Continued Logarithms and Associated Continued Fractions

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Abstract

We investigate some connections between continued fractions and binary continued logarithms as introduced by Bill Gosper in 1972 and explore three generalizations (Type I, II and III) to base $b \geq 2$.

- We show convergence for each using equivalent forms of their corresponding continued fractions.

Experimentally, we obtain the distribution of Type I continued logs.

- Moreover, the exponent terms have finite arithmetic means for almost all real numbers. These logarithmic Khintchine constants, have a pleasing relationship with geometric means of the corresponding continued fraction terms.

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We investigate some connections between continued fractions and binary continued logarithms as introduced by Bill Gosper in 1972 and explore three generalizations (Type I, II and III) to base $b \geq 2$.

- We show convergence for each using equivalent forms of their corresponding continued fractions.

Experimentally, we obtain the distribution of Type I continued logs.

- Moreover, the exponent terms have finite arithmetic means for almost all real numbers. These logarithmic Khintchine constants, have a pleasing relationship with geometric means of the corresponding continued fraction terms.
- While the classical Khintchine constant is believed unrelated to known numbers, we find surprisingly that the Type I distribution and Khintchine numbers are elementary.

We also conjecture Type II – and III – distributions and associated Khintchine constants.

Outline

1. The Binary Case
   - Continued Fractions
   - Continued Logarithms
   - Recurrences
   - Convergence and Equivalence
   - Gauss-Kuzmin Distribution and Khintchine Constant
2. Other Bases I
   - Construction
   - Basic Properties
   - Distributions and Khintchine Constants
   - Other Possibilities
3. Other Bases II and III
   - Type II Construction
   - Type II Properties
   - Type II Distribution
   - The Type II Ternary Case
   - Type III Construction and Properties
4. The Role of Experimental Computation
   - Experimental Discovery
   - Quadratic Irrationals: a Method
   - Quadratic Irrationals: Periodicity
5. Conclusion
   - Open Questions

Simple Continued Fractions

Given a positive real number x, write $a_0 = \lfloor x \rfloor$ (floor),:

$$x = \alpha_0 + \{x\}$$

(integer part plus fractional part). Terminate if $\{x\} = 0$. Otherwise, set $y = \frac{1}{\{x\}}$, and write $\alpha_1 = \lfloor y \rfloor$ so that

$$x = \alpha_0 + \frac{1}{\alpha_1 + \{y\}}$$

If $\{y\} = 0$, terminate. Otherwise continue in like fashion:

$$x = \alpha_0 + \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \ldots}}.$$

Example

- Lagrange: numbers with aperiodic decimal expansions may have periodic continued logarithms. For example (iff x is a quadratic irrationality):

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}.$$

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- Either the fraction never terminates, or the fractional part will at some point be zero, in which case

$$x = \alpha_0 + \cfrac{1}{\cdots + \cfrac{1}{\alpha_n}}.$$

Continued Fractions: Another Perspective

Consider the dynamical system f on $[0, \infty)$:

$$f(x) = \begin{cases} x - 1 & \text{if } x \geq 1 \\ \frac{1}{x} & \text{if } 0 < x < 1 \\ \text{terminate} & \text{if } x = 0. \end{cases} \tag{1}$$

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Count the number of times we encounter $x \to x - 1$ before we either reciprocate or terminate. These counts are the α_n. We will denote by $[\alpha_0; \alpha_1; \ldots]_{\text{cf}}$ the simple continued fraction

$$x = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \ldots}}}.$$

Binary Continued Logarithms

Define a similar dynamical system g on $[1, \infty)$:

$$g(x) = \begin{cases} x/2 & \text{if } x \geq 2 \\ \frac{1}{x-1} & \text{if } 1 < x < 2 \\ \text{terminate} & \text{if } x = 1. \end{cases} \tag{2}$$

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We count how many times we divide by 2 before we subtract and reciprocate or terminate. This gives values $a_0, a_1, a_2, \ldots$.
We denote the binary continued logarithm of x by $[a_0, a_1, a_2, \ldots]_{\text{cl}(2)}$ and may write

$$x = 2^{a_0} + \cfrac{2^{a_0}}{2^{a_1} + \cfrac{2^{a_1}}{2^{a_2} + \ldots}}. \tag{3}$$

Example: $x = 19$

We count how many times we divide by 2.

$$19 \to \frac{19}{2} \to \frac{19}{4} \to \frac{19}{8} \to \frac{19}{16}$$

so $a_0 = 4$.

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- The continued logarithm terms are the exponents on the continued fraction terms – hence much smaller.

Irregular continued fractions

Consider the continued fraction

$$x = \alpha_0 + \cfrac{\beta_1}{\alpha_1 + \cfrac{\beta_2}{\alpha_2 + \cfrac{\beta 3}{\alpha_3 + \ldots}}}.$$

Notation

We may, for the sake of simplicity, write with $\alpha_j, \beta_j > 0$

$$x = \alpha_0 + \frac{\beta_1|}{|\alpha_1} + \frac{\beta_2|}{|\alpha_2} + \frac{\beta_3|}{|\alpha_3} + \ldots.$$

Continued Fraction Recurrences I

Remark 1

Suppose x has the irregular continued fraction

$$x = \alpha_0 + \frac{\beta_1|}{|\alpha_1} + \frac{\beta_2|}{|\alpha_2} + \frac{\beta_3|}{|\alpha_3} + \ldots$$

Let x_n be the nth approximant whose continued logarithm is

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$$x = \alpha_0 + \frac{\beta_1|}{|\alpha_1} + \frac{\beta_2|}{|\alpha_2} + \cdots + \frac{\beta_n|}{|\alpha_n}.$$

Then $x_n = \frac{r_n}{s_n}$ where $r_{-1} = 1, s_{-1} = 0, r_0 = \alpha_0, s_0 = 1,$

And $$r_{n+1} = \alpha_{n+1} r_n + \beta_{n+1} r_{n-1}$$
$$s_{n+1} = \alpha_{n+1} s_n + \beta_{n+1} s_{n-1}.$$

Corresponding Binary Continued Logarithm Recurrence

Remark 1 leads to:

Theorem 1 (Recursion for approximants)

Suppose x *has continued logarithm* $[a_0, a_1, a_2, \ldots]$. *Let* x_n *be the nth continued logarithm approximant: the number whose continued logarithm is* $[a_0, a_1, a_2, \ldots, a_n]_{\mathrm{cl}(2)}$. *Then*

$$x_n = \frac{r_n}{s_n}$$

where $r_{-1} = 1$, $s_{-1} = 0$, $r_0 = 2^{a_0}$, $s_0 = 1$, *and*

$$r_{n+1} = 2^{a_{n+1}} r_n + 2^{a_n} r_{n-1}$$

$$s_{n+1} = 2^{a_{n+1}} s_n + 2^{a_n} s_{n-1}.$$

Continued Fraction Recurrences II

Remark 2 (Determinant)

We also have that

$$r_n s_{n-1} - r_{n-1} s_n = (-1)^{n+1} \prod_{k=1}^{n} \beta_k .$$

In the case of a simple continued fraction, of course, the product is always one.

Corresponding Binary Continued Logarithm Recurrence

Remark 2 leads to:

Theorem 2 (Continued Logarithm Differences)

Suppose x has continued logarithm $[a_0, a_1, a_2, \ldots]$. *Let* x_n *be the nth continued logarithm approximant: the number whose continued logarithm is* $[a_0, a_1, a_2, \ldots, a_n]_{\mathrm{cl}(2)}$.
Then $x_n = \frac{r_n}{s_n}$, *where*

$$\frac{r_n}{s_n} - \frac{r_{n-1}}{s_{n-1}} = \frac{(-1)^{n+1} 2^{a_0+a_1+\ldots a_{n-1}}}{s_n s_{n-1}}.$$

Equivalent Continued Fractions

Two (irregular) continued fractions

$$x = \alpha_0 + \frac{\beta_1|}{|\alpha_1} + \frac{\beta_2|}{|\alpha_2} + \frac{\beta_3|}{|\alpha_3} + \dots$$

$$\text{and} \quad x' = \alpha'_0 + \frac{\beta'_1|}{|\alpha'_1} + \frac{\beta'_2|}{|\alpha'_2} + \frac{\beta'_3|}{|\alpha'_3} + \dots$$

are equivalent if there exists a sequence of nonzero real numbers $\{c_n\}_{n=1}^{\infty}$ with $c_0 = 1$ such that

$$\alpha'_n = c_n \alpha_n \text{ and } \beta'_n = c_n c_{n-1} \beta_n, \quad n = 1, 2, \dots .$$

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$$\alpha'_n = c_n \alpha_n \text{ and } \beta'_n = c_n c_{n-1} \beta_n, \quad n = 1, 2, \dots.$$

- The c_n terms may be thought of as constants scaled by both numerators and denominators of successive terms.

Equivalent Binary Continued Logarithms

- The binary continued logarithm $[a_0, a_1, a_2, \dots]_{\mathrm{cl}(2)}$ is equivalent to each of the two continued fractions below: the reduced form and the denominator reduced form respectively.

Reduced Form and *Denominator Reduced Form*

$$2^{a_0} + \frac{1}{2^{a_1 - a_0}} \,\bigg|\, + \frac{1}{2^{a_2 - a_1 + a_0}} \,\bigg|\, + \cdots + \frac{1}{2^{\sum_{k=0}^{n} (-1)^{n-k} a_k}} \,\bigg|\, + \dots$$

$$2^{a_0} + \frac{2^{-a_1 + a_0}}{1} \,\bigg|\, + \frac{2^{-a_2}}{1} \,\bigg|\, + \frac{2^{-a_3}}{1} \,\bigg|\, + \cdots + \frac{2^{-a_n}}{1} \,\bigg|\, + \dots$$

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Reduced Form and Denominator Reduced Form

$$2^{a_0} + \frac{1|}{|2^{a_1-a_0}} + \frac{1|}{|2^{a_2-a_1+a_0}} + \cdots + \frac{1|}{|2^{\sum_{k=0}^{n}(-1)^{n-k}a_k}} + \ldots$$

$$2^{a_0} + \frac{2^{-a_1+a_0}|}{|1} + \frac{2^{-a_2}|}{|1} + \frac{2^{-a_3}|}{|1} + \cdots + \frac{2^{-a_n}|}{|1} + \ldots$$

- The *denominator reduced form* shows finite termination for the binary continued logarithm of every rational.

Convergence Theory

Theorem 3 (Convergence)

Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\beta_n\}_{n=1}^{\infty}$ *are real sequences such that* $\alpha_n > 0$ *and* $\beta_n > 0$ *for all n. The continued fraction*

$$x = \alpha_0 + \frac{\beta_1|}{|\alpha_1} + \frac{\beta_2|}{|\alpha_2} + \frac{\beta_3|}{|\alpha_3} + \ldots$$

converges if $\sum_{n=1}^{\infty} \frac{\alpha_n \alpha_{n+1}}{\beta_{n+1}} = \infty$. *If* x_n *is the nth approximant, then*

$$x_0 < x_2 < \cdots < x_{2k} < \cdots < x < \cdots < x_{2k+1} < \ldots x_3 < x_1$$

and so the limit is x *whenever* x_n *converges. (Proof: see [6].)*

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- We used this result to show convergence for continued logarithms of all bases for both constructions later shown.

Gauss-Kuzmin Distribution for Continued Fractions

Theorem 4 (Gauss, Kuzmin, Lévy)

Let $\mathcal{M}(A)$ denote the Lebesgue measure of a set A. For $x \in (0,1)$ let $\alpha_n(x)$ denote the nth denominator term of the simple continued fraction for x. Then we have that

$$\mathcal{P}(k) := \lim_{n\to\infty} \mathcal{M}\left(\{x : \alpha_{n+1}(x) = k\}\right) = \log_2\left(1 + \frac{1}{k(k+2)}\right).$$

(For a proof, see [3, Theorem 3.23 (Gauss, Kuzmin, Lévy)].)

Khintchine Constant for Continued Fractions

Corollary 5 (Khintchine Constant)

For almost all real numbers x, where the α_k are the denominator values of a simple continued fraction for x,

$$\mathcal{K} = \lim_{n\to\infty} \sqrt[n]{\alpha_1 \cdot \alpha_2 \cdot \ldots \alpha_n} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\log_2 k}$$
$$= 2.6854520010653\ldots.$$

(Proof. See [3, Remark 3.6]*.)*

- The extended numerical computation of $\mathcal{K}$ is difficult directly from the definition, see [1].

Gauss-Kuzmin Distribution (GKD) for Binary Continued Logarithms

Theorem 6

For $x \in (0,1)$, $\mathcal{M}(A)$ denoting the measure of a set A, and $\alpha_n(x)$ the nth continued logarithm term,

$$\mathcal{P}(k) := \lim_{n\to\infty} \mathcal{M}\left(\{x \in (0,1) : \alpha_n(x) = 2^k\}\right) = \frac{\log\left(1 + \frac{2^k}{(1+2^{k+1})^2}\right)}{\log(\frac{4}{3})}.$$

This was recently proven in the seemingly entirely different context of random Fibonacci numbers [5].

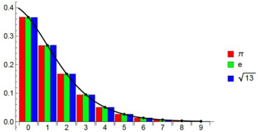

Figure: GKD and continued logarithm distribution for three presumably aperiodic irrationals ($\pi, e, \sqrt{13}$) computed to one million terms.

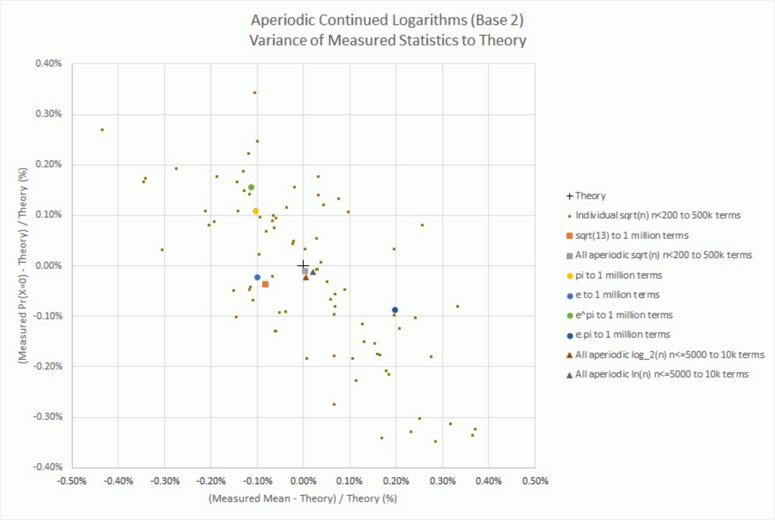

Figure: Deviation from expectation for a selection of aperiodic numbers.

Khintchine Constant for Continued Logarithms

Remark 3 (Existence of Khintchine Logarithmic Constant)

As a consequence of Theorem 6, we obtain the existence of a constant $\mathcal{KL}_2$, the predicted arithmetic mean of the continued logarithm terms. If $x = [a_0, a_1, \ldots]_{\mathrm{cl}(2)}$, then

$$\mathcal{KL}_2 := \lim_{N\to\infty} \left(\frac{1}{N}\right) \sum_{k=0}^{N} a_k. \tag{4}$$

Specifically: almost all numbers greater than one, satisfy

$$\mathcal{KL}_2 = \frac{\log\left(\frac{3}{2}\right)}{\log\left(\frac{4}{3}\right)} = 1.4094208396532. \tag{5}$$

Geometric and Arithmetic Means

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- Indeed, if $\mathcal{KL}_2$ denotes the arithmetic mean of the binary continued logarithm terms, the expected geometric mean of the continued fraction terms is

$$\mathcal{G}_2 = \lim_{N\to\infty}\left(\prod_{k=0}^{N-1} 2^{\mathcal{KL}_2}\right)^{\frac{1}{N}} = \lim_{N\to\infty}\left(2^{N\cdot\mathcal{KL}_2}\right)^{\frac{1}{N}} = 2^{\mathcal{KL}_2} \quad (6)$$

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- Note that $\mathcal{G}_2$, unlike $\mathcal{K}$ (presumably), is a (known) elementary constant.

Other Bases: The Challenge

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- All we are guaranteed after the final division by three is that $x \in [1, 3)$. The map

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  takes this interval to $[\frac{1}{2}, \infty)$ rather than $[1, \infty)$.
- A solution to this is the following: after dividing out powers of b, we replace the second map by

$$x \to \frac{b-1}{x-1}.$$

Other Bases: Type I

We may describe the process with the following dynamical system.

Type I Dynamical System

$$g_b(x) = \begin{cases} x/b & \text{if } x \geq b \\ \frac{b-1}{x-1} & \text{if } 1 < x < b \\ \text{terminate} & \text{if } x = 1 \end{cases}. \tag{7}$$

Type I Construction

We may describe this construction in a manner analogous to our binary construction. Letting $x = y_0$, we have

$$y_0 = b^{a_0} + (y_0 - b^{a_0}) = b^{a_0} + \cfrac{b-1}{\cfrac{b-1}{(y_0 - b^{a_0})}} = b^{a_0} + \cfrac{(b-1)\cdot b^{a_0}}{\cfrac{(b-1)\cdot b^{a_0}}{y_0 - b^{a_0}}}.$$

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Dividing the highest largest power of b out of the numerator and denominator of the lower fraction, we obtain

$$y_0 = b^{a_0} + \cfrac{(b-1)\cdot b^{a_0}}{\cfrac{b-1}{\frac{y_0}{b^{a_0}} - 1}} = b^{a_0} + \frac{(b-1)\cdot b^{a_0}}{y_1} \quad \text{where} \quad y_1 = \frac{b-1}{\frac{y_0}{b^{a_0}} - 1}.$$

We continue on in similar fashion.

Type I: Fractional Form

- The representation of this type of continued logarithm in continued fraction form is as follows:

Fractional Form

$$x = b^{a_0} + \cfrac{(b-1)b^{a_0}}{b^{a_1} + \cfrac{(b-1)b^{a_1}}{b^{a_2} + \cfrac{(b-1)b^{a_2}}{b^{a_3} + \dots}}}.$$

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- For $b = 2$, this is just Gosper's original formulation

Type I: Example

Example 7

Consider $\frac{1233}{47}$ which has ternary continued logarithm $l_3(\frac{1233}{47}) = [2, 0, 3, 1]_{\mathrm{cl}(3)}$. The corresponding continued fraction is as follows.

$$\frac{1233}{47} = 3^2 + \cfrac{2 \cdot 3^2}{3^0 + \cfrac{2 \cdot 3^0}{3^3 + \cfrac{2 \cdot 3^3}{3^1}}}.$$

This example will be useful for comparing this Type I formulation of the base b logarithm with the Type II formulation given below. Specifically, compare this example with Example 9.

Type I: Convergence and Loss of Rational Finiteness

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Question 1 (Finite Termination)

Given an integer base b, and especially in the case of $b = 3$, determine which rationals (indeed, even which integers) have finite continued logarithms to base b.

Type I: Distribution of Log Terms

Our binary Gauss-Kuzmin result has a natural extension to the general base b case. For almost any real number x, the expected probability of k being the continued logarithm exponent is

$$\mathcal{P}_b(k) = \frac{\log\left(1 + \frac{(b-1)^3 \cdot b^k}{((b-1)+b^{k+1})^2}\right)}{\log(\frac{b^2}{2b-1})}.$$

(Also implicitly proven in [5].)

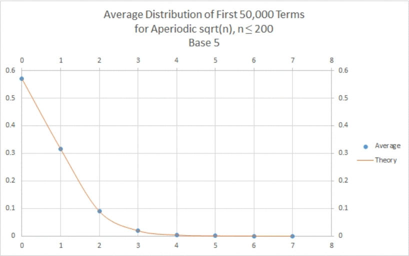

Figure: Distribution of the first 200, 000 terms of the base 5 continued logarithm for aperiodic $\sqrt{n}$ for $n \leq 200$.

Type I: Khintchine Logarithmic Constant

Corollary 8 (Khintchine Constant $\mathcal{KL}_b$)

For almost all real numbers exceeding $x > 1$, where $x = [a_0, a_1, \ldots]_{\mathrm{cl}(b)}$, the arithmetic mean of the continued logarithm terms is given by

$$\mathcal{KL}_b = \lim_{N\to\infty} \left(\frac{1}{N}\right) \sum_{k=0}^{N} a_k = \frac{\log(b)}{\log\left(\frac{b^2}{2b-1}\right)} - 1$$
$$= -\frac{\log_b(2\,b-1)-1}{\log_b(2\,b-1)-2}.$$

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Corollary 8 (Khintchine Constant $\mathcal{KL}_b$)

For almost all real numbers exceeding $x > 1$, where $x = [a_0, a_1, \ldots]_{\mathrm{cl}(b)}$, the arithmetic mean of the continued logarithm terms is given by

$$\mathcal{KL}_b = \lim_{N\to\infty} \left(\frac{1}{N}\right) \sum_{k=0}^{N} a_k = \frac{\log(b)}{\log\left(\frac{b^2}{2b-1}\right)} - 1$$
$$= -\frac{\log_b(2\,b-1)-1}{\log_b(2\,b-1)-2}.$$

- As with the binary case, $\mathcal{KL}_b$ has an elementary closed form.

Other Base Possibilities

- The factor $b-1$ in the numerator of $\frac{b-1}{x-1}$ comes because when we divide by the final factor of b, we end up with a value in an interval of length $b-1$.

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$$x \to \begin{cases} \frac{x}{n!} & \text{if } n! \leq x < (n+1)! \\ \frac{n}{x-1} & \text{if } x \in [1, n+1) \\ \text{terminate} & \text{if } x = 1 \end{cases} .$$

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- If at the k^{th} step we divide by $m_k!$, we could express the continued factorial logarithm as $[n_0, n_1, n_2, \ldots]_!$.

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$$x \to \begin{cases} \frac{x}{\omega_n} & \text{if } \omega_n \leq x < \omega_{n+1} \\ \frac{\left(\frac{\omega_{n+1}}{\omega_n} - 1\right)}{x-1} & \text{if } x \in [1, \frac{\omega_{n+1}}{\omega_n}) \\ \text{terminate} & \text{if } x = 1 \end{cases} .$$

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- Note that the second map takes 1 to ∞, and sends ω_{n+1}/ω_n to 1.

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- Refer to this for now as the continued logarithm with respect to the sequential base ω_n.
- We could even complicate things even further, by taking a different sequence $\omega_{k,n}$ at each iteration k.
- We have not yet decided if this is worth naming.

Other Bases: Type II

We consider another natural construction for the base b continued logarithm. Fix $b = 3$ and $x = 89$.
Let $89 = y_0$ and examine its base 3 expansion:

$$y_0 = 1 \cdot 3^4 + 0 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0.$$

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We set aside the trailing terms and use only the leading term to begin building a continued fraction in the usual way:

$$y_0 = 1 \cdot 3^4 + (y_0 - 1 \cdot 3^4) = 1 \cdot 3^4 + \cfrac{1}{\cfrac{1}{(y_0 - 1 \cdot 3^4)}} = 1 \cdot 3^4 + \cfrac{1 \cdot 3^4}{\cfrac{1 \cdot 3^4}{y_0 - 1 \cdot 3^4}}.$$

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Dividing the highest largest power of three out of the numerator and denominator of the lower fraction, we obtain

$$y_0 = 1\cdot 3^4 + \cfrac{1\cdot 3^4}{\cfrac{1}{\frac{y_0}{3^4} - 1}} = 1\cdot 3^4 + \frac{1\cdot 3^4}{y_1} \quad \text{where} \quad y_1 = \frac{1}{\frac{y_0}{3^4} - 1} = \frac{81}{8}.$$

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We repeat for y_1 what we did for y_0, taking its base expansion

$$y_1 = 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 + 0 \cdot 3^{-1} + 1 \cdot 3^{-2} + \ldots$$

and likewise using its leading term to build the continued fraction

$$y_1 = 1 \cdot 3^2 + \frac{1 \cdot 3^2}{y_2} \text{ where } y_2 = \frac{1}{\frac{y_1}{3^2} - 1} = 8.$$

Type II Construction

Finally, we have

$$y_2 = 2 \cdot 3^1 + \frac{2 \cdot 3^1}{y_3} \text{ where } y_3 = \frac{2}{\frac{y_2}{3^1} - 2} = 3.$$

Type II Construction

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$$y_2 = 2 \cdot 3^1 + \frac{2 \cdot 3^1}{y_3} \text{ where } y_3 = \frac{2}{\frac{y_2}{3^1} - 2} = 3.$$

This yields the continued fraction

$$89 = 1 \cdot 3^4 + \cfrac{1 \cdot 3^4}{1 \cdot 3^2 + \cfrac{1 \cdot 3^2}{2 \cdot 3^1 + \cfrac{2 \cdot 3^1}{1 \cdot 3^1}}} = [1 \cdot 3^4, 1 \cdot 3^2, 2 \cdot 3^1, 1 \cdot 3^1]_{\mathrm{cl}(3)}. \tag{8}$$

Type II Construction

We may formalise this as a dynamical system:

Type II Dynamical System on $[1,\infty)$

$$x \mapsto g_b(x) := \frac{\lfloor b^{\{\log_b x\}} \rfloor}{\{b^{\{\log_b x\}}\}} = \frac{\lfloor b^{\{\log_b x\}} \rfloor}{b^{\{\log_b x\}} - \lfloor b^{\{\log_b x\}} \rfloor}. \tag{9}$$

We associate to the sequence $y_{n+1} := g_b(y_n)$ the Type II continued logarithm $[p_0 \cdot b^{a_0}, p_1 \cdot b^{a_1}, p_2 \cdot b^{a_2}, \ldots]_{\mathrm{cl}(b)}$ where

$$p_n := \lfloor b^{\{\log_b y_n\}} \rfloor, \quad a_n = \lfloor \log_b y_n \rfloor.$$

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$$p_n := \lfloor b^{\{\log_b y_n\}} \rfloor, \quad a_n = \lfloor \log_b y_n \rfloor.$$

- With finite termination if some y_n is integer.

Type II Fractional Representation

The continued logarithm with this construction has a continued fraction which contains more number theory than the Type I construction.

Type II: Corresponding Continued Fraction

$$x = p_0 \cdot b^{a_0} + \cfrac{p_0 \cdot b^{a_0}}{p_1 \cdot b^{a_1} + \cfrac{p_1 \cdot b^{a_1}}{p_2 \cdot b^{a_2} + \dots}} \tag{10}$$

where each p_n is an integer in the interval $[1, b-1]$, and each $a_n \geq 0$ is integer.

Type II Distribution

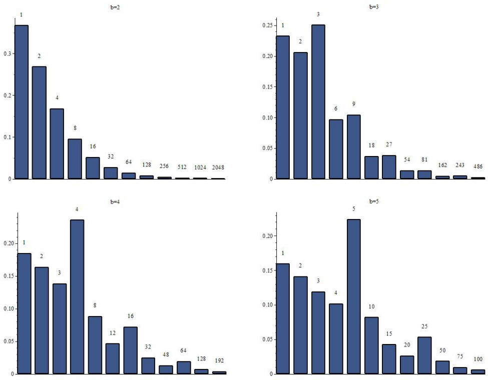

Figure: Type II probability function for $2 \leq b \leq 5$.

A Type II Example

Example 9

Let $b = 3$ and $x = \frac{1233}{47}$.

$$x = 2 \cdot 3^2 + \cfrac{2 \cdot 3^2}{2 \cdot 3^0 + \cfrac{2 \cdot 3^0}{1 \cdot 3^2 + \cfrac{1 \cdot 3^2}{1 \cdot 3^1 + \cfrac{1 \cdot 3^1}{1 \cdot 3^0 + \cfrac{1 \cdot 3^0}{2 \cdot 3^0 + \cfrac{2 \cdot 3^0}{1 \cdot 3^1 + \cfrac{1 \cdot 3^1}{1 \cdot 3^1}}}}}}$$

This is the same number used for Example 7.

Type II Properties

We obtain different properties.

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1. Still reduces to Gosper's binary case
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4. Originally we could not identify the type II distribution.
   - We returned in early 2016 with Jason Lynch. This led to the discovery of a recursive closed form.

Type II Equivalence

Lemma 10 (Equivalence)

The Type II continued logarithm $[p_0 \cdot b^{a_0}, p_1 \cdot b^{a_1}, p_2 \cdot b^{a_2}, \ldots]_{\mathrm{cl}(b)}$ *is equivalent to the denominator reduced continued fraction*

$$p_0 \cdot b^{a_0} + \frac{p_0 \cdot p_1^{-1} \cdot b^{-a_1+a_0}|}{|1} + \frac{p_2^{-1} \cdot b^{-a_2}|}{|1} + \frac{p_3^{-1} \cdot b^{-a_3}|}{|1} + \ldots. \tag{11}$$

- This equivalence was instrumental in showing finiteness of this continued logarithm formation for all rationals in all bases.

Type II Distribution

We conjecture that the distribution of log terms for $b \geq 2$ is given by a recursive process based on the binary case.

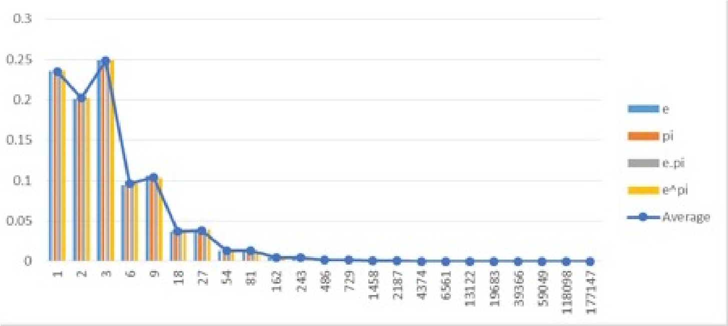

Figure: Non-monotonic distribution for Type II ternary logarithm.

Type II Distribution

- Originally we could not identify the Type II distribution. We returned to this in early 2016 with Jason Lynch.

Theorem 11 (Type II distribution)

Let X be the limiting distribution of the terms $p_n b^{a_n}$ in a Type II continued logarithm base b. Then

$$\mathcal{P}(X = pb^k) = \mu_b(1 + p^{-1}b^{-k}) - \mu_b(1 + (p+1)^{-1}b^{-k}).$$

Here $\mu_b^{(n)}(\alpha)$ denotes measure of $\{y \in (1,2) : x_n < \alpha\}$, x_n is the nth tail of the corresponding continued fraction, and $\mu_b = \lim_{n\to\infty} \mu_b^n$.

- $\mathcal{P}$ is a indeed a probability density function.
- We then sought a recursive form for the μ_b functions.

Finding the Type II Recursion

- We obtain good convergence of $\mu_b^n(x)$ – as described in the next section – after around 10 iterations.
- The graphics for μ_b show it is piecewise smooth, this ultimately lead to our conjectured recursion.

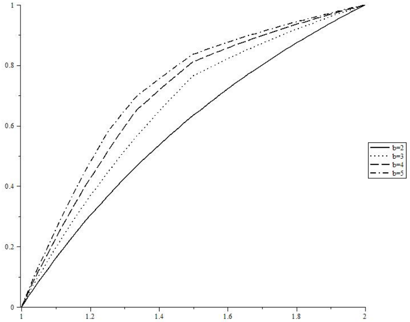

Figure: Type II $\mu_b^{10}(x)$ for $2 \leq b \leq 5$.

Type II Conjectured Recursion

We conjecture the following form for the μ_b function:

Conjectured Recursion

$$\mu_2(x) = \frac{\log \frac{2x}{x+1}}{\log \frac{4}{3}}$$

$$\mu_b(x) = \begin{cases} c_b \mu_{b-1}(x) & 1 \leq x \leq \frac{b}{b-1} \\ d_b(\mu_{b-1}(x) - 1) + 1 & \frac{b}{b-1} < x \leq 2 \end{cases}$$

where

$$d_b = \frac{c_b \mu_{b-1}\left(\frac{b}{b-1}\right) - 1}{\mu_{b-1}\left(\frac{b}{b-1}\right)}.$$

Example: Type II Distribution

We provide the explicit distribution for the case $b = 3$.

- As $1 + 1/(pb^k) > b/(b-1)$ iff $pb^k < b - 1$ iff $1 \leq p \leq b-1, k = 0$:

Example 12 ($\mathcal{P}$ for $b = 3$)

The conjectured recursion leads to:

$$\mathcal{P}(p \cdot 3^k) \stackrel{?}{=} \mu_3\left(1 + (p \cdot 3^k)^{-1}\right) - \mu_3\left(1 + ((p+1)3^k)^{-1}\right)$$

where

$$\mu_3(x) = \begin{cases} \frac{c_3}{\log\left(\frac{4}{3}\right)} \log\left(\frac{2x}{x+1}\right) & 1 \leq x \leq \frac{3}{2} \\ 1 & x = 2 \end{cases}$$

Type II Ternary Distribution Example

- This allows us to simplify to

Probability for $b = 3$

$$\mathcal{P}(p \cdot 3^k) = \begin{cases} 1 - \frac{c_3}{\log\left(\frac{4}{3}\right)} \log\left(\frac{6}{5}\right) & p \cdot 3^k = 1 \\ \frac{c_3}{\log\left(\frac{4}{3}\right)} \log\left(\frac{\left(p3^k+1\right)\left(2\,(p+1)3^k+1\right)}{\left(2\,p3^k+1\right)\left((p+1)3^k+1\right)}\right) & \textit{otherwise} \end{cases} \tag{12}$$

- From this, we may compute a nearly "closed form" for the corresponding Khintchine constant $\mathcal{KL}_3$.

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- We originally conjectured $c_b = (1 + 1/b)^{2/3}$.
- Now we doubt this.

Numerical confirmation? a Gibbs phenomenon or failure?

- First 4 bars in each group show 10,000 terms for $e, \pi, e \cdot \pi$ and e^{π}. Blue line (dots at vertices) is the average. The last two show 100,000 terms for π, e^{π}. Green line is the theoretical distribution.
- Comparing 2nd to 5th bar (π to 10,000 vs 100,000 terms), the experimental distribution is trending in right direction (towards green line). Similarly, for bars 4 and 6.

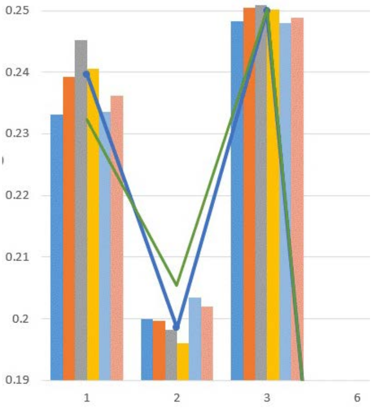

Figure: Type II $k = 1, 2, 3$ for $b = 3$.

Expressing the Ternary Khintchine Constant

- Peeling off the first term gives a relatively rapidly convergent series for $\mathcal{KL}_3 = \log_3 \mathcal{G}_3$ as two sums of logs.

A series for $\mathcal{KL}_3$

$$\mathcal{KL}_3 \stackrel{?}{=} \frac{\left(\frac{4}{3}\right)^{\frac{2}{3}}}{\log\left(\frac{4}{3}\right)} \left[\log\left(\frac{21}{20}\right)\frac{\log 2}{\log 3} + \sum_{k=1}^{\infty}\log\left(1+\frac{1}{2\cdot 3^k+1}\right) + \frac{\log 2}{\log 3}\sum_{k=1}^{\infty}\log\left(1+\frac{3^k}{(3^{k+1}+1)(4\cdot 3^k+1)}\right)\right] \tag{13}$$

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- Numerically: $\mathcal{K}L_3 = 1.1181949509488983\ldots$

$$\mathcal{G}_3 = 3.4159741693740855\ldots$$

- Also the first sum in (13) is $\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{\infty} 1/\left(3^k+1\right)^n}{n2^n}$ and the second is $\sum_{n=1}^{\infty} \frac{\sum_{k=1}^{\infty} 1/\left(2\,3^k+1\right)^n}{n2^n} - \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{\infty} 1/\left(3\,3^k+1\right)^n}{n2^n}$.

Discovering the Type III Fraction

- Our difficulties in resolving the Type II distribution led us to investigate other options.
- This led to the discovery of a Type III generalization
- This Type III construction retains the best qualities of both the Type I and Type II constructions, namely:
  1. Finite termination for rationals
  2. Distribution has an elementary closed form
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  1. Finite termination for rationals
  2. Distribution has an elementary closed form
  3. An explicit Khintchine constant
- For such reasons, perhaps this ought to be called the *Type 0* continued logarithm or the *natural* continued logarithm.

Type III Construction

Where $p_0 \cdot b^{a_0}$ is the leading term of the base b expansion of y_0, set

$$y_0 = p_0 \cdot b^{a_0} + (y_0 - p_0 \cdot b^{a_0}) = p_0 \cdot b^{a_0} + \cfrac{1}{\cfrac{1}{(y_0 - p_0 \cdot b^{a_0})}} = b^{a_0} + \cfrac{b^{a_0}}{\cfrac{b^{a_0}}{y_0 - p_0 \cdot b^{a_0}}}.$$

If $y_n - p_n b^{a_n} = 0$ then terminate. Otherwise, set

$$y_{n+1} = \frac{b^{a_n}}{y_n - p_n b^{a_n}}$$

The corresponding continued logarithm is of the form

$$y_0 = p_0 b^{a_0} + \cfrac{b^{a_0}}{p_1 b^{a_1} + \cfrac{b^{a_1}}{p_2 b^{a_2} + \cfrac{b^{a_2}}{\ddots}}}$$

Type III: A Probability Distribution

- The probability distribution was discovered by a similar recursive process to that of the Type II continued logarithm.
- Surprisingly, it turns out to be elementary. Let $\mu_b = \lim_{n\to\infty} \mu_b^{(n)}$ denote the limiting distribution. Then

$$\mu_b(x) = \frac{\log \frac{x+b-1}{bx}}{\log \frac{b+1}{2b}}. \tag{14}$$

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Type III distribution

For $p = 1, 2, \ldots, b-1$ and $k = 0, 1, \ldots$

$$\begin{aligned}\mathcal{P}(X = p \cdot b^k) &= \mu_b\left(1 + p^{-1}b^{-1}\right) - \mu_b\left(1 + (p+1)^{-1}b^{-k}\right)\\ &= \frac{1}{\log \frac{b+1}{2b}}\left(\log \frac{1 + p^{-1}b^{-k-1}}{1 + p^{-1}b^{-k}} - \log \frac{1 + (p+1)^{-1}b^{-k-1}}{1 + (p+1)^{-1}b^{-k}}\right)\end{aligned}$$

Type III: Distribution (still non-monotonic)

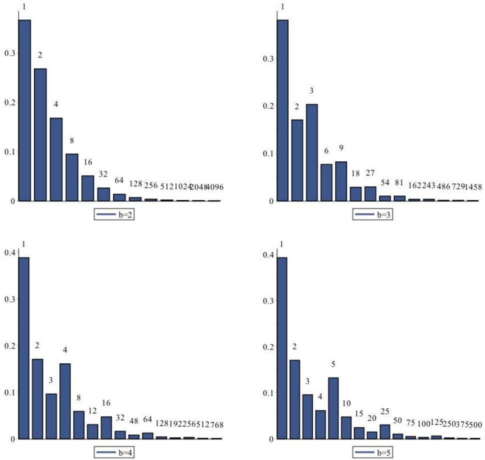

Figure: Type III probability function for $2 \leq b \leq 5$

A Surprising Result: Type III Khintchine Constant

Type III Khinchine Constant

For $b = 2, 3, \cdots$ the Type III constant is given by

$$\mathcal{KL}_b = \frac{1}{\log_b \frac{b+1}{2b}} \sum_{p=2}^{b} \log_b \left(1 + \frac{1}{p}\right) \log_b \left(1 - \frac{1}{p}\right).$$

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- Using *Maple* and the Inverse Symbolic Calculator, we found that the limit of the geometric constants $\mathcal{G}3_b := b^{\mathcal{KL}_b}$ turns out to be exactly Khintchine's original constant 2.685452001065306445 . . . . (Proven by [1, Lemma 1a].)
- Moreover $\mathcal{G}3_3 = 8/3$.
- As b goes to infinity the distribution converges to the classical Gauss-Kuzmin distribution.

Finding the Functional Relation

- We next indicate the experimental mathematics [2] process used to find the Type I base b distribution. Similar more subtle steps led to discovery of the Type II and III distributions.
- Let $x \in \mathbb{R}, x > 1$ have the (aperiodic) continued logarithm $[a_0, a_1, \ldots]_{\mathrm{cl}(2)}$. Let x_n be the nth tail of the equivalent denominator reduced continued fraction. Then we have

$$x = 2^{a_0} \cdot \left(1 + \cfrac{2^{-a_1}}{1 + \cfrac{2^{-a_2}}{\cdots + \cfrac{2^{-a_n}}{x_n}}} \right). \tag{15}$$

Finding the Functional Relation

Consider the Lebesgue measure $\mu_n(\zeta)$ of $\{x \in (1,2) : x_n < \zeta\}$.
Setting $x_{n-1} = 1 + \frac{2^{-a_n}}{x_n}$ it follows that $x_n < x$ if and only if

$$\frac{2^{-a_n}}{x_{n-1} - 1} < x \text{ which is just } x_{n-1} > 1 + \frac{2^{-a_n}}{x}.$$

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Thus $\mu_0(x) = x - 1$ and

$$\mu_n(x) = \sum_{k=0}^{\infty} \left(\mu_{n-1}\left(1 + 2^{-k}\right) - \mu_{n-1}\left(1 + \frac{2^{-k}}{x}\right)\right). \quad (16)$$

Finding the Closed Form

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3. We found good convergence of $\mu(x)$ after 10 iterations.[1]
4. We used the 101 data points to seek the best fit to a function of the form

$$\mu(x) = C \log_2 \left(\frac{ax + b}{cx + d} \right)$$

where C, a, b, c, and d are constants to be determined by the fitting process.

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Finding the Closed Form

- To meet the boundary conditions, it is necessary that

$$\begin{aligned} \mu(1) &= 0 \\ \mu(2) &= 1 \\ d &= a+b-c \\ C &= \frac{1}{\log_2\left(\frac{2a+b}{a+b+c}\right)}. \end{aligned}$$

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- Motivated by the case of a simple continued fraction, we had originally considered the form $C\log_2(ax+b)$ and, when that failed, we considered a superposition of two such terms.

- To eliminate any common factor between the numerator and denominator of $\frac{ax+b}{cx+d}$, we set $c = 1$, leaving the functional form to be fitted as

$$\mu(x) = \frac{\log_2\left(\frac{ax+b}{x+a+b-1}\right)}{\log_2\left(\frac{2a+b}{a+b+1}\right)}. \tag{17}$$

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- This result *suggests* candidate values of $a = \frac{1}{2}$ and $b = \frac{1}{2}$.
- Thus we obtained

$$\mu_2(x) = \mu(x) = \frac{\log\left(\frac{2x}{x+1}\right)}{\log\left(\frac{4}{3}\right)}. \tag{18}$$

Binary Probability Distribution

This suggested the probability distribution

$$\mathcal{P}(X=k) = \mu\left(1+2^{-k}\right) - \mu\left(1+2^{-k-1}\right) = \frac{\log\left(1+\frac{2^k}{\left(2^{k+1}+1\right)^2}\right)}{\log\left(\frac{4}{3}\right)}.$$

We then computed the mean:

$$E(X) = \sum_{k=0}^{\infty} k \cdot \mathcal{P}(X=k) = 1.4094208397\ldots.$$

k	$\mathcal{P}(X=k)$
0	0.3662394210...
1	0.2675211579...
2	0.1675533738...
3	0.0949153712...
4	0.0507000346...
5	0.0262283498...
6	0.0133430145...
7	0.0067299284...

Figure: Distribution of first eight binary continued logarithm terms.

Quadratic Irrationals

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- For example, $\sqrt{13}$ appears to be aperiodic, as do $\sqrt{14}$ and $\sqrt{15}$. However, $\sqrt{17}$, has a nice continued logarithm (periodic constant).
- Similarly, $\sqrt{19}$, $\sqrt{21}$ and $\sqrt{23}$ are likewise periodic. We again find aperiodic $\sqrt{n}$ for n values $31, 35, 39, 41, 43, 46, 47, 55, 57, 59, 61, 62, 63, 67, 71, 79, 85,$ $91, 94, 97, 99, 101, 103, 106, 107, 109, 113, 114, 115, 116, 119,$ and so on.

Quadratic Irrationals: Method

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- Even in the case of aperiodic surds (e.g., $\sqrt{13}$) this method is roughly an order of magnitude faster than a conventional approach using fixed-precision, floating-point arithmetic.
- This method applies to the Type I base b continued logarithm.

Quadratic Irrationals: Method of Computation

- Recall the dynamical system g on $[1, \infty)$:

$$g(x) = \begin{cases} x/b & \text{if } x \geq b \\ \dfrac{b-1}{x-1} & \text{if } x = b \\ \text{terminate} & \text{if } x = 1. \end{cases} \tag{19}$$

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$$x = \frac{p}{q}(c + d\sqrt{n}) \tag{20}$$

where p, q, c, d and n are all integers with $p, q > 0$ and $n > 1$.

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- To implement this dynamical system efficiently, there are two cases to be considered.

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- From this simplified definition it follows that

$$\begin{aligned} x \geq b \quad &\text{iff} \quad p \geq bq \\ x = 1 \quad &\text{iff} \quad p = q. \end{aligned} \tag{21}$$

Case I continued

Given the current value of x, represented by integers (p, q), we evaluate $g(x)$, represented by integers (p', q'), as follows.

$$\begin{aligned} p' &= p, \qquad \text{for } x \geq b \\ q' &= bq \\ \\ p' &= q, \qquad \text{for } 1 < x < b \\ q' &= p - q \end{aligned} \tag{22}$$

Case II: $d \neq 0$

- The way these tests are performed depends on the sign of d and the sign of $bq - cp$ or $q - cp$ as follows:

Condition	d	$bq - cp$	True iff
$x \geq b$	$+$	$+$	$nd^2p^2 \geq (bq - cp)^2$
	$+$	$-$	Always
	$-$	$+$	Never
	$-$	$-$	$nd^2p^2 \leq (bq - cp)^2$

Condition	d	$q - cp$	True iff
$x = 1$	$+$	$+$	$nd^2p^2 = (q - cp)^2$
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- The above depends on p, q being positive, so, at each iteration, the sign of c and d should be reversed as needed.

Case II continued

Given the current value of x, represented by integers (p, q, c, d), we evaluate $g(x)$, represented by integers (p', q', c', d'), as follows.

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\begin{aligned}
p' &= p, && \text{for } x \geq b\\
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c' &= c\\
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$$

$$
\begin{aligned}
p' &= (b-1)q, && \text{for } 1 < x < b\\
q' &= (cp-q)^2 - nd^2p^2\\
c' &= cp - q\\
d' &= -dp
\end{aligned}
$$

Periodicity of Quadratics

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- The longest period found was 293 for $n = 16,813,731$.
- While we might be missing some periodic roots with very long periods, we should have detected any with periods up to $3,000$ for $n < 2,000,000$ and periods up to 600 thereafter.

Method of Detection

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- In other words, the prefix had to be shorter than $3,333$ terms and the maximum detectable period is $3,333$.
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  - For $2,000,000 < n < 12 \cdot 10^8$, we only computed $2,000$ terms, so the upper limit on the period detectable is now 666.
- We *conjecture* that for periodic clogs of $\sqrt{n}$ the prefix has exactly two terms. If so, $10,000$ computed terms would detect periods up to $4,999$
  - as mentioned, we found no period greater than 300 for any n in the range studied.

A Possible Upper Boundary of Growth

- The $\{n, \text{period}\}$-tuples that appear to define the upper boundary of growth, for n values less than one thousand, are

$$\{2, 1\}, \{23, 20\}, \{37, 26\}, \{167, 66\}, \{531, 134\}, \{819, 178\}.$$

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$$\{2, 1\}, \{23, 20\}, \{37, 26\}, \{167, 66\}, \{531, 134\}, \{819, 178\}.$$

- These are consistent with an upper bound on growth of $1.4 \cdot n^{1/2.27} \log n$
  - but this seems an overestimate for larger n.

Density of Periodics

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- This has been checked by looking at the first 200,000 terms for n up to 50,000.
- While there are 237 periodic roots for $n \leq 1000$, there are only 1,262 periodic roots in the first 50,000.

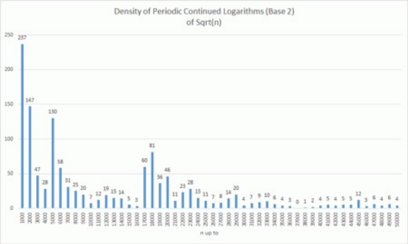

Figure: Density of periodic binary continued logs for $2 \leq \sqrt{n} \leq 50,000$.

Aperiodic Case

- In every aperiodic case tested, $\sqrt{n}$ appears to satisfy the limiting distribution of Theorem 6.

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- This leads to the *conjecture* that

> Each $\sqrt{n}$ is either eventually periodic or obeys the limiting distribution and the corresponding Khintchine constant.

Period Length

- The period length seems tied to the fundamental solution of the corresponding Pell equation as with simple continued fractions [7].

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- The period can vary widely in length as also true of the simple continued fraction.

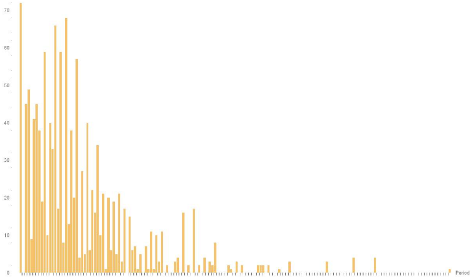

Figure: Distributionof periods of binary continued logs of periodic $\sqrt{n}$ up to $5,000$.

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- For $\sqrt{10}$ we have

$$[1, 0, 0, 1, \overline{1, 0, 1}, 1, 0, 1, \ldots]_{\mathrm{cl}(2)}.$$

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- For $\sqrt{10}$ we have

$$[1, 0, 0, 1, \overline{1, 0, 1}, 1, 0, 1, \ldots]_{\mathrm{cl}(2)}.$$

- For $\sqrt{11}$ we have

$$[1, 0, 0, 0, 3, 0, 0, 1, \overline{1, 0, 0, 3, 0, 0, 1}, \ldots]_{\mathrm{cl}(2)}.$$

Open Questions I

Question 2

Are there any nice representations for elementary or special functions arising from continued logarithms in a manner analogous to the irregular continued fraction for $\tan^{-1}$ [3]?

Question 3

Are there nice homographic methods to implement arithmetic to a single base b for either the Type I or Type II continued logarithms?

Question 4

Are there nice homographic methods to implement arithmetic to a sequential base a_n? Are there nice homographic methods to implement arithmetic between two different types of bases, returning the result with respect to a third type of base?

Open Questions II

Question 5

Can one characterise when the binary log of a quadratic irrational – or just of $\sqrt{n}$ – is eventually periodic?

Question 6

Can one bound the maximum length of a period in the periodic case of $\sqrt{n}$ using of the fundamental solution to the corresponding Pell equation as in the continued fraction case [7]? Can one thereby prove that $\sqrt{13}$ say is aperiodic.

Question 7

Can one find a complete closed form for the Gauss-Kuzmin distribution for continued logarithms of Type II for $b \geq 4$?

Open Questions III

Question 8

Can we rigorously prove the validity of our conjectured recursion for the distributions for Type II base b continued logarithms?

Question 9

Is there a closed form for the Khintchine constants resulting from the conjectured recursion for the distributions for Type II base b continued logarithms?

Thank You


BOOLE ORDERS LUNCH
NO, NO, YES, NO, NO, YES, YES, NO, NO, NO, YES...
Menu
harris

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