

Continued Logarithms And Associated Continued Fractions

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Abstract

We investigate some of the connections between continued fractions and continued logarithms. We study the binary continued logarithms as introduced by Bill Gosper and explore two generalizations of the continued logarithm to base b . We show convergence for them using equivalent forms of their corresponding continued fractions. Through numerical experimentation we discover that, for one such formulation, the exponent terms have finite arithmetic means for almost all real numbers. This set of means, which we call the logarithmic Khintchine numbers, has a pleasing relationship with the geometric means of the corresponding continued fraction terms. While the classical Khintchine's constant is believed not to be related to any naturally occurring number, we find surprisingly that the logarithmic Khintchine numbers are elementary.

1 Introduction

The history of continued fractions dates back, in at least some sense, to Euclid's elements: the greatest common divisor algorithm generates a continued fraction as a byproduct [2]. Continued (binary) logarithms, by contrast, appear to have been introduced explicitly around 1972 by Gosper in his appendix on Continued Fraction Arithmetic [4]. His motivation for introducing this concept was that continued logarithms can be used to represent numbers in which continued fractions are ineffective: for example, Avogadro's number is 23 digits long, but only the first six digits are known: hence even the integer part of its continued fraction is unknown. As Gosper puts it, continued logarithms provide "a sort of recursive version of scientific notation".

1.1 The structure of this paper

In Section 1, we will give a brief overview of Gosper's original construction for binary (base 2) continued logarithms and consider some informative examples

which suggest useful connections between the continued logarithms and their corresponding continued fractions. In Section 2, we recall some simple results which are well known in the case of continued fractions though less well known in the case of continued logarithms. We go on to develop the analogous continued logarithm results in Section 3, as well as showing the convergence of approximants, including showing that, in the binary case, rationals have finite continued logarithms.

In Section 4, we introduce a Gauss-Kuzmin-type Distribution for binary continued logarithms and consequently obtain an analogue to Khintchine's Constant for continued logarithms, which we discuss in Section 5. In Section 7, we consider some continued logarithms with interesting continued fractions, and we go on to explore in greater depth the fascinating properties which emerge from studying quadratic irrationals in Section 8.

In Sections 9 and 10, we generalize the notion of continued **logarithms** from the binary case to the base b case, exploring various ways of defining the construction with different dynamical systems. We elect to explore two in particular: Type I in Section 9 and Type II in Section 10. In each case, we discover properties unique to the construction. The Type I construction possesses a corresponding Gauss-Kuzmin distribution and elementary Khintchine constant for all base integers $b \geq 2$; however, for $b > 2$ this construction may be infinite for rationals. By contrast, the Type II construction has a more exotic distribution but terminates finitely for all rationals. Finally, in Section 11, we go on to describe some methods for computing with continued logarithms in an efficient way.

1.2 Continued Fractions

The standard description of *regular (simple) continued fractions* proceeds as follows: given a positive real number x , write $\alpha_0 = \lfloor x \rfloor$:

$$x = \alpha_0 + \{x\}$$

(the integer part of x plus the fractional part of x). Terminate if $\{x\} = 0$. Otherwise, set $y = \frac{1}{\{x\}}$, and write $\alpha_1 = \lfloor y \rfloor$, so that

$$x = \alpha_0 + \frac{1}{\alpha_1 + \{y\}}$$

If $\{y\} = 0$ then terminate: otherwise continue in this fashion until either

$$x = \alpha_0 + \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \dots}}}$$

in which the continued fraction never terminates, or at some point the fractional part will be zero, in which case

$$x = \alpha_0 + \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \frac{1}{\dots + \frac{1}{\alpha_n}}}}}$$

Another way to view this is the following: we will consider a dynamical system f on $[0, \infty)$ in which $f(x)$ is given by

$$f(x) = \begin{cases} x - 1 & \text{if } x \geq 1 \\ \frac{1}{x} & \text{if } 0 < x < 1 \\ \text{terminate} & \text{if } x = 0. \end{cases} \quad (1)$$

Now iterate f starting at x : count the number of times we encounter $x \rightarrow x - 1$ before we either reciprocate or terminate. These counts gives us our values $\alpha_0, \alpha_1, \alpha_2, \dots$. We will denote by $[\alpha_0; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots]_{cf}$ the continued fraction

$$x = \alpha_0 + \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \dots}}}$$

1.3 Continued Logarithms

We start with the binary case. For continued logarithms we can define a similar dynamical system g on $[1, \infty)$:

$$g(x) = \begin{cases} x/2 & \text{if } x \geq 2 \\ \frac{1}{x-1} & \text{if } 1 < x < 2 \\ \text{terminate} & \text{if } x = 1. \end{cases} \quad (2)$$

As with continued fractions where we count how many times we subtract one, we now count how many times we divide by 2 before we subtract and reciprocate or terminate. This gives the values a_0, a_1, a_2, \dots . We denote the binary continued logarithm of x by $[a_0, a_1, a_2, \dots]_{cl(2)}$.

Example 1. Let us start with $x = 19$:

$$19 \rightarrow \frac{19}{2} \rightarrow \frac{19}{4} \rightarrow \frac{19}{8} \rightarrow \frac{19}{16}$$

so $a_0 = 4$:

$$\frac{19}{16} \rightarrow \frac{1}{\frac{19}{16} - 1} = \frac{16}{3}.$$

$$\frac{16}{3} \rightarrow \frac{8}{3} \rightarrow \frac{4}{3}$$

so $a_1 = 2$:

$$\frac{4}{3} \rightarrow \frac{1}{\frac{4}{3} - 1} = 3.$$

$$3 \rightarrow \frac{3}{2}$$

so $a_3 = 1$.

$$\frac{3}{2} \rightarrow \frac{1}{\frac{3}{2} - 1} = 2.$$

And finally,

$$2 \rightarrow 1$$

after which we terminate, so $b_4 = 1$. Hence the continued logarithm for 19 is $[4, 2, 1, 1]_{\text{cl}(2)}$. \diamond

Remark 1. We may explicitly express this continued logarithm in classical continued fraction form as follows:

$$19 = 2^4 + \frac{2^4}{2^2 + \frac{2^2}{2^1 + \frac{2^1}{2^1}}}. \quad (3)$$

More generally, where x has continued logarithm $[a_0, a_1, a_2, \dots]_{\text{cl}(2)}$,

$$x = 2^{a_0} + \frac{2^{a_0}}{2^{a_1} + \frac{2^{a_1}}{2^{a_2} + \dots}} \quad (4)$$

which is the original formulation found in [4]. To see why this is so, recall the previous example where $y_0 = 19$ and consider its binary expansion:

$$y_0 = 1 \cdot 2^4 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0.$$

We set aside the trailing terms and use only the leading term to begin building a continued fraction in the usual way:

$$y_0 = 2^4 + (y_0 - 2^4) = 2^4 + \frac{1}{\frac{1}{(y_0 - 2^4)}} = 2^4 + \frac{2^4}{\frac{2^4}{y_0 - 2^4}}.$$

Dividing the numerator and denominator of the lower fraction in order to make the numerator 1, we obtain

$$y_0 = 2^4 + \frac{2^4}{\frac{1}{\frac{y_0}{2^4} - 1}} = 2^4 + \frac{2^4}{y_1} \quad \text{where } y_1 = \frac{1}{\frac{y_0}{2^4} - 1} = \frac{16}{3}.$$

We repeat the same process for y_1 as we did for y_0 , taking its base expansion

$$y_1 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + \dots$$

and likewise using its leading term to build the continued fraction

$$y_1 = 2^2 + \frac{2^2}{y_2} \text{ where } y_2 = \frac{1}{\frac{y_1}{2^2} - 1} = 3.$$

Finally, we have

$$y_2 = 2^1 + \frac{2^1}{y_3} \text{ where } y_3 = \frac{1}{\frac{y_2}{2^1} - 1} = 2.$$

This yields the continued fraction in Equation (3). It is clear that, when the process terminates finitely, it is equal to the desired rational number. While we will be concerned mainly with exploring the properties for continued logarithms for numbers in $(1, \infty)$, Remark 11, near the end of Section 10, illustrates that the techniques we explore may also be used for numbers in $(0, 1)$. \diamond

2 Continued Fraction Recurrences

The continued fraction recurrences are well known, see for example [2, Ch. 9] or [6, Ch. 7] for the general (irregular) case. We restate some of them here in order to justify the forthcoming continued logarithm recurrences.

Remark 2. Consider the continued fraction

$$x = \alpha_0 + \frac{\beta_1}{\alpha_1 + \frac{\beta_2}{\alpha_2 + \frac{\beta_3}{\alpha_3 + \dots}}}$$

We may instead, for the sake of simplicity, write

$$x = \alpha_0 + \left\lfloor \frac{\beta_1}{\alpha_1} \right\rfloor + \left\lfloor \frac{\beta_2}{\alpha_2} \right\rfloor + \left\lfloor \frac{\beta_3}{\alpha_3} \right\rfloor + \dots$$

which is definitely more compact. \diamond

Remark 3. Where x is the irregular continued fraction

$$x = \alpha_0 + \left\lfloor \frac{\beta_1}{\alpha_1} \right\rfloor + \left\lfloor \frac{\beta_2}{\alpha_2} \right\rfloor + \left\lfloor \frac{\beta_3}{\alpha_3} \right\rfloor + \dots$$

Let x_n be the n th continued fraction approximant, which is the number whose continued logarithm is

$$x = \alpha_0 + \left\lfloor \frac{\beta_1}{\alpha_1} \right\rfloor + \left\lfloor \frac{\beta_2}{\alpha_2} \right\rfloor + \dots + \left\lfloor \frac{\beta_n}{\alpha_n} \right\rfloor.$$

Then

$$x_n = \frac{r_n}{s_n}$$

where $r_{-1} = 1$, $s_{-1} = 0$, $r_0 = \alpha_0$, $s_0 = 1$, and

$$r_{n+1} = \alpha_{n+1}r_n + \beta_{n+1}r_{n-1}$$

and

$$s_{n+1} = \alpha_{n+1}s_n + \beta_{n+1}s_{n-1}.$$

Derivations of this formula may be found in [6] and [2]. This makes it easy to compute the quotient. \diamond

Remark 4. We also have that

$$r_n s_{n-1} - r_{n-1} s_n = (-1)^{n+1} \prod_{k=1}^n \beta_k.$$

In the case of a simple continued fraction, of course, the product is always one. \diamond

We will use the following two classical continued fraction results, from [6, 8.12] and [2, 9.5] respectively, to show convergence for continued logarithms and finite termination for rationals respectively.

Remark 5. Suppose that $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are sequences such that $\alpha_n > 0$ and $\beta_n > 0$ for all n . Then the continued fraction

$$x = \alpha_0 + \cfrac{\beta_1}{\alpha_1} + \cfrac{\beta_2}{\alpha_2} + \cfrac{\beta_3}{\alpha_3} + \dots$$

converges if $\sum_{n=1}^\infty \frac{\alpha_n \alpha_{n+1}}{\beta_{n+1}} = \infty$. Moreover, when the x_n is the n th approximant as in Theorem 3, then

$$x_0 < x_2 < \dots < x_{2k} < \dots < x < \dots < x_{2k+1} < \dots < x_3 < x_1$$

and so the limit is x in the case where x_n converges. \diamond

Definition 1. Two continued fractions

$$x = \alpha_0 + \cfrac{\beta_1}{\alpha_1} + \cfrac{\beta_2}{\alpha_2} + \cfrac{\beta_3}{\alpha_3} + \dots \quad \text{and} \quad x' = \alpha'_0 + \cfrac{\beta'_1}{\alpha'_1} + \cfrac{\beta'_2}{\alpha'_2} + \cfrac{\beta'_3}{\alpha'_3} + \dots$$

are *equivalent* if there exists a sequence of nonzero real numbers $\{c_n\}_{n=1}^\infty$ with $c_0 = 1$ such that

$$\alpha'_n = c_n \alpha_n \quad \text{and} \quad \beta'_n = c_n c_{n-1} \beta_n, \quad n = 1, 2, \dots$$

The c_n terms may be thought of as constants multiplied by both numerators and denominators of successive terms. We will exploit equivalence in several ways. \diamond

3 Continued Logarithm Recurrences

The recurrences for continued logarithms are less familiar.

Theorem 1. Let x be the real number with continued logarithm $[a_0, a_1, a_2, \dots]$, and let x_n be the n th continued logarithm approximant, that is, the number which terminates on 1 after the continued logarithm sequence $[a_0, a_1, a_2, \dots, a_n]$. Then

$$x_n = \frac{r_n}{s_n}$$

where $r_{-1} = 1$, $s_{-1} = 0$, $r_0 = 2^{a_0}$, $s_0 = 1$, and

$$r_{n+1} = 2^{a_{n+1}}r_n + 2^{a_n}r_{n-1}$$

and

$$s_{n+1} = 2^{a_{n+1}}s_n + 2^{a_n}s_{n-1}.$$

Proof. The proof of this follows immediately from the classical recursion in Remark 3 for irregular continued fractions. Simply notice that $2^{a_0} = \alpha_0$, $2^{a_{n+1}} = \alpha_{n+1}$, and $2^{a_n} = \beta_{n+1}$ for all n . \square

Note that r_n and s_n are no longer necessarily relatively prime; indeed, they usually will not be! Furthermore, consecutive approximants differ nicely, just as with continued fractions.

Theorem 2. With $x_n = \frac{r_n}{s_n}$ as above,

$$\frac{r_n}{s_n} - \frac{r_{n-1}}{s_{n-1}} = \frac{(-1)^{n+1}2^{a_0+a_1+\dots+a_{n-1}}}{s_n s_{n-1}}.$$

Proof. It follows immediately from Remark 4 that

$$\frac{r_n}{s_n} - \frac{r_{n-1}}{s_{n-1}} = \frac{(-1)^{n+1}\beta_1\beta_2\dots\beta_n}{s_n s_{n-1}}.$$

Simply notice that, for the continued logarithm, we set $\beta_k = 2^{\alpha_{k-1}}$. \square

Definition 2. A continued fraction where the numerator terms are all 1 and the denominator terms are rationals is called *reduced*.

We shall call a continued fraction where the denominator terms are all 1 and the numerator terms are rationals *denominator reduced*.

A *reduced* continued fraction where the denominator terms are all integers is called *simple*.

Lemma 3. The binary continued logarithm $[a_0, a_1, a_2, \dots]_{cl(2)}$ is equivalent to each of the two continued fractions below: the reduced form and the denominator reduced form respectively.

$$2^{a_0} + \cfrac{1}{\sqrt{2^{a_1-a_0}}} + \cfrac{1}{\sqrt{2^{a_2-a_1+a_0}}} + \cfrac{1}{\sqrt{2^{a_3-a_2+a_1-a_0}}} + \dots + \cfrac{1}{\sqrt{2^{\sum_{k=0}^n (-1)^{n-k} a_k}}} + \dots \quad (5)$$

$$2^{a_0} + \cfrac{2^{-a_1+a_0}}{\sqrt{1}} + \cfrac{2^{-a_2}}{\sqrt{1}} + \cfrac{2^{-a_3}}{\sqrt{1}} + \dots + \cfrac{2^{-a_n}}{\sqrt{1}} + \dots \quad (6)$$

Proof. Recall from Equation (4) that

$$[a_0, a_1, a_2, \dots]_{cl(2)} = 2^{a_0} + \cfrac{2^{a_0}}{\sqrt{2^{a_1}}} + \cfrac{2^{a_1}}{\sqrt{2^{a_2}}} + \dots$$

In terms of Definition 1, this is just $\alpha_n = 2^{a_n}$, $\beta_n = 2^{a_{n-1}}$ for all n . We will construct a sequence $\{c_n\}_{n=0}^{\infty}$:

$$c_0 = 1, \quad c_1 = 2^{-a_0}, \quad c_2 = 2^{-a_1+a_0}, \quad \dots, \quad c_n = 2^{\sum_{k=0}^{n-1} (-1)^{n-k} a_k} \dots$$

Then, with $\alpha'_n = 2^{\sum_{k=0}^n (-1)^{n-k} a_k}$ and $\beta'_n = 1$ for all n , the requirements of Definition 1 are satisfied, showing equivalence for Equation (5). Now let

$$c_0 = 1, c_1 = 2^{-a_1}, \dots, c_n = 2^{-a_n}, \dots$$

Then, with $\alpha'_0 = 2^{a_0}$, $\alpha'_n = 1$ for all $n > 0$, $\beta'_1 = 2^{-a_1+a_0}$, $\beta'_n = 2^{-a_n}$ for all $n > 1$, the requirements of Definition 1 are again satisfied, showing equivalence for Equation (6). \square

Theorem 4 (Binary Convergence). *The binary continued logarithm for a number $x \geq 1$ converges to x .*

Proof. Suppose that the binary continued logarithm for x terminates and is $[a_1, \dots, a_n]_{\text{cl}(2)}$. From the construction in Remark 1,

$$\begin{aligned} x = y_0 &= 2^{a_0} + \frac{2^{a_0}}{y_1} \\ y_1 &= 2^{a_1} + \frac{2^{a_1}}{y_2} \\ &\dots \end{aligned}$$

From Equation (2), finite termination for the continued logarithm implies that

$$y_{n-1} = \frac{1}{\frac{y_{n-1}}{2^{a_{n-1}}} - 1} = 2^{a_n}.$$

Thus we simply have

$$x = 2^{a_0} + \frac{2^{a_0}}{2^{a_1} + \frac{2^{a_1}}{2^{a_2} + \dots + \frac{2^{a_{n-1}}}{2^{a_n}}}}.$$

This shows convergence in the case of finite termination. We now consider the case where it does not terminate finitely. Let α_n and β_n be the n th denominator and numerator terms of the continued fraction corresponding to the continued logarithm for a number x . Then each of the α_n, β_n are positive and

$$\sum_{n=1}^{\infty} \frac{\alpha_n \alpha_{n+1}}{\beta_{n+1}} = \sum_{n=1}^{\infty} 2^{a_n} = \infty, \text{ since each } a_n \text{ is a nonnegative integer.}$$

The convergence to x then follows immediately from Remark 5. \square

Theorem 5 (Finite representation of rationals). *The binary continued logarithm for a rational number will always terminate finitely (and conversely).*

Proof. Again we use the reduced form of Equation (6) with $\alpha_n = 1$, and $\beta_n = 2^{-a_n}$ for all $n \geq 2$. Suppose y is rational and let y_n be the ‘tail’ of the continued fraction. Then we may write

$$y_n = \frac{2^{-a_n}}{1 + y_{n+1}},$$

so that

$$y_{n+1} + 1 = \frac{1}{2^{a_n} y_n}.$$

Since each $0 < y_n$ is rational we may write $y_n = u_n/v_n$ for positive relatively prime integers u_n, v_n . Thence

$$y_{n+1} + 1 = \frac{u_{n+1} + v_{n+1}}{v_{n+1}} = \frac{v_n}{2^{a_n} u_n}.$$

Hence

$$2^{a_n} u_n (u_{n+1} + v_{n+1}) = v_n v_{n+1}.$$

Since $u_{n+1} + v_{n+1}$ and v_{n+1} are relatively prime we deduce that $u_{n+1} + v_{n+1}$ is a divisor of v_n . In particular, $u_{n+1} + v_{n+1} \leq v_n$ and so v_n is a sequence of strictly decreasing natural numbers and so must terminate. \square

We conclude this section with the first few terms of the binary continued logarithms for π and e .

$$\begin{aligned} \pi &= [1, 0, 0, 1, 0, 0, 3, 0, 3, 0, 2, 0, 0, 2, 5, \dots]_{\text{cl}(2)} \\ e &= [1, 1, 1, 1, 0, 2, 2, 0, 2, 0, 0, 0, 1, 1, 0, \dots]_{\text{cl}(2)} \end{aligned}$$

While the terms 299 and 20177 appear as the 5th and 432nd terms of the continued fraction expansion for π [9], the largest terms in the first 100 and 500 continued logarithm terms for π are 8 and 15 respectively. Similarly for e , we find that early large terms are not present in the continued logarithm representation. Weisstein also notes that, in 1977, the record computation of the continued fraction for π was due to Gosper.[9].

4 A Gauss-Kuzmin Distribution for Continued Logarithms

In this section we recall the celebrated Gauss-Kuzmin distribution for simple continued fractions and the related Khintchine constant. Through intensive numerical computation which we will describe in Section 6, we discovered a distribution governing the continued logarithm terms which is analogous to the Gauss-Kuzmin Distribution. Conveniently, the arithmetic mean of the log terms relates to the geometric mean of the corresponding continued fraction terms in a very desirable way. We will begin by restating some classical results.

4.1 The Continued Fraction Case

Theorem 6 (Gauss, Kuzmin, Lévy). *Let $x \in (0, 1)$ and $\mathcal{M}(A)$ denote the measure of a set A . Let $\alpha_n(x)$ denote the n th denominator term of the simple continued fraction for x . Then we have that*

$$\lim_{n \rightarrow \infty} \mathcal{M}(\{x \in (0, 1) : \alpha_{n+1}(x) = k\}) = \log_2 \left(1 + \frac{1}{k(k+2)} \right).$$

Proof. See [2, Theorem 3.23 (Gauss, Kuzmin, Lévy)]. \square

From this one may derive the following.

Corollary 7 (Khinchine's Constant). *For almost all real numbers x , where the α_k are the denominator values of a simple continued fraction $x = [\alpha_1, \alpha_2, \dots]_{cf}$,*

$$\mathcal{K} = \lim_{n \rightarrow \infty} \sqrt[n]{\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\log_2 k} = 2.6854520010653. \quad (7)$$

Proof. See [2, Remark 3.6]. \square

Recall that the reason we do not generally study the arithmetic mean is that it is infinite for almost all real numbers [2]. There are, however, well-defined harmonic and other means [1]. To the best of our knowledge, **the number \mathcal{K}** is a transcendental constant unrelated to classical ones. We call this **the Khinchine (geometric mean) constant** and denote it by \mathcal{K} .

It is natural to ask if Gauss and Kuzmin's result has a version for continued logarithms. We investigated by intensive numerical computation and discovered the following result.

4.2 The Continued Logarithm Case

We discovered quite remarkably that the arithmetic means for continued logarithms have closed form.

Theorem 8. *Where x and \mathcal{M} are as defined in Remark 6 and α_n is the n th continued logarithm term,*

$$\mathcal{P}(k) := \lim_{n \rightarrow \infty} \mathcal{M}(\{x \in (0, 1) : \alpha_n(x) = 2^k\}) = \frac{\log\left(1 + \frac{2^k}{(1+2^{k+1})^2}\right)}{\log\left(\frac{4}{3}\right)}.$$

Proof. We subsequently discovered that this result has actually been proven in the seemingly different context of [5]. The adherence to the distribution of three presumably aperiodic irrationals is shown in Figure 1. **The accuracy for a broader selection of aperiodic numbers is illustrated further in Figure 2. The vertical axis shows the deviation of the actual percentage of 0 terms in their respective continued logarithms from the proportion theory would suggest. The horizontal axis shows the deviation of the mean of the continued logarithm terms from what the theory would suggest.** \square

Before verifying that \mathcal{P} is, in fact, a probability density function, we will begin with two useful propositions.

Proposition 9. *Let $h : \mathbb{N} \rightarrow \mathbb{R}$. The following equality holds:*

$$\prod_{k=0}^N \left(\frac{h(k) \cdot h(k+2)}{h(k+1)^2}\right) = \frac{h(0) \cdot h(N+2)}{h(1) \cdot h(N+1)}.$$

Proof. This follows from straightforward induction once found. \square

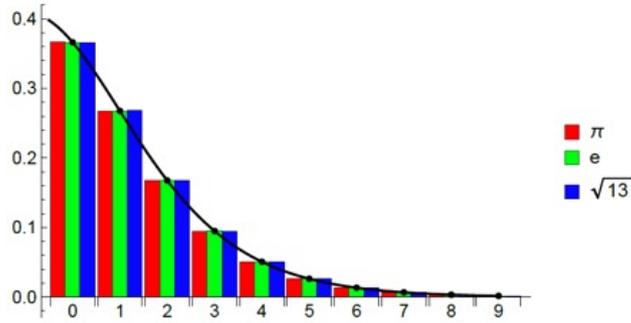


Figure 1: Distribution of continued logarithm terms shown for three presumably aperiodic irrationals computed to one million terms.

Proposition 10. Let $h : \mathbb{N} \rightarrow \mathbb{R}$. The following equality also holds:

$$\prod_{k=0}^N \left(\frac{h(k)^k \cdot h(k+2)^k}{h(k+1)^{2k}} \right) = \frac{h(1) \cdot h(N+2)^N}{h(N+1)^{N+1}}.$$

Proof. The proof by induction is again straightforward. It can also be deduced directly from Proposition 9. \square

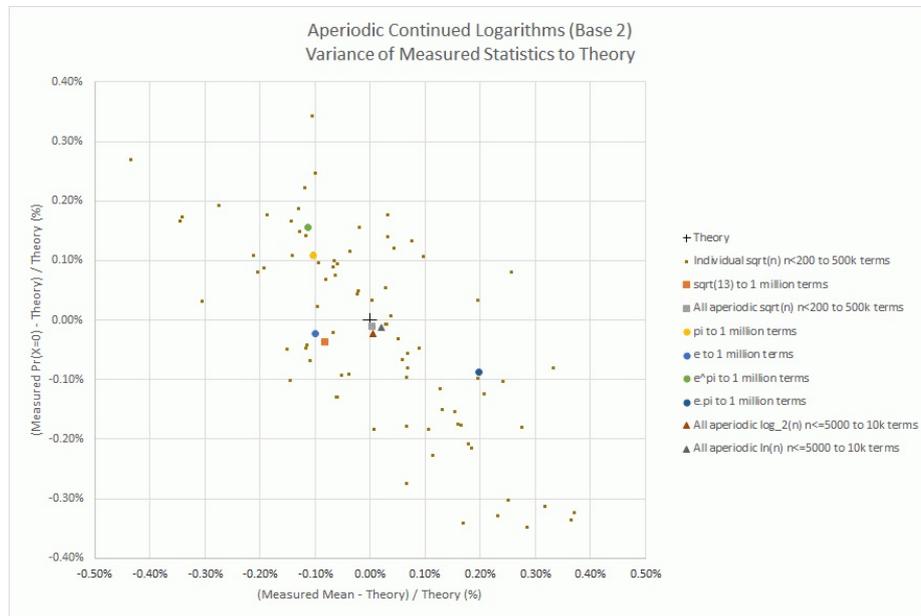


Figure 2: Deviation from expectation for a selection of aperiodic numbers.

Theorem 11. For almost all real numbers $x > 1$ and any integer k , the probability that k is any given digit of the binary continued logarithm for x is given by the probability density function $\mathcal{P} : \mathbb{N} \rightarrow (0, 1)$ where

$$\mathcal{P}(k) = \frac{\log\left(1 + \frac{2^k}{(1+2^{k+1})^2}\right)}{\log\left(\frac{4}{3}\right)}.$$

Proof. To show that this is, in fact, a probability density function, we will show that

$$\sum_{k=0}^{\infty} \log\left(1 + \frac{2^k}{(1+2^{k+1})^2}\right) = \log\left(\frac{4}{3}\right)$$

by showing that $\lim_{N \rightarrow \infty} S_N = \log\left(\frac{4}{3}\right)$ where $S_N = \sum_{k=0}^N \log\left(1 + \frac{2^k}{(1+2^{k+1})^2}\right)$. Since the sum of the logs is the log of the product of their inner terms, we may write

$$\sum_{k=0}^N \log\left(1 + \frac{2^k}{(1+2^{k+1})^2}\right) = \log\left(\prod_{k=0}^N \left(1 + \frac{2^k}{(1+2^{k+1})^2}\right)\right).$$

Thus it is sufficient to consider only the behavior of the product inside of the right hand side. Accordingly, we will show that

$$\lim_{N \rightarrow \infty} \prod_{k=0}^N \left(1 + \frac{2^k}{(1+2^{k+1})^2}\right) = \frac{4}{3}.$$

To that end, let $d, h : \mathbb{N} \rightarrow \mathbb{R}$ by

$$d(x) = 1 + \frac{x}{(1+2x)^2} \tag{8}$$

$$h(x) = 1 + 2^x. \tag{9}$$

Then we have that

$$\prod_{k=0}^N \left(1 + \frac{2^k}{(1+2^{k+1})^2}\right) = \prod_{k=0}^N d(2^k).$$

Factoring and simplifying d , we just have

$$d(x) = \frac{(x+1)(4x+1)}{(1+2x)^2}. \tag{10}$$

Thus, we have that

$$d(2^k) = \frac{(2^k+1)(4 \cdot 2^k+1)}{(1+2 \cdot 2^k)^2} = \frac{(2^k+1)(2^{k+2}+1)}{(1+2^{k+1})^2} = \frac{h(k) \cdot h(k+2)}{h(k+1)^2}.$$

Written this way, it becomes apparent from Proposition 9 that we actually have a telescoping product. Specifically,

$$\prod_{k=0}^N \left(1 + \frac{2^k}{(1+2^{k+1})^2}\right) = \prod_{k=0}^N \left(\frac{h(k) \cdot h(k+2)}{h(k+1)^2}\right) = \frac{h(0) \cdot h(N+2)}{h(1) \cdot h(N+1)}.$$

Simplifying and taking the limit, we have

$$\lim_{N \rightarrow \infty} \frac{h(0) \cdot h(N+2)}{h(1) \cdot h(N+1)} = \lim_{N \rightarrow \infty} \frac{(2)(1+2^{N+2})}{(3)(1+2^{N+1})} = \lim_{N \rightarrow \infty} \frac{2+2^{N+3}}{3+3 \cdot 2^{N+1}} = \frac{4}{3}.$$

Thus the convergence is shown. \square

5 Khintchine's Constant for Continued Logarithms

As a corollary of Theorem 11, we obtain a constant we call \mathcal{KL}_2 which gives the predicted arithmetic mean of the continued logarithm terms. That is to say: where $x = [a_0, a_1, \dots]_{\text{cl}(2)}$, we obtain a constant which gives us the mean of the exponent terms on the 2s of the continued fraction:

$$\mathcal{KL}_2 := \lim_{N \rightarrow \infty} \left(\frac{1}{N} \right) \sum_{k=0}^N a_k. \quad (11)$$

It is a consequence of Theorem 8 that this mean exists for almost all real numbers.

Corollary 12 (Khintchine's Binary Logarithm Constant). *For almost all numbers greater than one the arithmetic mean of the binary continued logarithm terms is*

$$\mathcal{KL}_2 = \frac{\log\left(\frac{3}{2}\right)}{\log\left(\frac{4}{3}\right)} = 1.4094208396532. \quad (12)$$

Worthy of note is that, while Khintchine's Constant is believed not to be related to any naturally occurring number, \mathcal{KL}_2 is actually elementary and has closed form.

Proof. Let d, h be as in Theorem 11. Since the arithmetic mean is precisely

$$\sum_{k=0}^{\infty} k \cdot \mathcal{P}(k) = \sum_{k=0}^{\infty} k \cdot \frac{\log(d(2^k))}{\log\left(\frac{4}{3}\right)} = \frac{1}{\log\left(\frac{4}{3}\right)} \sum_{k=0}^{\infty} \log((d(2^k))^k),$$

We may, as we did in Theorem 11, restrict to showing that the product inside of the log on the right hand side below converges, in this case to $\frac{3}{2}$:

$$\lim_{N \rightarrow \infty} \sum_{k=0}^N \log(d(2^k)^k) = \lim_{N \rightarrow \infty} \log\left(\prod_{k=0}^N d(2^k)^k\right).$$

Considering this product, we see from Proposition 10 that it telescopes nicely:

$$\prod_{k=0}^N d(2^k)^k = \prod_{k=0}^N \frac{h(k)^k \cdot h(k+2)^k}{h(k+1)^{2k}} = \frac{h(1) \cdot h(N+2)^N}{h(N+1)^{N+1}}.$$

Simplifying and evaluating,

$$\lim_{N \rightarrow \infty} \frac{h(1) \cdot h(N+2)^N}{h(N+1)^{N+1}} = \lim_{N \rightarrow \infty} \frac{3 \cdot (1+2^{N+2})^N}{(1+2^{N+1})^{N+1}} = \frac{3}{2}.$$

This shows the desired convergence. \square

Remark 6. The predicted geometric mean of the first N continued fraction terms is just ‘2’ raised to the arithmetic mean of the log terms. To see why this is the case, just notice that, where \mathcal{KL}_2 denotes the arithmetic mean of the binary continued logarithm terms, the expected geometric mean of the continued fraction terms is given by

$$\mathcal{G} = \lim_{N \rightarrow \infty} \left(\prod_{k=0}^{N-1} 2^{\mathcal{KL}_2} \right)^{\frac{1}{N}} = \lim_{N \rightarrow \infty} (2^{N \cdot \mathcal{KL}_2})^{\frac{1}{N}} = 2^{\mathcal{KL}_2} = 2.6563050580919\dots \quad (13)$$

Note that \mathcal{G} , unlike \mathcal{K} , is an elementary constant. \diamond

6 Experimental Discovery of the Distribution

We give an overview of the experimental methods that were employed in the discovery of the distribution of Theorem 8.

6.1 Finding the Functional Relation

Let $x \in \mathbb{R}, x > 1$ such that x has the (aperiodic) continued logarithm $[a_0, a_1, \dots]_{\text{cl}(2)}$. Let x_n be the n th tail of the equivalent denominator reduced continued fraction. Then we have that

$$x = 2^{a_0} \cdot \left(1 + \frac{2^{-a_1}}{1 + \frac{2^{-a_2}}{\dots + \frac{2^{-a_n}}{x_n}}} \right). \quad (14)$$

Consider the Lebesgue measure $\mu_n(\zeta)$ of $\{x \in (1, 2) : x_n < \zeta\}$. Given $x_{n-1} = 1 + \frac{2^{-a_n}}{x_n}$ it follows that $x_n < x$ if and only if

$$\frac{2^{-a_n}}{x_{n-1} - 1} < x. \quad (15)$$

This is just the same as

$$x_{n-1} > 1 + \frac{2^{-a_n}}{x}. \quad (16)$$

But, in order to produce the term, a_n , we must have

$$x_{n-1} < 1 + 2^{-a_n}. \quad (17)$$

Combining these two inequalities, we obtain

$$1 + \frac{2^{-a_n}}{x} < x_{n-1} < 1 + 2^{-a_n}. \quad (18)$$

Thus we have that

$$\mu_0(x) = x - 1 \quad (19)$$

$$\mu_n(x) = \sum_{k=0}^{\infty} \left(\mu_{n-1}(1 + 2^{-k}) - \mu_{n-1}\left(1 + \frac{2^{-k}}{x}\right) \right). \quad (20)$$

6.2 Finding the Closed Form of $\mu(x)$

We investigated the form of $\mu(x)$ by iterating the recurrence relation in Equation (19) at points evenly spaced over the interval $[1, 2]$. We began with $\mu_0(x) = x - 1$, fitting a spline to these points at each iteration. This may be seen in the *Mathematica* code in Figure 3 which evaluates each “infinite” sum to 100 terms and breaks the interval $[1, 2]$ into 100 sub-intervals.

```
pd[n_]:= Module[{i, u, v, x, f},
  u = ListInterpolation[{1, 2}], {{1, 2}},
  InterpolationOrder -> 1];
  For[i=0, i<n, i++,
    v=Table[Sum[u[1+2^-k]-u[1+2^-k/x],
      {k, 0, 100}], {x, 1, 2, .01}];
    u=ListInterpolation[v, {{1, 2}}, Method -> "Spline"];
  ];
  Table[{x, u[x]}, {x, 1, 2, .01}]
]
```

Figure 3: Mathematica code for fitting the spline to the points at each iteration.

We found good apparent convergence of $\mu(x)$ after about 10 iterations. We used the 101 data points from this process to seek the best fit to a function of the form

$$\mu(x) = C \log_2 \left(\frac{ax + b}{cx + d} \right)$$

where C, a, b, c , and d are constants to be determined by the fitting process. Note that, in order to meet the boundary conditions, it is necessary that

$$\begin{aligned} \mu(1) &= 0 \\ \mu(2) &= 1 \\ d &= a + b - c \\ C &= \frac{1}{\log_2 \left(\frac{2a+b}{a+b+c} \right)}. \end{aligned}$$

Motivated by the case of a simple continued fraction, we had originally considered the form $C \log_2(ax + b)$ and, when that failed, we considered a superposition of two such terms.

To eliminate any common factor between the numerator and denominator of $\frac{ax+b}{cx+d}$, we set $c = 1$, leaving the functional form to be fitted as

$$\mu(x) = \frac{\log_2 \left(\frac{ax+b}{x+a+b-1} \right)}{\log_2 \left(\frac{2a+b}{a+b+1} \right)}. \quad (21)$$

The code to perform this fit and its output are shown in Figure 4. Mathematica gives the output

```

data=pd[10];
FindFit[data,Re[Log[2,(a-x+b)/(x+a+b-1)]/
Log[2,(2a+b)/(a+b+1)]],{{a,0.9},{b,1}},x]

```

Figure 4: Code showing the fit.

```
{a -> 0.49999983502291956, b -> 0.5000002651575332}
```

This result suggests candidate values of $a = \frac{1}{2}$ and $b = \frac{1}{2}$. It follows that

$$\mu(x) = \frac{\log\left(\frac{2x}{x+1}\right)}{\log\left(\frac{4}{3}\right)}. \quad (22)$$

Equation (22) gives us the following probability distribution.

6.3 Probability Distribution

From Equation (22), the probability distribution of the terms of the continued logarithm in the aperiodic case where the approximants are presumed to be randomly distributed on the interval $(1, 2)$ is given by

$$\Pr(X = k) = \mu(1 + 2^{-k}) - \mu(1 + 2^{-k-1}) = \frac{\log\left(1 + \frac{2^k}{(2^{k+1}+1)^2}\right)}{\log\left(\frac{4}{3}\right)}. \quad (23)$$

We then computed the mean:

$$E(X) = \sum_{k=0}^{\infty} k \cdot \Pr(X = k) = 1.4094208397\dots$$

The distribution for the first several terms is given in Figure 5.

k	Pr(X=k)
0	0.3662394210...
1	0.2675211579...
2	0.1675533738...
3	0.0949153712...
4	0.0507000346...
5	0.0262283498...
6	0.0133430145...
7	0.0067299284...

Figure 5: Distribution for the first eight binary continued logarithm terms.

The curve of the distribution is drawn in Figure 1.

7 Continued Logarithms with Interesting Continued Fractions

Let us say that a sequence $a = [a_0, a_1, \dots, a_n]$ has the *alternating sum property* if for each k ,

$$b_k = \sum_{i=0}^k (-1)^{k-i} a_i \geq 0.$$

It is easy to prove the following:

Theorem 13. *let $a = [a_0, a_1, \dots, a_n]$ have the alternating sum property. Then if x has continued logarithm a then the simple continued fraction for x is $[2^{b_0}, 2^{b_1}, \dots, 2^{b_n}]$ (where the final term could be $2^{b_n} + 1$ depending on the form of the continued fraction (does it end in a final $+1/1$ term)).*

Proof. This is an immediate consequence of Equation (5) in Lemma 3. \square

Question 1. Are there any nice representations for elementary or special functions arising from continued logarithms in a manner analogous to the irregular continued fraction for \tan^{-1} [2]?

8 Continued Logarithms of Quadratic Irrationals

We recall the *Euler-Lagrange theorem*, that for simple continued fractions, x has an ultimately periodic simple continued fraction if and only if x is a quadratic irrational. See for example [2, Thm 2.48].

By contrast, for continued logarithms, it becomes clear that \sqrt{n} sometimes has a nice periodic continued logarithm but not always. For example, $\sqrt{13}$ appears to be aperiodic, as do $\sqrt{14}$ and $\sqrt{15}$. However, when we consider $\sqrt{17}$, we discover that it has a nice continued logarithm (periodic constant). Similarly, $\sqrt{19}$, $\sqrt{21}$ and $\sqrt{23}$ are likewise periodic. We again find aperiodic \sqrt{n} for n values 31, 35, 39, 41, 43, 46, 47, 55, 57, 59, 61, 62, 63, 67, 71, 79, 85, 91, 94, 97, 99, 101, 103, 106, 107, 109, 113, 114, 115, 116, 119, and so on.

We present a method of computation of continued logarithms of quadratic irrationals which uses integer arithmetic only. Even in the case of surds which are aperiodic (e.g. $\sqrt{13}$) we find this method to be roughly an order of magnitude faster than a conventional approach using fixed-precision, floating-point arithmetic.

We define a dynamical system g on $[1, \infty)$:

$$g(x) = \begin{cases} x/b & \text{if } x \geq b \\ \frac{b-1}{x-1} & \text{if } 1 < x < b \\ \text{terminate} & \text{if } x = 1. \end{cases} \quad (24)$$

We count how many times we divide by the base, b , before we subtract and reciprocate or terminate. This count provides the terms of the continued logarithm, a_0, a_1, \dots

We consider the general case where x can be expressed in the form

$$x = \frac{p}{q}(c + d\sqrt{n}). \quad (25)$$

where p, q, c, d and n are all integers with $p, q > 0$ and $n > 1$. To implement this dynamical system efficiently, there are two cases to be considered.

Case 1: $d = 0$

This circumstance arises either because we seek the continued logarithm of a rational number, p/q , or n is a square number. In the former case, we start with $p = cp, c = 1, d = 0$, and, in the latter case, we set $p = c + d\sqrt{n}, c = 1$, and $d = 0$. Henceforth, we may ignore c and d since $x = p/q$. From this simplified definition it follows that

$$\begin{aligned} x \geq b & \quad \text{iff} \quad p \geq bq \\ x = 1 & \quad \text{iff} \quad p = q. \end{aligned} \quad (26)$$

Given the current value of x , represented by integers (p, q) , we evaluate $g(x)$, represented by integers (p', q') , as follows.

$$\begin{aligned} p' = p, & & \text{for } x \geq b \\ q' = bq & & \\ \\ p' = q, & & \text{for } 1 < x < b \\ q' = p - q & & \end{aligned} \quad (27)$$

Case 2: $d \neq 0$

In this general circumstance, more care is needed to evaluate the conditions governing the dynamical system using integer arithmetic. The way these tests are performed depends on the sign of d and the sign of $bq - cp$ or $q - cp$ as follows:

Condition	d	$bq - cp$	True iff
$x \geq b$	+	+	$nd^2p^2 \geq (bq - cp)^2$
	+	-	Always
	-	+	Never
	-	-	$nd^2p^2 \leq (bq - cp)^2$
Condition	d	$q - cp$	True iff
$x = 1$	+	+	$nd^2p^2 = (q - cp)^2$
	+	-	Never
	-	+	Never
	-	-	$nd^2p^2 = (q - cp)^2$

(28)

Note that the above formulation depends on p and q being positive numbers. For this reason the sign of c and d should be reversed, where necessary, at each iteration to satisfy this requirement.

Given the current value of x , represented by integers (p, q, c, d) , we evaluate $g(x)$, represented by integers (p', q', c', d') , as follows.

$$\begin{aligned}
 p' &= p, \\
 q' &= bq \\
 c' &= c \\
 d' &= d
 \end{aligned}
 \qquad \text{for } x \geq b$$

$$\begin{aligned}
 p' &= (b-1)q, \\
 q' &= (cp-q)^2 - nd^2p^2 \\
 c' &= cp-q \\
 d' &= -dp
 \end{aligned}
 \qquad \text{for } 1 < x < b$$
(29)

8.1 Empirical study of quadratic numbers

Given this method of computing the continued logarithm for a quadratic number, we were able to perform some extensive calculations. We computed 200,000 terms for $2 \leq n \leq 50,000$, 20,000 terms up to two million, and 2,000 terms for $n \leq 1.2 \cdot 10^9$. The longest period found was 293 for $n = 16,813,731$. While we might be missing some periodic roots with very long periods, we should have detected any with periods up to length 3,000 for $n < 2,000,000$ and periods up to 600 thereafter.

Remark 7 (Detecting periodicity). For this purpose, we computed the first 10,000 terms of the binary continued logarithm of \sqrt{n} for $n \leq 2,000,000$. Then starting at the last term, we compare to previous terms until there is a mismatch, then last two terms to previous two, etc. As long as we found a sequence of repeating terms that covered the last two thirds of the entire sequence, we were confident that we had found a periodic clog. In other words, the prefix had to be shorter than 3,333 terms and the maximum detectable period is 3,333. For $2,000,000 < n < 1.2 \cdot 10^9$, We only computed 2,000 terms, so the upper limit on the period detectable is now 666.

We *conjecture* that for periodic clogs of \sqrt{n} the prefix must be exactly two terms. If this is so, then 10,000 computed terms would detect periods up to 4,999. As previously mentioned, we never found a period greater than 300 for any n in the range studied. \diamond

The $\{n, \text{period}\}$ -tuples that appear to define the upper boundary of growth, for n values less than one thousand, are

$$\{2, 1\}, \{23, 20\}, \{37, 26\}, \{167, 66\}, \{531, 134\}, \{819, 178\}.$$

These are consistent with an upper bound on growth of $1.4 \cdot n^{1/2.27} \log n$, but this seems like an overestimate for larger n .

It appears, see Figure 6, that asymptotically the number of periodic quadratics is very small and can largely be explained by ad hoc arguments, as in the case of 5,17,... This has been checked by looking at the first 200,000 continued logarithm terms for n up to 50,000. While there are 237 periodic roots for n up to 1000, there are only 1,262 periodic roots in the first 50,000.

More interestingly, in every aperiodic case tested, \sqrt{n} appears to satisfy the limiting distribution of Theorem 8. (See Figure 8). This leads to the *conjecture*

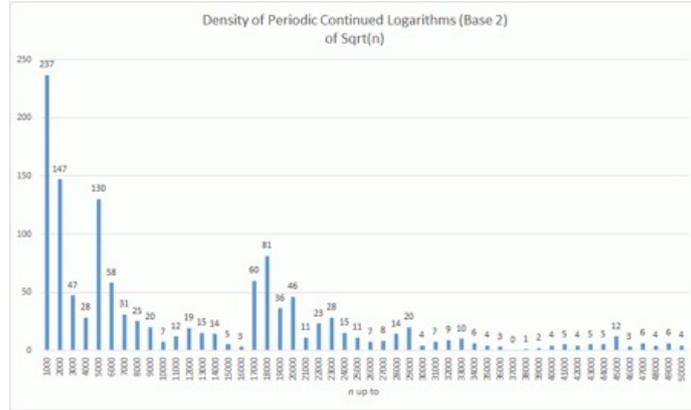


Figure 6: Density of periodic binary continued logarithms for $2 \leq \sqrt{n} \leq 50,000$.

that \sqrt{n} is either eventually periodic or satisfies the limiting distribution and the corresponding Khintchine constant.

The length of the period seems related to the fundamental solution of the corresponding Pell's equation as in the case of simple continued fractions [7], but we have been unable to obtain a precise conjecture. As Figure 8 shows, the period can vary widely in length as also true of the simple continued fraction.

What we have also observed - and can presumably prove by the classical method - is that the continued logarithm of a periodic square root is palindromic (after an initial segment). For example, for $\sqrt{10}$ we have

$$[1, 0, 0, 1, \overline{1, 0, 1}, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, \dots]_{cl(2)}.$$

For $\sqrt{11}$ we have

$$[1, 0, 0, 0, 3, 0, 0, 1, \overline{1, 0, 0, 3, 0, 0, 1}, 1, 0, 0, 3, 0, 0, 1, 1, 0, 0, 3, 0, 0, 1, \dots]_{cl(2)}.$$

9 Continued Logarithms to Other Bases: Type I

If we try to construct continued logarithms to a different base, say by dividing by '3' instead of by '2,' then we run into problems with the second type of map. All we are guaranteed after the final division by three is that $x \in [1, 3)$. The map

$$x \rightarrow \frac{1}{x-1}$$

takes this interval to $[\frac{1}{2}, \infty)$ rather than $[1, \infty)$. A solution to this is the following: if we are repeatedly dividing by b , then we replace the second map by

$$x \rightarrow \frac{b-1}{x-1}.$$

Thus we may describe the process with the dynamical system

$$g_b(x) = \begin{cases} x/b & \text{if } x \geq b \\ \frac{b-1}{x-1} & \text{if } 1 < x < b \\ \text{terminate} & \text{if } x = 1 \end{cases} \quad (30)$$

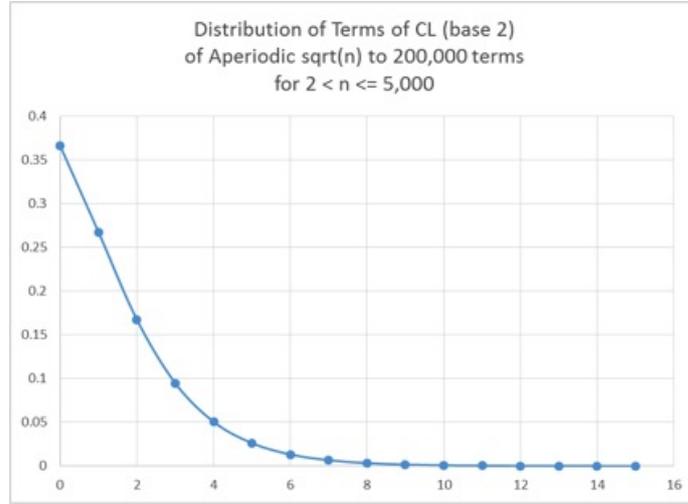


Figure 7: Distribution of the first 200,000 terms of the binary continued logarithm for aperiodic \sqrt{n} for $2 < n \leq 5,000$.

Remark 8. We may describe this construction in a manner analogous to our binary construction in Remark 1. Letting $x = y_0$, we have

$$y_0 = b^{a_0} + (y_0 - b^{a_0}) = b^{a_0} + \frac{b-1}{\frac{y_0 - b^{a_0}}{b-1}} = b^{a_0} + \frac{(b-1) \cdot b^{a_0}}{\frac{y_0 - b^{a_0}}{b-1}}$$

Dividing the highest largest power of b out of the numerator and denominator of the lower fraction, we obtain

$$y_0 = b^{a_0} + \frac{(b-1) \cdot b^{a_0}}{\frac{y_0}{b^{a_0}} - 1} = b^{a_0} + \frac{(b-1) \cdot b^{a_0}}{y_1} \text{ where } y_1 = \frac{y_0}{b^{a_0}} - 1.$$

We continue on in similar fashion. For a fixed base b we denote $l_b(x)$ the continued logarithm of x taken to base b . The representation of this type of continued logarithm in continued fraction form is as follows:

$$x = b^{a_0} + \frac{(b-1)b^{a_0}}{b^{a_1} + \frac{(b-1)b^{a_1}}{b^{a_2} + \frac{(b-1)b^{a_2}}{b^{a_3} + \dots}}}. \quad (31)$$

For $b = 2$, this is just Gosper's original formulation from Equation (4).

Example 2. Consider $\frac{1233}{47}$ which has ternary continued logarithm $l_3(\frac{1233}{47}) =$

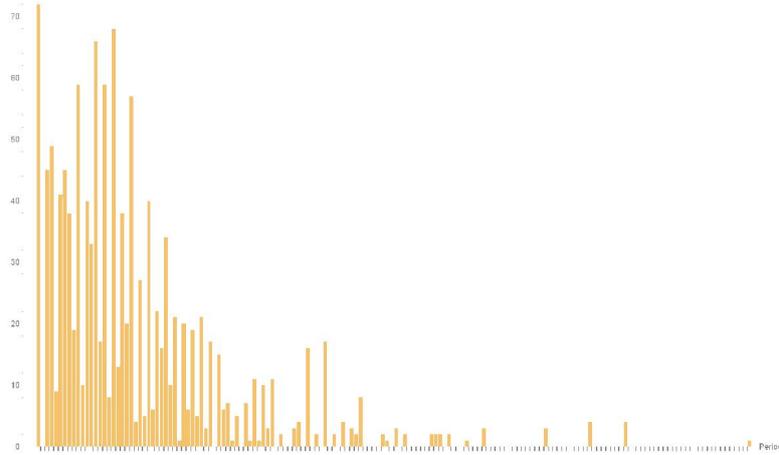


Figure 8: Distribution of periods of binary continued logarithms for periodic \sqrt{n} for $2 < n \leq 5,000$.

$[2, 0, 3, 1]_{\text{cl}(3)}$. The corresponding continued fraction is as follows.

$$\frac{1233}{47} = 3^2 + \frac{2 \cdot 3^2}{3^0 + \frac{2 \cdot 3^0}{3^3 + \frac{2 \cdot 3^3}{3^1}}} \quad (32)$$

This example will be useful for comparing this Type I formulation of the base b logarithm with the Type II formulation in Section 10. Specifically, compare this example with Example 3. \diamond

Worthy of note is that, with this formulation, rationals do not necessarily have finite continued logarithms. Indeed, to base three, we have $l_3(2) = [0, 0, 0, 0, \dots]$. One might reasonably wonder why the proof in Theorem 5 of finiteness for rationals in the case for $b = 2$ does not carry over to the case $b > 2$. The reason for this is explored in Remark 10. We can and will, however, demonstrate convergence in much the same way as for the $b = 2$ case.

Lemma 14 (Denominator Reduced Equivalence). *The continued logarithm $[a_0, a_1, a_2, \dots]_{\text{cl}(b)}$ is equivalent to the denominator reduced continued fraction*

$$3^{a_0} + \frac{(b-1)b^{-a_1+a_0}}{1} + \frac{(b-1)b^{-a_2}}{1} + \frac{(b-1)b^{-a_3}}{1} + \dots \quad (33)$$

Proof. Recall from Equation (31) that

$$[a_0, a_1, a_2, \dots]_b = b^{a_0} + \frac{(b-1)b^{a_0}}{b^{a_1}} + \frac{(b-1)b^{a_1}}{b^{a_2}} + \dots$$

In terms of Definition 1, this is just $\alpha_n = b^{a_n}, \beta_n = (b-1)b^{a_n-1}$ for all n . We will again construct a sequence $\{c_n\}_{n=0}^\infty$ showing equivalence. Let

$$c_0 = 1, c_1 = b^{-a_1}, \dots, c_n = b^{-a_n}, \dots$$

Then, where $\alpha'_0 = b^{a_0}$, $\alpha'_n = 1$ for all $n > 0$, $\beta'_1 = (b-1)b^{-a_1+a_0}$, $\beta'_n = (b-1)b^{-a_n}$ for all $n > 1$, the requirements of Definition 1 are again satisfied, showing equivalence for Equation (33). \square

Theorem 15 (Type I Convergence). *The Type I continued logarithm of base b for a number $x \geq 1$ converges to x .*

Proof. Suppose that the Type I base b continued logarithm for x terminates finitely and is $[a_1, \dots, a_n]_{\text{cl}(b)}$. From the construction in Remark 8,

$$\begin{aligned} x = y_0 &= b^{a_0} + \frac{(b-1) \cdot b^{a_0}}{y_1} \\ y_1 &= b^{a_1} + \frac{(b-1) \cdot b^{a_1}}{y_2} \\ &\dots \end{aligned}$$

From Equation (30), finite termination for the continued logarithm implies that

$$y_{n-1} = \frac{b-1}{\frac{y_{n-1}}{b^{a_{n-1}}} - 1} = b^{a_n}.$$

Thus we simply have

$$x = b^{a_0} + \frac{(b-1) \cdot b^{a_0}}{b^{a_1} + \frac{(b-1) \cdot b^{a_1}}{b^{a_2} + \dots + \frac{(b-1) \cdot b^{a_{n-1}}}{b^{a_n}}}}.$$

This shows convergence in the case of finite termination. We again turn our attention to the infinite case. Let α_n, β_n again be the denominators and numerators of the continued fraction corresponding to the Type I continued logarithm for a number x . Then each of the α_n, β_n are positive and

$$\sum_{n=1}^{\infty} \frac{\alpha_n \alpha_{n+1}}{\beta_{n+1}} = \sum_{n=1}^{\infty} \frac{b^{a_n}}{(b-1)} = \infty \text{ because each } a_n \text{ is a nonnegative integer.}$$

As was true in the $b = 2$ case, the convergence again follows immediately from Remark 5. \square

Question 2. Given an integer base b , and especially in the case of $b = 3$, determine which rationals (indeed, even which integers) have finite continued logarithms to base b .

We point out that the factor $b-1$ in the numerator of $\frac{b-1}{x-1}$ comes about because when we divide a number greater than b by the final factor of b , we end up with a value in an interval of length $b-1$.

Because of this, we are not restricted to continued logarithms to fixed bases: indeed, we could take the sequence $\omega_n = n!$, and our map becomes the iteration of the following pair of maps: determine n so that $n! \leq x < (n+1)!$. Then let

$$x \longrightarrow \frac{x}{n!} \longrightarrow \begin{cases} \frac{n}{x-1} & \text{if } x \in (1, n+1) \\ \text{terminate} & \text{if } x = 1 \end{cases}.$$

replacing

$$x \rightarrow \begin{cases} \frac{x}{n!} & \text{if } n! \leq x < (n+1)! \\ \frac{n}{x-1} & \text{if } x \in [1, n+1) \\ \text{terminate} & \text{if } x = 1 \end{cases} .$$

If at the k^{th} step we divide by $n_k!$, we could express the continued factorial logarithm (for want of a better name) as $[n_0, n_1, n_2, \dots]$.

In general, if we have a strictly monotonic sequence ω_n , with $\omega_n \rightarrow \infty$, (here ω_n corresponds to b^n , so that ω_{n+1}/ω_n will play a role similar to that of b in the preceding sections) then we can write down a corresponding type of continued logarithm using pairs of maps as follows: determine n so that $\omega_n \leq x < \omega_{n+1}$: then

$$x \rightarrow \frac{x}{\omega_n} \rightarrow \begin{cases} \frac{(\frac{\omega_{n+1}}{\omega_n} - 1)}{x-1} & \text{if } x \in (1, \frac{\omega_{n+1}}{\omega_n}) \\ \text{terminate} & \text{if } x = 1 \end{cases} .$$

replacing

$$x \rightarrow \begin{cases} \frac{x}{\omega_n} & \text{if } \omega_n \leq x < \omega_{n+1} \\ \frac{(\frac{\omega_{n+1}}{\omega_n} - 1)}{x-1} & \text{if } x \in [1, \frac{\omega_{n+1}}{\omega_n}) \\ \text{terminate} & \text{if } x = 1 \end{cases} .$$

Note that the second map takes 1 to ∞ , and sends ω_{n+1}/ω_n to 1.

Again, if x is such that we use the n_k^{th} map at the k^{th} step, we can compactly represent this continued log as $[n_0, n_1, n_2, \dots]$. We'll refer to this for now as the *continued logarithm with respect to the sequential base ω_n* .

We could even complicate things even further, by taking a different sequence $\omega_{k,n}$ at each iteration k . We have not yet decided whether this is worth dignifying with a name.

Remark 9 (Base b distribution). It is again natural to ask if our Gauss-Kuzmin result from the base 2 case has a natural extension in the base b case. Indeed, for almost any real number x , the expected probability of k being the continued logarithm exponent is given by

$$\mathcal{P}_b(k) = \frac{\log\left(1 + \frac{(b-1)^3 \cdot b^k}{((b-1) + b^{k+1})^2}\right)}{\log\left(\frac{b^2}{2b-1}\right)} .$$

This is implicitly proven in [5]. It was found by the methods described for binary logarithms. Adherence to this distribution for several aperiodic irrationals may be seen in Figure 9. \diamond

Theorem 16. *The function $\mathcal{P}_b : \mathbb{N} \rightarrow (0, 1)$ is a probability density function.*

Proof. We will follow the pattern of Theorem 11 and show that $\sum_{k=0}^{\infty} \mathcal{P}_b(k) = 1$ by showing that

$$\sum_{k=0}^{\infty} \log\left(1 + \frac{(b-1)^3 \cdot b^k}{((b-1) + b^{k+1})^2}\right) = \log\left(\frac{b^2}{2b-1}\right) .$$

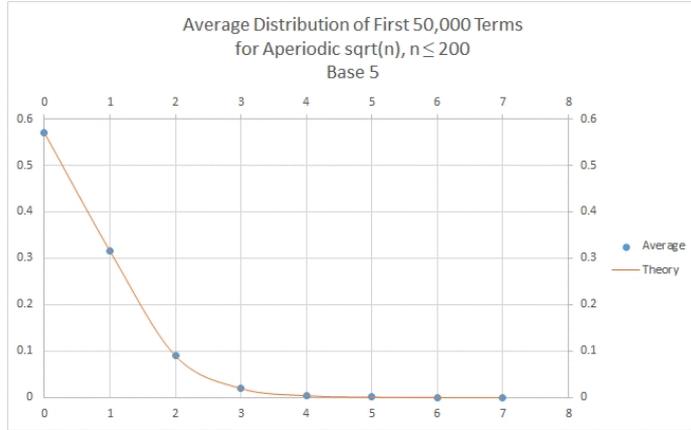


Figure 9: Distribution of the first 200,000 terms of the base 5 continued logarithm for aperiodic \sqrt{n} for $n \leq 200$.

Again it suffices to rewrite the sum of the logs as the log of a product,

$$\sum_{k=0}^N \log \left(1 + \frac{(b-1)^3 \cdot b^k}{((b-1) + b^{k+1})^2} \right) = \log \left(\prod_{k=0}^N \left(1 + \frac{(b-1)^3 \cdot b^k}{((b-1) + b^{k+1})^2} \right) \right),$$

and consider the behavior of that product. Let $d_b, h_b : \mathbb{N} \rightarrow \mathbb{R}$ by

$$d_b(x) = 1 + \frac{(b-1)^3 \cdot b^x}{((b-1) + b^{x+1})^2} \quad (34)$$

$$h_b(x) = b^x + b - 1. \quad (35)$$

Factoring $d_b(k)$, we obtain

$$d_b(k) = \frac{(b^k + b - 1) \cdot (b^{k+2} + b - 1)}{(b - 1 + b^{k+1})^2} = \frac{h_b(k) \cdot h_b(k + 2)}{h_b(k + 1)^2}.$$

Thus our problem simplifies to the telescoping product in Proposition 9 and we obtain

$$\prod_{k=0}^N d_b(k) = \prod_{k=0}^N \frac{h_b(k) \cdot h_b(k + 2)}{h_b(k + 1)^2} = \frac{h_b(0) \cdot h_b(N + 2)}{h_b(1) \cdot h_b(N + 1)}.$$

Simplifying and taking the limit,

$$\lim_{N \rightarrow \infty} \frac{h_b(0) \cdot h_b(N + 2)}{h_b(1) \cdot h_b(N + 1)} = \lim_{N \rightarrow \infty} \frac{(b^0 + b - 1) \cdot (b^{N+2} + b - 1)}{(b^1 + b - 1) \cdot (b^{N+1} + b - 1)} = \frac{b^2}{2b - 1},$$

we obtain the desired result. \square

As a corollary of Theorem 16, we obtain a constant which gives the predicted arithmetic mean of the continued logarithm terms.

Corollary 17 (Khinchine's Constant \mathcal{KL}_b). *For almost all real numbers exceeding 1, where $x = [a_0, a_1, \dots]_{\text{cl}(b)}$, the arithmetic mean of the continued logarithm terms is given by*

$$\mathcal{KL}_b = \lim_{N \rightarrow \infty} \left(\frac{1}{N} \right) \sum_{k=0}^N a_k = \frac{\log(b)}{\log\left(\frac{b^2}{2b-1}\right)} - 1 \quad (36)$$

$$= -\frac{\log_b(2b-1) - 1}{\log_b(2b-1) - 2}. \quad (37)$$

Notice that, as with the binary case, \mathcal{KL}_b is also elementary and has a closed form.

Proof. This proof will be very similar to that of Corollary 12. Let d_b, h_b be as defined in Theorem 16. Then the arithmetic mean of the exponent terms for the continued base b logarithm is given by

$$\sum_{k=0}^{\infty} k \cdot \mathcal{P}_b(k) = \sum_{k=0}^{\infty} k \cdot \frac{\log(d_b(b^k))}{\log\left(\frac{b^2}{2b-1}\right)} = \frac{1}{\log\left(\frac{b^2}{2b-1}\right)} \sum_{k=0}^{\infty} \log(d_b(b^k)^k).$$

We may, as we did in Theorem 16, restrict to showing that the product inside of the log on the right hand side below converges, in this case to $\frac{2b-1}{b}$.

$$\lim_{N \rightarrow \infty} \sum_{k=0}^N \log(d_b(b^k)^k) = \lim_{N \rightarrow \infty} \log\left(\prod_{k=0}^N d_b(b^k)^k\right).$$

Considering this product, we see from Proposition 10 that it telescopes nicely:

$$\prod_{k=0}^N d_b(b^k)^k = \prod_{k=0}^N \frac{h_b(k)^k \cdot h_b(k+2)^k}{h_b(k+1)^{2k}} = \frac{h_b(1) \cdot h_b(N+2)^N}{h_b(N+1)^{N+1}}.$$

Simplifying and evaluating,

$$\lim_{N \rightarrow \infty} \frac{h(1) \cdot h(N+2)^N}{h(N+1)^{N+1}} = \lim_{N \rightarrow \infty} \frac{(b^1 + b - 1)(b^{N+2} + b - 1)^N}{(b^{N+1} + b - 1)^{N+1}} = \frac{2b-1}{b}.$$

Thus we have that

$$\sum_{k=0}^{\infty} k \cdot \mathcal{P}_b(k) = \frac{\log\left(\frac{2b-1}{b}\right)}{\log\left(\frac{b^2}{2b-1}\right)} = \frac{\log(b)}{\log\left(\frac{b^2}{2b-1}\right)} - 1.$$

Thus \mathcal{KL}_b is as claimed. \square

Question 3. How many of the results about recurrences and differences between consecutive convergents carry through to more generalized continued logarithms where the radix representation changes from term to term?

We conclude this section with the first few terms of Type I base 3 continued logarithms for π and e .

$$\begin{aligned}\pi &= [1, 3, 1, 2, 1, 0, 1, 4, 2, 1, 1, 0, 0, 0, \dots]_{\text{cl}(3)} \\ e &= [0, 0, 2, 1, 0, 0, 1, 2, 2, 1, 2, 3, 1, 0, 0, \dots]_{\text{cl}(3)}\end{aligned}$$

10 Continued Logarithms to Other Bases: Type II

There is yet another useful construction for the base b continued logarithm of a number. Suppose $b = 3$ and the number for which we desire to build a continued logarithm is 89. Let $89 = y_0$ and consider its base 3 expansion:

$$y_0 = 1 \cdot 3^4 + 0 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0.$$

We set aside the trailing terms and use only the leading term to begin building a continued fraction in the usual way:

$$y_0 = 1 \cdot 3^4 + (y_0 - 1 \cdot 3^4) = 1 \cdot 3^4 + \frac{1}{\frac{1}{(y_0 - 1 \cdot 3^4)}} = 1 \cdot 3^4 + \frac{1 \cdot 3^4}{\frac{1 \cdot 3^4}{y_0 - 1 \cdot 3^4}}.$$

Dividing the highest largest power of three out of the numerator and denominator of the lower fraction, we obtain

$$y_0 = 1 \cdot 3^4 + \frac{1 \cdot 3^4}{\frac{y_0}{3^4} - 1} = 1 \cdot 3^4 + \frac{1 \cdot 3^4}{y_1} \quad \text{where} \quad y_1 = \frac{1}{\frac{y_0}{3^4} - 1} = \frac{81}{8}.$$

We repeat the same process for y_1 as we did for y_0 , taking its base expansion

$$y_1 = 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 + 0 \cdot 3^{-1} + 1 \cdot 3^{-2} + \dots$$

and likewise using its leading term to build the continued fraction

$$y_1 = 1 \cdot 3^2 + \frac{1 \cdot 3^2}{y_2} \quad \text{where} \quad y_2 = \frac{1}{\frac{y_1}{3^2} - 1} = 8.$$

Finally, we have

$$y_2 = 2 \cdot 3^1 + \frac{2 \cdot 3^1}{y_3} \quad \text{where} \quad y_3 = \frac{2}{\frac{y_2}{3^1} - 2} = 3.$$

This yields the continued fraction

$$89 = 1 \cdot 3^4 + \frac{1 \cdot 3^4}{1 \cdot 3^2 + \frac{1 \cdot 3^2}{2 \cdot 3^1 + \frac{2 \cdot 3^1}{1 \cdot 3^1}}} = [1 \cdot 3^4, 1 \cdot 3^2, 2 \cdot 3^1, 1 \cdot 3^1]_{\text{cl}(3)}. \quad (38)$$

We may formalize this construction in the following way.

Definition 3. Let $\lfloor \cdot \rfloor$ and $\{\cdot\}$ denote the floor and fractional part respectively of a positive number. Fix any integer $b \geq 2$ and consider the dynamical system on $[1, \infty)$ given by

$$x \mapsto g_b(x) := \frac{\lfloor b^{\{\log_b x\}} \rfloor}{\{b^{\{\log_b x\}}\}} = \frac{\lfloor b^{\{\log_b x\}} \rfloor}{b^{\{\log_b x\}} - \lfloor b^{\{\log_b x\}} \rfloor} \quad (39)$$

and associate to the sequence $y_{n+1} := g_b(y_n)$ the continued logarithm sequence $[p_0 \cdot b^{a_0}, p_1 \cdot b^{a_1}, p_2 \cdot b^{a_2}, \dots]_{\text{cl}(b)}$ by

$$p_n := \lfloor b^{\{\log_b y_n\}} \rfloor, \quad a_n = \lfloor \log_b y_n \rfloor.$$

Then we have that

$$x = p_0 \cdot b^{a_0} + \frac{p_0 \cdot b^{a_0}}{p_1 \cdot b^{a_1} + \frac{p_1 \cdot b^{a_1}}{p_2 \cdot b^{a_2} + \dots}} \quad (40)$$

where each p_n is an integer in the interval $[1, b-1]$. If for some \hat{n} , $y_{\hat{n}}$ is an integer less than or equal b , then terminate the sequences y_n and p_n after \hat{n} . The last term of the continued logarithm is just $y_{\hat{n}} \cdot 3^0$. \diamond

To see more explicitly how this relates to our example above, simply build the continued fraction in the usual way while keeping the general form:

$$\begin{aligned} y_0 &= \lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor} + (y_0 - \lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor}) \\ &= \lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor} + \frac{1}{\left(\frac{1}{y_0 - \lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor}}\right)} \\ &= \lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor} + \frac{\lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor}}{\frac{\lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor}}{y_0 - \lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor}}} \\ &= \lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor} + \frac{\lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor}}{\frac{\lfloor b^{\{\log_b y_0\}} \rfloor}{b^{\{\log_b y_0\}} - \lfloor b^{\{\log_b y_0\}} \rfloor}} \\ &= \lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor} + \frac{\lfloor b^{\{\log_b y_0\}} \rfloor \cdot b^{\lfloor \log_b y_0 \rfloor}}{y_1}. \end{aligned}$$

A moment's reflection shows that for $b = 2$, each of the p_n in Equation (40) is precisely 1 and so this method is just the previous construction of Gosper from Equation (4). Because this was also the case for our Type I formulation in Section 9, it helps to have an example illustrating the difference between the two for case $b > 2$.

Example 3. To that end, let $b = 3$ and consider the case $x = \frac{1233}{47}$.

$$\frac{1233}{47} = 2 \cdot 3^2 + \frac{2 \cdot 3^2}{2 \cdot 3^0 + \frac{1 \cdot 3^2}{1 \cdot 3^1 + \frac{1 \cdot 3^1}{1 \cdot 3^0 + \frac{2 \cdot 3^0}{2 \cdot 3^0 + \frac{1 \cdot 3^1}{1 \cdot 3^1}}}}}$$

This is the same number used for Example 2. \diamond

In particular, the corresponding continued fraction for continued logarithms of this new form has the property that $\alpha_n = \beta_{n+1}$ while, for the Type I formulation in Section 9, it instead holds that $(b-1)\alpha_n = \beta_{n+1}$.

Example 4. We may generally describe a Type II continued logarithm in the same manner shown in Equation (38). Written thusly,

$$e^\pi \approx [2 \cdot 3^2, 3, 3, 1, 2 \cdot 3^4, 2 \cdot 3^2, 3, 3, 1, 3, \dots]_{\text{cl}(3)}.$$

From the differences in the representations, it will be clear which type (Type I or Type II) we are using. \diamond

Lemma 18 (Equivalence). *The continued logarithm $[p_0 \cdot b^{a_0}, p_1 \cdot b^{a_1}, p_2 \cdot b^{a_2}, \dots]_{\text{cl}(b)}$ is equivalent to the reduced continued fraction*

$$p_0 \cdot b^{a_0} + \frac{p_0 \cdot p_1^{-1} \cdot b^{-a_1 + a_0}}{1} + \frac{p_2^{-1} \cdot b^{-a_2}}{1} + \frac{p_3^{-1} \cdot b^{-a_3}}{1} + \dots \quad (41)$$

Proof. Recall from Equation (40) that

$$[p_0 \cdot b^{a_0}, p_1 \cdot b^{a_1}, p_2 \cdot b^{a_2}, \dots]_{\text{cl}(b)} = p_0 \cdot b^{a_0} + \frac{p_0 \cdot b^{a_0}}{p_1 \cdot b^{a_1}} + \frac{p_1 \cdot b^{a_1}}{p_2 \cdot b^{a_2}} + \dots$$

In terms of Definition 1, this is just $\alpha_n = p_n \cdot b^{a_n}$, $\beta_n = p_{n-1} \cdot b^{a_{n-1}}$ for all n . We again construct a sequence $\{c_n\}_{n=0}^\infty$. Let

$$c_0 = 1, c_1 = p_1^{-1} \cdot b^{-a_1}, \dots, c_n = p_n^{-1} \cdot b^{-a_n}, \dots$$

Then, where $\alpha'_0 = p_0 \cdot b^{a_0}$, $\alpha'_n = 1$ for all $n > 0$, $\beta'_1 = p_0 \cdot p_1^{-1} \cdot b^{-a_1 + a_0}$, $\beta'_n = p_n^{-1} \cdot b^{-a_n}$ for all $n > 1$, the requirements of Definition 1 are again satisfied, showing equivalence for Equation (41). \square

Theorem 19 (Type II Convergence). *The Type II continued logarithm of base b for a number $x \geq 1$ converges to x .*

Proof. As before, the case of finite termination for the continued logarithm follows readily from the construction, so we restrict our consideration to the infinite case. Let α_n, β_n again be respectively the denominator and numerator terms of the continued fraction corresponding to the Type II continued logarithm for x .

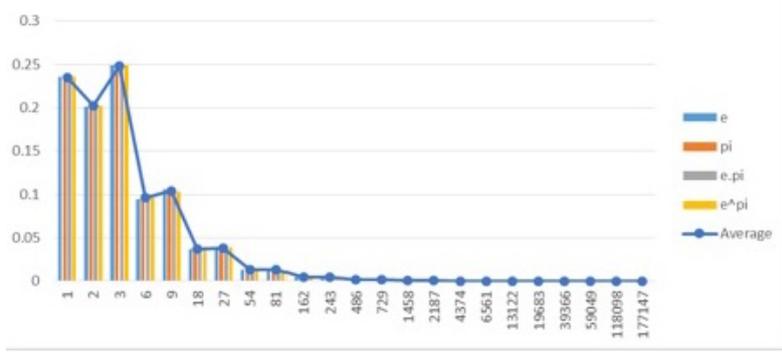


Figure 10: Non-monotonic distribution for the Type II base 3 continued logarithm terms is shown.

$$\sum_{n=1}^{\infty} \frac{\alpha_n \alpha_{n+1}}{\beta_{n+1}} = \sum_{n=1}^{\infty} p_n \cdot b^{a_n} = \infty \text{ because each } a_n, p_n \text{ is a nonnegative integer.}$$

The convergence follows yet again from Remark 5. □

Question 4. While we have been able to compute the distribution of terms for the Type I continued logarithm for base 3 and other bases, a closed form for the distribution of terms for the Type II continued logarithm for other bases escapes us. Figure 10 is strong evidence that a limiting distribution exists, though it is not monotonic. It displays 100,000 terms for $e, \pi, e \cdot \pi$ and e^π . The blue line shows the average.

Theorem 20 (Type II Rational Finiteness). *For a rational number, the continued Type I logarithm of base b will terminate finitely.*

Proof. We will follow a method of proof similar to that in Theorem 5. Again we use the reduced form from Equation (6) with $\alpha_n = 1$, and $\beta_n = p_n^{-1} \cdot b^{-a_n}$ for all $n \geq 2$. Suppose y is rational and let y_n be the ‘tail’ of the continued fraction. Then we may write

$$y_n = \frac{p_n^{-1} \cdot b^{-a_n}}{1 + y_{n+1}}$$

so that

$$y_{n+1} + 1 = \frac{1}{p_n \cdot b^{a_n} y_n}.$$

Since each $0 < y_n$ is rational we may write $y_n = u_n/v_n$ for positive relatively prime integers u_n, v_n . Thence

$$y_{n+1} + 1 = \frac{u_{n+1} + v_{n+1}}{v_{n+1}} = \frac{v_n}{p_n \cdot b^{a_n} u_n}.$$

Thus we have that

$$p_n \cdot b^{a_n} u_n (u_{n+1} + v_{n+1}) = v_n v_{n+1}.$$

Since $u_{n+1} + v_{n+1}$ and v_{n+1} are relatively prime we deduce that $u_{n+1} + v_{n+1}$ is a divisor of v_n . In particular, $u_{n+1} + v_{n+1} \leq v_n$ and so v_n is a sequence of strictly decreasing natural numbers and so must terminate. \square

Remark 10. We have already given an example to show that the Type I formulation from Section 9 is not guaranteed to be finite for all rationals (recall that $l_3(2) = [0, 0, 0, 0, \dots]$). It is reasonable to consider what might be learned from attempting to prove finite termination as in Theorem 20 above and seeing what we might learn. Using the reduced form of Equation (33) and proceeding as before, we obtain:

$$b^{a_n} u_n (u_{n+1} + v_{n+1}) = v_n v_{n+1} (b - 1).$$

This is sufficient only to show that $u_{n+1} + v_{n+1}$ is a divisor of $(b - 1)v_n$. \diamond

Remark 11. It is worth noting that, while we have only been interested in the continued logarithms for numbers greater than 1, this second method also works for numbers in $(0, 1)$. The continued fraction terms all remain positive, so the proof of convergence still holds. Likewise, finiteness for rationals holds for the Type II construction. Slightly different is that the first logarithm exponent term will be negative. For example,

$$\frac{1}{2} = 3^{-1} + \frac{3^{-1}}{2 \cdot 3^0} = [1 \cdot 3^{-1}, 2 \cdot 3^0]_{\text{cl}(3)}.$$

The rest of the continued logarithm exponent terms will still be positive as usual. \diamond

We conclude this section with the first few terms of Type II base 3 continued logarithms for π and e .

$$\begin{aligned} \pi &= [1 \cdot 3^1, 2 \cdot 3^2, 1 \cdot 3^1, 1 \cdot 3^0, 2 \cdot 3^1, 1 \cdot 3^1, 2 \cdot 3^0, 2 \cdot 3^0, 2 \cdot 3^3, 2 \cdot 3^0, \dots]_{\text{cl}(3)} \\ e &= [2 \cdot 3^0, 2 \cdot 3^0, 2 \cdot 3^0, 1 \cdot 3^1, 1 \cdot 3^1, 1 \cdot 3^0, 1 \cdot 3^0, 1 \cdot 3^1, 1 \cdot 3^2, 1 \cdot 3^2, \dots]_{\text{cl}(3)} \end{aligned}$$

11 Computing with Continued Logarithms

Building on the work of Gosper, we should like to investigate how to compute efficiently with continued logarithms as was described in Section 8 above for quadratic irrationals. By a homographic method, we mean one based on fractional linear transformations as discussed for continued fractions in [4]. Specifically, we ask:

Question 5. Are there nice homographic methods to implement arithmetic to a single base b for either the Type I or Type II continued logarithms?

Question 6. Are there nice homographic methods to implement arithmetic to a sequential base a_n ? Are there nice homographic methods to implement arithmetic between two different types of bases, returning the result with respect to a third type of base?

12 Conclusion

Given that Khintchine’s Constant is, to the best of our knowledge, unrelated to any other known transcendentals, it is a surprising finding that Khintchine Continued Logarithmic Constants are elementary. Much remains unknown, and many lines of possible inquiry stand out to us. For the approximants, many of the familiar best approximations for rationals are still present with continued logarithms. The properties determining exactly which these are remain unexplored. For the golden ratio, they remain precisely the same since the continued logarithm of the golden ratio with any base simply induces the classical continued fraction. **Almost all results or questions about simple continued fractions, both easy or hard such as Zaremba’s conjecture [3], have natural analogues for continued logarithms. We continue to explore such matters.** Given the useful results already discovered, it is likely that continued logarithms have far greater utility than has so far been uncovered. Finally, we recall several open questions in addition to those in Section 11.

Question 7. Can one characterise when the binary log of a quadratic irrational or just of \sqrt{n} is eventually periodic?

Question 8. Can one bound the maximum length of a period in the periodic case of \sqrt{n} using of the fundamental solution to the corresponding Pell equation as in the continued fraction case [7]? Can one thereby prove that $\sqrt{13}$ say is aperiodic.

Question 9. Can one find a closed form for the Gauss-Kuzmin distribution for continued logarithms of type II?

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