k$ decrease geometrically as $2^{-k}$.
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## Integral Representations

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu t - z\sin t)dt - \frac{\sin \nu\pi}{\pi}\int_0^\infty e^{-\nu t - z\sinh t}dt,$$

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin t - \nu t)dt - \frac{1}{\pi}\int_0^\infty (e^{\nu t} + e^{-\nu t}\cos \nu\pi)e^{-z\sinh t}d$$

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos t}\cos \nu t\ dt - \frac{\sin \nu\pi}{\pi}\int_0^\infty e^{-z\cosh t - \nu t}dt,$$

and

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh \nu t\ dt = \frac{1}{2}\int_{-\infty}^\infty e^{-z\cosh t - \nu t}dt.$$

A quick change of variable allows us to express the finite integrals in terms of $\mathcal{I}(p, q)$.

For integral order, the infinite integrals in $J$ and $I$ disappear due to the $\sin \nu\pi$. Thus we have

$$J_n(z) = \frac{1}{2\pi}\left(e^{-in\pi/2}\mathcal{I}(iz, n) + e^{in\pi/2}\mathcal{I}(-iz, n)\right),$$

and

$$I_n(z) = \frac{1}{2\pi}\left(\mathcal{I}(z, n) + e^{\pi in}\mathcal{I}(-z, n)\right).$$

A quick change of variable allows us to express the finite integrals in terms of $\mathcal{I}(p, q)$.

For integral order, the infinite integrals in $J$ and $I$ disappear due to the $\sin \nu\pi$. Thus we have

$$J_n(z) = \frac{1}{2\pi}\left(e^{-in\pi/2}\mathcal{I}(iz, n) + e^{in\pi/2}\mathcal{I}(-iz, n)\right),$$

and

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To deal with the general case, we need to evaluate the infinite integrals. A change of variables plus integration by parts gives us

$$\int_0^\infty e^{-\nu t - z \sinh t} dt = \frac{1}{\nu} - \frac{z}{\nu} \int_0^\infty e^{-zs} e^{-\nu \operatorname{arcsinh} s} ds.$$

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# Expand about a point other than zero!

For example, the expansion at infinity is

$$s^{\nu} e^{-\nu \operatorname{arcsinh} s} = \sum_{n=0}^{\infty} \frac{A_n(\nu)}{s^{2n}},$$

where $A_0(\nu) = 2^{-\nu}$ and for $n \geq 1$,

$$A_n(\nu) = \frac{(-1)^n \nu 2^{-\nu} (\nu + n + 1)_{n-1}}{2^{2n} n!},$$

This expansion is valid for $|s| > 1$. The coefficients satisfy

$$A_n = -\frac{(\nu + 2n - 2)(\nu + 2n - 1)}{4n(n + \nu)} A_{n-1}.$$

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## How about at other points?

No nice formula for coefficients, but that's okay.

For fixed $k$, we have the expansion

$$e^{-\nu \operatorname{arcsinh}(k+s)} = \sum_{n=0}^{\infty} \frac{a_n(k,\nu)}{n!} s^n$$

where for $n \geq 0$,

$$a_{n+2} = \frac{1}{k^2+1}\left((\nu^2 - n^2)a_n - k(2n+1)a_{n+1}\right),$$

and

$$a_0 = (k + \sqrt{k^2+1})^{-\nu}, \qquad a_1 = -\frac{\nu a_0}{\sqrt{k^2+1}}.$$

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Thus for any positive integer $N$, we have

$$\int_0^\infty e^{-zs}e^{-\nu \operatorname{arcsinh} s}ds = \sum_{n=0}^\infty \Bigg( \frac{a_n(0,\nu)}{n!}\alpha_n(z) + \beta_n(z)\sum_{k=1}^N e^{-kz}\frac{a_n(k,\nu)}{n!} + A_n(\nu)G_n(N+\tfrac{1}{2},z,\nu)\Bigg),$$

where

$$\alpha_n(z) := \int_0^{1/2} e^{-zs}s^n ds = -\frac{e^{-z/2}}{2^n z} + \frac{n}{z}\alpha_{n-1}(z),$$

$$\beta_n(z) := \int_{-1/2}^{1/2} e^{-zs}s^n ds = \frac{e^{z/2}}{(-2)^n z} - \frac{e^{-z/2}}{2^n z} + \frac{n}{z}\beta_{n-1}(z),$$

and

$$G_n(N,z,\nu) := \frac{e^{-Nz}}{N^{2n+\nu-1}} \int_0^\infty e^{-Nzs}(1+s)^{-2n-\nu} ds$$

$$= \frac{1}{(\nu+2n-1)(\nu+2n-2)} \times \left( \frac{e^{-Nz}(2(n-z-1)+\nu)}{N^{2n+\nu-1}} + z^2 G_{n-1}(N,z,\nu) \right).$$

Key points:

- All the summands are easily computable via recursion
- The recursions only involve elementary operations. The initial conditions $B_0$ (from $\mathcal{I}$) and $G_0$ each require one evaluation of incomplete gamma, which can be done via continued fraction.
- Each series converges geometrically, like $2^{-N}$ (as opposed to the Hadamard expansion for $I_\nu$, which is like $N^{-\nu}$)
- Can choose $N$ large to avoid the $A_n G_n$ sum if we want. In this way, we can pre-compute summands involving only $\nu$ and summands involving only $z$ for "one-$z$ many-$\nu$" or one-$\nu$ many-$z$" evaluations.

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