Convex analysis in groups and semigroups: a sampler

Jonathan M. Borwein · Ohad Giladi

This paper is dedicated to R. Tyrell Rockafellar on the occasion of his eightieth birthday

Abstract We define convexity canonically in the setting of monoids. We show that many classical results from convex analysis hold for functions defined on such groups and semigroups, rather than only vector spaces. Some examples and counter-examples are also discussed.

Part I: Basic convex analysis

1 Introduction

The notion of convexity is classical [30], and heavily used in diverse contexts [9, Chapter 1]. While normally considered in the concrete setting of vector spaces — either $\mathbb{R}^d$ or infinite dimensional — it has often been examined in very general axiomatic form, see [11] and [33]. In the vector space case, x is said to be a *convex combination* of $x_1, \ldots, x_n$ if there exist $\alpha_1, \ldots, \alpha_n \in (0, 1)$

J. M. Borwein
Centre for Computer-assisted Research Mathematics and its Applications (CARMA), School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
E-mail: jonathan.borwein@newcastle.edu.au

O. Giladi.
Centre for Computer-assisted Research Mathematics and its Applications (CARMA), School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
Tel.: +61 2 49261808
Fax: +61 2 49216898
E-mail: ohad.giladi@newcastle.edu.au

such that

$$x = \sum_{i=1}^{n} \alpha_i x_i, \quad \sum_{i=1}^{n} \alpha_i = 1. \tag{1}$$

If we assume for a moment that α_i is of the form $\alpha_i = \frac{m_i}{\sum_{i=1}^{n} m_i}$ where $m_1, \ldots, m_n \in \mathbb{N} = \{1, 2, \ldots\}$, then (1) becomes

$$mx = \sum_{i=1}^{n} m_i x_i, \quad m = \sum_{i=1}^{n} m_i. \tag{2}$$

In (1) we must be able to define αx for $\alpha \in \mathbb{R}$ and $x \in X$. More generally, (1) can be used whenever X is a module. On the other hand, in (2) we use *only* the additive structure of X, i.e., we may assume that X is merely an additive semigroup. (See Section 2 for the exact definitions.) Using (2), we show how one can build a *canonical* theory of convexity for additive groups and semigroups. We refer the reader to [27, 33] for more information on abstract convexity in all its manifestations. Some aspects of convex analysis in a more abstract setting have also been studied in [16, 18, 24]. Note that in [24] for example, it is only required that a function is convex over geodesic curves (in this case, in the Heisenberg group). Thus, the various notions of convexity do not always coincide. See also Remark 1 in [24].

In a similar fashion to (2), one can define convex functions on additive groups and semigroups (again, see Section 2). It is then natural to ask whether one may obtain useful analogues of known results for convex functions. It turns out that under only minimal assumptions on the underlying monoid or group, it is possible to reconstruct many classical results from the theory of convex functions such as Hahn-Banach type theorems, Fenchel duality, certain constrained optimisation results, and more. We dedicate Section 3 to exhibiting concrete examples of groups and their convex sets and convex hulls. It turns out that even in simple examples, the structure of convex sets is subtle and can differ significantly from the structure of convex sets in vector spaces.

The rest of the paper is dedicated to generalising classical results of the theory of convexity to more general settings. While many of the results presented here hold when we assume that the underlying space is a module (see Section 2.2), for the sake of concreteness we formulate most of the results for groups and semigroups. In Section 4 we discuss the interpolation of subadditive and convex function. In short, the question (say, in the convex case) is: given two functions f and g with $g \leq f$ and f, and $-g$ are convex, can we find an affine function a such that $g \leq a \leq f$. Such questions were studied in [26] and generalised in [19]. We show that interpolation is possible for convex functions on semigroups which are semidivisible (see Section 2.2).

Part II of this paper (Sections 5 and 6) is dedicated to the study of convex operators between (semi)groups. We define some well known and widely used notions, such as directional derivatives and conjugate functions in the groups setting. In Section 5, we show that some of the best known results, such as the

the max formula, sandwich theorems and Fenchel type duality theorems extend to this general setting. Finally, in Section 6 we briefly discuss optimisation over groups before making some concluding remarks in Section 7.

2 Convex basics

We define convex sets and functions and examine some basic properties.

2.1 Convexity in algebraic structures

First, we begin by recalling some some basic definitions from algebra.

Definition 1 (Semigroup) A set X equipped with a binary operation $\cdot$ is said to be a semigroup, if it satisfies the associative law $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in X$. If $a \cdot b = b \cdot a$ for all $a, b \in X$, then X is said to be commutative. If there exists and element $x_0 \in X$ satisfying $x_0 \cdot x = x \cdot x_0 = x$ for all $x \in X$, then x_0 is said to be a unit element in X.

Definition 2 (Semiring) A set R with two binary operations $+$ and $\cdot$ is said to be a semiring if $(R, +)$ is a commutative semigroup with unit element 0, and $(R, \cdot)$ is a semigroup (not necessarily commutative) with unit element 1. The two operations are connected by the distributive laws. Also, we assume that for all $r \in R$, we have $r0 = 0r = 0$.

Definition 3 (Semimodule) A set X equipped with a binary operation $+$ is said to be a left semimodule over a semiring R if $(X, +)$ is a commutative semigroup with unit element 0, and there exists a map $R \times X \to X, (r, x) \mapsto rx$ satisfying for all $r, r_1, r_2 \in R$ and $x, x_1, x_2 \in X$,

1. $(r_1 r_2)x = r_1(r_2 x)$;
2. $r(x_1 + x_2) = rx_1 + rx_2$;
3. $(r_1 + r_2)x = r_1 x + r_2 x$;
4. $1x = x$;
5. $r0 = 0 = 0x$.

Remark 1 Note that in the equation $r0 = 0 = 0x$, the first two zero elements refer to the unit element of the semimodule X, while the last zero refers to the additive unit element of the semiring R. ◇

Next, we have the following definition.

Definition 4 (Monoid) A set X equipped with a binary operation $+$ is said to be a monoid if $(X, +)$ is a commutative semigroup with unit element.

In particular, we have that every monoid is a semimodule over the non-negative integers $\mathbb{Z}_+$. Next, for the sake of completeness, we also include the definitions of rings and modules.

Definition 5 (Ring) A set R equipped with two binary operations $+$ and $\cdot$ is said to be a ring if $(R, +)$ is a commutative group with unit 0 and $(R, \cdot)$ is a semigroup with unit 1. The two operations are connected by the distributive laws.

Definition 6 (Module) A set X equipped with a binary operation $+$ is said to be a left module over a ring R, if $(X, +)$ is a commutative group with unit element 0, and there exists a map $R \times X \to X$, $(r, x) \mapsto rx$ satisfying for all $r, r_1, r_2 \in R$ and $x, x_1, x_2 \in X$,

1. $(r_1 r_2)x = r_1(r_2 x)$;
2. $r(x_1 + x_2) = rx_1 + rx_2$;
3. $(r_1 + r_2)x = r_1 x + r_2 x$;
4. $1x = x$.

In particular, every commutative group is a module over the ring $\mathbb{Z}$. We are now in a position to define convex sets in semimodules.

Definition 7 (Convex set in a semimodule) Assume that X is a semimodule over a semiring R, and $A \subseteq X$. Let $r_1, \ldots, r_n \in R \backslash \{0\}$, and $x_1, \ldots, x_n \in A$. Assume that there exists $x \in X$ satisfying

$$rx = \sum_{i=1}^{n} r_i x_i, \quad r = \sum_{i=1}^{n} r_i.$$

If $x \in A$ for every choice of $n \in \mathbb{N}$, $r_1, \ldots, r_n \in R \setminus \{0\}$ and $x_1, \ldots, x_n \in A$, then A is said to be convex.

If R is assumed to be a ring, not just a semiring, then we assume it is equipped with a *compatible* partial order, that is, we have $r + r_1 \leq r + r_2$ whenever $r_1 \leq r_2$ and $r \cdot r_1 \leq r \cdot r_2$ whenever $r_1 \leq r_2$ and $r \geq 0$. Then, in Definition 7, we take only elements that are strictly positive. In particular, if R is a field with a compatible partial order, R_+ is the collection of all positive elements, and $r_1, \ldots, r_n \in R_+ \setminus \{0\}$, then we have

$$\sum_{i=1}^{n} r_i = r \implies \sum_{i=1}^{n} \frac{r_i}{r} = 1, \quad rx = \sum_{i=1}^{n} r_i x_i \implies x = \sum_{i=1}^{n} \frac{r_i}{r} x_i,$$

which gives the standard definition of convexity (for example, over $\mathbb{R}$ or $\mathbb{Q}$). To illustrate the fact that convexity in groups can behave differently than in vector spaces, consider the following example.

Example 1 (Finite groups) If X is a finite group then by the pigeon hole principle there exists $m \in \mathbb{N}$ such that $mx = 0 = m \cdot 0$. Thus $x \in \mathrm{conv}(\{0\})$ for every $x \in X$. Hence, X and $\emptyset$ are the only convex sets in X. $\diamond$

Additional examples are discussed in Section 3. Next, as in vector spaces, we can also define convex cones.

Definition 8 (Convex cone in a semimodule) Assume that X is a semimodule over a semiring R. Let $r_1, \ldots, r_n \in R \setminus \{0\}$ and $x_1, \ldots, x_n \in A$. Assume that there exists $x \in X$ and $r \in R \setminus \{0\}$ such that

$$rx = \sum_{i=1}^{n} r_i x_i.$$

If $x \in X$ for every choice $n \in \mathbb{N}$, $r_1, \ldots, r_n \in R \setminus \{0\}$ and $x_1, \ldots, x_n \in A$, then A is said to a convex cone.

Remark 2 Note that the only difference between Definition 7 and Definition 8 is that in Definition 8 we omit the condition $r = \sum_{i=1}^{n} r_i$. $\diamond$

Herein, we will focus on additive groups and semigroups. As noted in [16], a monoid with a nontrivial idempotent element cannot be embedded in a group. Also, as noted above, every monoid is a semimodule over the semiring $\mathbb{Z}_+$. Thus, the elements in Definition 7 are positive integers, denoted m_j instead of r_j. For a general commutative group, one cannot always solve the equation

$$\left(\sum_{i=1}^{n} m_i \right) x = \sum_{i=1}^{n} m_i x_i. \tag{3}$$

Yet, equation (3) is very useful in some cases. Thus, we recall the following.

Definition 9 (Divisible group) An additive group X is said to be divisible if for every $n \in \mathbb{N}$, $nX = X$. Alternatively, X is divisible if for every $y \in X$ and for every $n \in \mathbb{N}$, there exists $x \in X$ such that $nx = y$.

Definition 10 (Semidivisible group) An additive group is said to be p-semidivisible if there exists $p \in \mathbb{N}$ prime such that $pX = X$, and X is said to be semidivisible if it is p-semidivisible for some prime p.

We can similarly define *divisible* and *semidivisible monoids*, as well as *divisible* and *semidivisible semimodules*. In particular, all divisible submodules and divisible submonoids are convex cones. A notion which is stronger than the above two is the following.

Definition 11 (Uniquely divisible group) An additive group X is said to be uniquely divisible if for every $n \in \mathbb{N}$ and for every $y \in X$, there exists a unique $x \in X$ such that satisfies $nx = y$. Alternatively, X is said to be uniquely divisible if it is divisible and for every $n \in \mathbb{N}$, the map $x \mapsto nx$ is an injective map.

Similarly, we can consider the following notion.

Definition 12 (Uniquely divisible monoid) A monoid X is said to be uniquely divisible if it is divisible and for every $n \in \mathbb{N}$, the map $x \mapsto nx$ is an injective map.

Note that in monoids, singletons are convex if and only if the monoid is uniquely divisible, since we want $\sum_{i=1}^{n} m_i x = \left(\sum_{i=1}^{n} m_i\right) x$ to be the same as $\left(\sum_{i=1}^{n} m_i\right) y$ if and only if $x = y$. Divisibility and semidivisibility are important for the structure theory of infinite abelian groups. See for example [13]. See also [21, 22] for some more recent examples relating to divisible groups.

Remark 3 A subgroup of a divisible group need not be divisible, or even semidivisible. As a simple example, take $X = \mathbb{R}$ and $\mathbb{Z} \subseteq X$. $\diamond$

Remark 4 (Divisibility in abelian groups) It is known that every abelian group is a subgroup of a divisible group. Moreover, the quotient of a divisible group is again divisible, for example, $\mathbb{R}/\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$. Also, the torsion subgroup T_G (of all elements of finite order) is divisible and the quotient G/T_G is a $\mathbb{Q}$-vector space. Finally, the divisible groups are exactly the *injective* abelian groups. See for example [13, Chapter IV] for the exact definition of injective groups and how they are related to divisible groups. $\diamond$

Remark 5 If X is p-semidivisible, i.e., $pX = X$ then for every $l \in \mathbb{N}$ we have $p^l X = p^{l-1}(pX) = p^{l-1} X = \cdots = pX = X$. $\diamond$

Remark 6 Assume $X = nX$ for some $n \in \mathbb{N}$. Write $n = p_1^{m_1} \cdots p_l^{m_l}$. Then $X = p_1^{m_1} \cdots p_l^{m_l} X = p_1 \left(p_1^{m_1-1} \cdots p_l^{m_l}\right) X \subseteq p_1 X \subseteq X$ and so $X = p_1 X$. Thus, for us the assumption that p is a prime number in Definition 10 plays no significant rôle. $\diamond$

As mentioned above, convexity has an entirely axiomatic approach. We refer the reader to [33] for more information about this rich topic. We will present only the basic definitions and then return to the more concrete case of convexity in algebraic structures.

Definition 13 (Convexity) A collection $\mathcal{C}$ of subsets of a set X is said to be a convexity (also an *alignment*), if:

1. Contains the empty set: $\emptyset \in \mathcal{C}$;
2. Closed under intersections: if Λ is an index set and $C_\lambda \in \mathcal{C}$ for every $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} C_\lambda \in \mathcal{C}$;
3. Closed under nested unions: if Λ is a totally ordered index set and $C_\lambda \in \mathcal{C}$ for every $\lambda \in \Lambda$, and $C_{\lambda_1} \subseteq C_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$, then $\bigcup_{\lambda \in \Lambda} C_\lambda \in \mathcal{C}$.

It is straightforward to check the convex sets defined by Definition 7 form a convexity. Given Definition 13, we can also define the convex hull.

Definition 14 (Convex hull) If $A \subseteq X$, define

$$\text{conv}(A) = \bigcap_{\substack{A \subseteq B \\ B \text{ convex}}} B.$$

The convex hull is a *closure operator*, i.e., it satisfies the following: 1. $A \subseteq B \implies \text{conv}(A) \subseteq \text{conv}(B)$; 2. $A \subseteq \text{conv}(A)$; 3. $\text{conv}(\text{conv}(A)) = \text{conv}(A)$; 4. $\text{conv}(\emptyset) = \emptyset$; 5. Closure under intersections and directed unions.

In the case of monoids, we have the following concrete result.

Proposition 1 (Convex hull in monoid) *If X is a monoid and $A \subseteq X$, the convex hull of A is given by*

$$\text{conv}(A) = \left\{ x \in X \;\middle|\; mx = \sum_{i=1}^{n} m_i x_i, \; x_i \in A, \; m_i \in \mathbb{N}, \; m = \sum_{i=1}^{n} m_i \right\}. \quad (4)$$

Proof Clearly the set on the right side of (4) is convex and contains A. If $A \subseteq B$ and B is convex, then B contains the set on the right side of (4). $\square$

A map $T : X_1 \to X_2$ between two monoids is said to be *additive* if $T(x_1 + x_2) = Tx_1 + Tx_2$ for all $x_1, x_2 \in X_1$. It is well known that a linear image of a convex set in a vector space is again convex. We establish a similar fact for additive bijections between monoids.

Proposition 2 (Convexity under additive bijection) *Assume that X_1, X_2 are monoids and $T : X_1 \to X_2$ is an additive bijection. If $A \subseteq X_1$ is convex, then $TA \subseteq X_2$ is convex.*

Proof Assume that $m, m_1, \ldots, m_n \in \mathbb{N}$ and $y_1, \ldots, y_n \in TA$, $y \in X_2$ are such that $my = \sum_{i=1}^{n} m_i y_i$, $m = \sum_{i=1}^{n} m_i$. Since T is onto, there exists $x \in X_1$ such that $Tx = y$. Since $y_1, \ldots, y_n \in TA$, there exist $x_1, \ldots, x_n \in A$ such that $y_1 = Tx_1, \ldots, y_n = Tx_n$. Hence, we have $T(mx) = mTx = my = \sum_{i=1}^{n} m_i y_i = \sum_{i=1}^{n} m_i Tx_i = T\left(\sum_{i=1}^{n} m_i x_i\right)$. Since T is injective, we have $mx = \sum_{i=1}^{n} m_i x_i$. Since $x_1, \ldots, x_n \in A$ and A is convex, it follows that $x \in X$. Thus, $y = Tx \in TA$ and TA is convex. $\square$

Remark 7 If X_2 is divisible then in the proof of Proposition 2 we always have y such that $my = \sum_{i=1}^{n} m_i y_i$. If T is additive and A is convex, we must have $y \in TA$. Hence, in this case we need not assume that T is a bijection. $\diamond$

For the inverse image, we have a more general result.

Proposition 3 (Convexity under inverse additive map) *Assume that X_1 and X_2 are monoids and $T : X_1 \to X_2$ is additive. Assume that $A \subseteq X_2$ is convex. Then $T^{-1}A \subseteq X_1$ is convex.*

Proof Assume that $x_1, \ldots, x_n \in T^{-1}A = \{x \mid Tx \in A\}$, $m, m_1, \ldots, m_n \in \mathbb{N}$ and $x \in X_1$ are such that $mx = \sum_{i=1}^{n} m_i x_i$, $m = \sum_{i=1}^{n} m_i$. Since $x_1, \ldots, x_n \in T^{-1}A$, we have $Tx_1, \ldots, Tx_n \in A$. Since T is additive, we have $\sum_{i=1}^{n} m_i Tx_i = T\left(\sum_{i=1}^{n} m_i x_i\right) = T(mx) = mTx$. Since A is convex, we have $Tx \in A$. Thus, $x \in T^{-1}A$, which completes the proof. $\square$

As we shall see, studying convexity in such a general setting also brings about a better understanding of this notion in the standard setting of vector spaces. One complaint about convexities is that there are too many of them and that in different settings one has to adjoin many additional axioms. This is one more motivation for the current study.

2.2 Classes of functions

Here we consider several classes of functions defined on semimodules, particularly on monoids, classes which are well studied in the vector spaces setting. In order to define convex functions, we need to consider an *ordered* semimodule, i.e., a semimodule with a partial order $\leq$. We may also assume that the semiring over which the semimodule is defined is also equipped with a partial order and that $r \geq 0$ for every element in the semiring.

Definition 15 (Compatible partial order on a semimodule) Given a semimodule X over a semiring R, we say that a partial order $\leq$ is *compatible* with the module operations, if $rx_1 \leq rx_2$, $x + x_1 \leq x + x_2$ for all $x \in X$, $r \in R$, whenever $x_1 \leq x_2$.

Next, with Definition 15 in hand, we can define a convex function between two semimodules.

Definition 16 (Convex function) Let X, Y be a semimodules over a semiring R. Assume that Y is equipped with a compatible partial order $\leq$. A function $f : X \to Y$ is said to be convex if for every $n \in \mathbb{N}$, every $r_1, \ldots, r_n \in R \backslash \{0\}$ and every $x_1, \ldots, x_n \in X$,

$$rf(x) \leq \sum_{i=1}^{n} r_i f(x_i), \tag{5}$$

for every x satisfying,

$$rx = \sum_{i=1}^{n} r_i x_i, \quad r = \sum_{i=1}^{n} r_i. \tag{6}$$

$f : X \to Y$ is said to be concave if $-f$ is convex. Clearly the sum of two convex functions is convex.

Remark 8 Assume that X, Y are modules over a ring R, rather than semimodules over a semiring. Assume also that there is a compatible partial order on R (see the discussion following Definition 7 for a detailed explanation of a ring with a compatible partial order). The set $R_+ \setminus \{0\}$ is then the set of all strictly positive elements in R. In this case, we say that a function $f : X \to Y$ is convex if (5) and (6) hold for $r_1, \ldots, r_n, r \in R_+ \setminus \{0\}$. $\diamond$

Remark 9 In convex analysis of real functions, we often consider functions whose range is $(-\infty, \infty]$, $[-\infty, \infty)$, or $[-\infty, \infty]$, rather than $\mathbb{R}$. Similarly, in the more general case of convex mappings between semimodules, we often consider a maximal element in Y, ∞. Also, in the case where Y is a module, not just a semimodule, we may also consider a minimal element $-\infty$. In order for (5) to make sense, we assume that the following arithmetic identities involving the maximal and minimal element hold: $\infty - \infty = 0 \cdot \infty = \infty$. $\diamond$

Definition 17 (Affine function) Let X, Y be semimodules over a semiring R. Then $f : X \to Y$ is said to be affine if for every $n \in \mathbb{N}$, every $r_1, \ldots, r_n \in R \setminus \{0\}$ and every $x_1, \ldots, x_n \in X$,

$$r f(x) = \sum_{i=1}^{n} r_i f(x_i),$$

whenever $x \in X$ satisfies,

$$r x = \sum_{i=1}^{n} r_i x_i, \quad r = \sum_{i=1}^{n} r_i.$$

Clearly every affine function is both convex and concave. For an affine function, we again cannot allow it to attain $\pm\infty$.

We can, however, consider the following notion.

Definition 18 (Generalised affine function) Assume that X, Y are semimodules over a semiring R. It may happen that Y contains a maximal element ∞ or a minimal element $-\infty$. A function $f : X \to Y \cup \{\pm\infty\}$ is said to be generalized affine if it is both convex and concave.

Generalised affine functions are either affine or 'very' infinite.

Proposition 4 *Assume that X and Y are groups, and $a : X \to Y \cup \{\pm\infty\}$ is generalised affine. Then either a is everywhere finite, or $a = +\infty$, or $a = -\infty$, or a attains both values $+\infty$ and $-\infty$.*

Proof Assume that a is not everywhere finite, and that it is not identically $+\infty$ or $-\infty$. Assume for example that there exist $x_1, x_2 \in X$ such that $a(x_1) > \alpha$ for all $\alpha \in Y$ and $a(x_2)$ is finite. We have $2x_2 = \big(x_2 + (x_1 - x_2)\big) + \big(x_2 - (x_1 - x_2)\big) = x_1 + (2x_2 - x_1)$, and so since a is concave we have $2a(x_2) \geq a(x_1) + a(2x_2 - x_1) > \alpha + a(2x_2 - x_1)$. Therefore we must have $a(2x_2 - x_1) = -\infty$. If we assume $a(x_1) = -\infty$ rather than $+\infty$, the proof is similar. $\qquad\square$

Definition 19 (Subadditive function) Assume that X, Y are semimodules over a semiring R, and assume that Y is equipped with a partial order $\leq$. A function $f : X \to Y \cup \{\pm\infty\}$ is said to be subadditive if for every $x, y \in X$,

$$f(x + y) \leq f(x) + f(y).$$

The function $x \mapsto \sqrt{x}$ is subadditive on $[0, +\infty)$ but not convex. As we will mostly be concerned with groups and monoids, we now focus on functions with subadditive properties over $\mathbb{N}$.

Definition 20 ($\mathbb{N}$-sublinear functions) Assume that X, Y are semimodules over a semiring R, and assume that Y is equipped with a partial order $\leq$. A function $f : X \to Y \cup \{\pm\infty\}$ is said to be $\mathbb{N}$-sublinear if it is subadditive and in addition it is positively homogeneous, i.e., $f(mx) = mf(x)$ for every $x \in X$ and every $m \in \mathbb{N} \cup \{0\}$.

Definition 21 (Generalised $\mathbb{N}$-linear function) Assume that X, Y are as in Definition 20. A function $f : X \to Y \cup \{\pm\infty\}$ is said to be generalised $\mathbb{N}$-linear if both f and $-f$ are $\mathbb{N}$-sublinear.

If f is a generalised $\mathbb{N}$-linear function and f is finite, then for every choice of positive integers $m_1, \ldots, m_n \in \mathbb{N}$, we have $f\left(\sum_{i=1}^n m_i x_i\right) = \sum_{i=1}^n m_i f(x_i)$. The functions that satisfy this property are exactly the additive functions on semimodules over $\mathbb{Z}_+$.

If f is $\mathbb{N}$-sublinear and $mx = \sum_{i=1}^n m_i x_i$, where $m = \sum_{i=1}^n m_i$, then

$$mf(x) = f(mx) = f\left(\sum_{i=1}^n m_i x_i\right) \leq \sum_{i=1}^n f(m_i x_i) = \sum_{i=1}^n m_i f(x_i).$$

In particular, every $\mathbb{N}$-sublinear function on a monoid is convex.

Recall that a *lattice* order on a partially ordered set is a partial order in which every two elements have a unique suprmum and infimum. For such partially ordered monoids, we have the following.

Proposition 5 *Assume that X is a monoid, $(Y, \leq)$ a monoid with a compatible lattice order $\leq$, and $f_1, \ldots, f_k : X \to Y \cup \{\pm\infty\}$ are convex ($\mathbb{N}$-sublinear, subadditive). Then the function $\max\{f_1, \ldots, f_k\}$ is also convex ($\mathbb{N}$-sublinear, subadditive).*

Proof If $f_1, \ldots, f_k$ are convex and $m, m_1, \ldots, m_n \in \mathbb{N}$, $x, x_1, \ldots, x_n \in X$ are such that $mx = \sum_{i=1}^n m_i x_i$, $m = \sum_{i=1}^n m_i$, then

$$m \cdot \max_{1 \leq j \leq k}\{f_j(x)\} = \max_{1 \leq j \leq k}\{mf_j(x)\}$$

$$\leq \max_{1 \leq j \leq k}\left\{\sum_{i=1}^n m_i f_j(x_i)\right\}$$

$$\overset{(*)}{\leq} \sum_{i=1}^n m_i \cdot \max_{1 \leq j \leq k}\{f_j(x_i)\}.$$

In $(*)$ we used the fact that $\leq$ is a lattice order, compatible with the group operations on Y. The case of sublinear or subadditive functions is easy. We omit the proof. □

Proposition 6 *Assume that X, Y are monoids. Then it suffices in Definition 16 that $r = p^l$ for a fixed prime p and all $l \in \mathbb{N}$.*

Proof Indeed, if $r \neq p^l$, then there exists $l \in \mathbb{N}$ such that $r < p^l$. Thus,

$$\left(p^l - r\right) x + \sum_{i=1}^n r_i x_i = p^l x.$$

By the convexity property,

$$p^l f(x) \leq \left(p^l - r\right) f(x) + \sum_{i=1}^n r_i f(x_i),$$

which gives

$$rf(x) \leq \sum_{i=1}^{n} r_i f(x_i),$$

as required. □

Proposition 6 implies the following.

Proposition 7 *Assume that X, Y are monoids. Assume that $f : X \to Y$ is subadditive and there exists $p \in \mathbb{N}$ such that $f(px) = pf(x)$ for every $x \in X$, then f is convex. If Y is a group, then f is in fact $\mathbb{N}$-sublinear.*

Proof By Proposition 6, it is enough to assume in Definition 16 that $r = p^l$, $l \in \mathbb{N}$. Assume then that $p^l x = \sum_{i=1}^{n} m_i x_i$. We have,

$$p^l f(x) \overset{(*)}{=} f(p^l x) = f\left(\sum_{i=1}^{n} m_i x_i\right) \overset{(**)}{\leq} \sum_{i=1}^{n} m_i f(x_i),$$

where in $(*)$ we used the homogeneity assumption on f, and in $(**)$ we used the subadditivity of f. To prove the second assertion, let $m \in \mathbb{N}$. Then there exist $m', l \in \mathbb{N}$ such that $m + m' = p^l$. Thus, we have

$$(m + m')f(x) = f\big((m + m')x\big) \leq f(mx) + m'f(x) \leq (m + m')f(x).$$

Thus, we have

$$(m + m')f(x) = f(mx) + m'f(x),$$

and since Y is a group, this implies that $f(mx) = mf(x)$ for all $m \in \mathbb{N}$ and all $x \in X$. This complete the proof. □

2.3 Some properties of convex functions

In this section we study some of the basic properties of convex functions on groups and monoids, which are then used to prove that some known results regarding convex functions on topological vector spaces can be generalized to the class of topological groups.

It is well known that a convex function on a (semi)normed vector space is continuous at x_0 if and only if it is bounded from above in a neighbourhood of x_0. If the space is normed, we derive a Lipschitz condition. See [9,35]. In Corollary 1 below, we establish a similar fact for convex functions on topological groups into $[-\infty, \infty]$.

For a set $B \subseteq X$ in an additive group and $m \in \mathbb{N}$ define $\frac{1}{m}B = \{x \mid mx \in B\}$. It is straightforward to show that if B is convex, $\frac{1}{m}B$ is convex for all $m \in \mathbb{N}$. Also, a set $B \subseteq X$ is said to be *symmetric* if $-B = B$. Again, if B is symmetric, then $\frac{1}{m}B$ is symmetric. We have the following proposition.

Proposition 8 *Let X be an additive group, $f : X \to [-\infty, \infty]$ a convex function, and assume that there is a symmetric $B \subseteq X$ and $M \in \mathbb{R}$ such that $f(x) \leq f(x_0) + M$ for all $x \in x_0 + B$. Then for every $y \in \frac{1}{m}B$, we have $|f(x_0 + y) - f(x_0)| \leq \frac{M}{m}$.*

Proof First, note that if $u \in B$ then $-u \in B$ and by convexity we have $2f(x_0) \leq f(x_0 + u) + f(x_0 - u) \leq f(x_0 + u) + f(x_0) + M$ and so $f(x_0 + u) \geq f(x_0) - M$. If $f(x_0) = -\infty$ then $f = -\infty$ on $x_0 + B$. Assume then that $f(x_0) > -\infty$. Let $y \in \frac{1}{m}B$. Then there exists $u \in B$ such that $my = u$. Thus, we have $m(x_0 + y) = (x_0 + u) + (m-1)x_0$ and then using convexity of f gives $mf(x_0 + y) \leq f(x_0 + u) + (m-1)f(x_0) \leq M$. This gives $f(x_0 + y) - f(x_0) \leq \frac{1}{m}\big(f(x_0 + u) - f(x_0)\big) \leq \frac{M}{m}$. Also, by convexity, we have $f(x_0) - f(x_0 + y) \leq f(x_0 - y) - f(x_0) \leq \frac{M}{m}$, which completes the proof. $\square$

Recall that a topological group is a groups in which the group operations are continuous. In Corollary 1 and Corollary 2 below, we use Proposition 8 and show that fact which are known in the context of topological vector spaces, also hold in the more general setting of topological groups.

Corollary 1 (Continuity) *Assume that X is a topological group and $f : X \to [-\infty, \infty]$ is convex. Then f is bounded from above in around x_0 if and only if f is continuous at x_0.*

We next show convex minorants inherit continuity of a majorant.

Corollary 2 (Minorants) *Assume that X is a topological group and $f, g : X \to [-\infty, \infty]$. Suppose that g is bounded above in a neighbourhood of x_0, f is a convex minorant of g and $f(x_0)$ is finite. Then f is continuous at x_0.*

Next, we study further properties of convex functions on monoids.

Proposition 9 (Three-slope lemma for monoids) *Let X be a monoid, and $x, x_1, x_2 \in X$, $m_1, m_2 \in \mathbb{N}$ such that $(m_1 + m_2)x = m_1x_1 + m_2x_2$. Then for any convex function $f : X \to (-\infty, \infty]$ we have*

$$\frac{f(x) - f(x_1)}{m_2} \leq \frac{f(x_2) - f(x_1)}{m_1 + m_2} \leq \frac{f(x_2) - f(x_1)}{m_1}.$$

Proof By convexity, we have $(m_1 + m_2)f(x) \leq m_1f(x_1) + m_2f(x_2)$, from which both inequalities follow easily. $\square$

Except in a divisible setting we do not capture convexity using only three points – we can not induct.

Proposition 10 (Monotone composition) *Assume that X is a monoid. If $f : X \to (-\infty, \infty]$ is sublinear and increasing and $g : X \to (-\infty, +\infty)$ is convex and non-decreasing, then $f \circ g$ is also convex.*

Proof Assume that $mx = \sum_{i=1}^{n} m_i x_i$, $m_i \in \mathbb{N}$, $m = \sum_{i=1}^{n} m_i$. Then,

$$mf(g(x)) = f\big(mg(x)\big) \leq f\big(m_1 g(x_1) + \cdots + m_n g(x_n)\big) \leq \sum_{i=1}^{n} m_i f(g(x_i)),$$

as required. □

Remark 10 (Midpoint convexity and measurability) Assume that X is a vector space which is also a measure space, and $f : X \to \mathbb{R}$ is measurable. Then it is known that if f is *midpoint convex*, then it must be convex. It is also known that if f is measurable and additive, then it must be linear. There are certainly analogous results to be discovered in appropriate monoids, see [31]. ◇

2.4 Operations on functions

We next extend some well known vector operations on convex and subadditive functions.

Definition 22 (Subadditive and sublinear minorants) Assume that X is a monoid and $f : X \to (-\infty, \infty]$. Define

$$p(x) = \inf \left\{ \sum_{i=1}^{n} f(x_i) \ \bigg| \ \sum_{i=1}^{n} x_i = x, \ n \in \mathbb{N} \right\}.$$

Define also

$$po(x) = \inf \left\{ \frac{p(mx)}{m} \ \bigg| \ m \in \mathbb{N} \right\},$$

where p is defined as above.

We have the following.

Proposition 11 *Assume that X is a monoid, and $f : X \to (-\infty, \infty]$. Then p as defined in Definition 22 is the largest function satisfying $p \leq f$ and also $p(x + y) \leq p(x) + p(y)$.*

Proof By definition, we have $p \leq f$. Next, assume that $x_1, \ldots, x_n, x$ and $y_1, \ldots, y_{n'}, y$ are elements in X such that $x = \sum_{i=1}^{n} x_i$ and $y = \sum_{i=1}^{n'} x_i'$. Hence, we have $x + y = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n'} x_i'$, and so

$$p(x + y) \leq \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n'} f(x_i').$$

Taking the infimum over the right hand side shows that $p(x+y) \leq p(x)+p(y)$. To show that p is the largest subadditive minorant, assume that $q \leq f$ and q is subadditive. Assume that $x \in X$ is such that $x = \sum_{i=1}^{n} x_i$. Then we have

$$q(x) \leq \sum_{i=1}^{n} q(x_i) \leq \sum_{i=1}^{n} f(x_i).$$

Taking the infimum over the right side shows that $q \leq p$, and so indeed p is the largest subadditive minorant of f, which completes the proof.

Proposition 12 *Assume that X is a monoid and $f : X \to (-\infty, \infty]$. Then po as defined in Definition 22 is the largest $\mathbb{N}$-sublinear minorant of f.*

Proof We begin by showing that *po* is positively homogeneous. Indeed, we have

$$po(m_0 x) = m_0 \inf \left\{ \frac{1}{m_0 m} \sum_{i=1}^{n} f(x_i) \;\middle|\; m \in \mathbb{N}, \; \sum_{i=1}^{n} x_i = m_0 x \right\}$$

$$= m_0 \inf \left\{ \frac{1}{m} \sum_{i=1}^{n} f(x_i) \;\middle|\; m \in \mathbb{N}, \; \sum_{i=1}^{n} x_i = x \right\},$$

where the last equality holds since for every $x_1, \ldots, x_n \in X$, we can choose $x'_1, \ldots, x'_{n'} \in X$ satisfying $\frac{1}{m} \sum_{i=1}^{n} f(x_i) = \frac{1}{m_0 m} \sum_{i=1}^{n'} f(x'_i)$. Next, we show that *po* is subadditive. Indeed, we have

$$\frac{1}{m_1} \sum_{i=1}^{n} f(x_i) + \frac{1}{m_2} \sum_{i=1}^{n'} f(x'_i) = \frac{1}{m_1 m_2} \left(\sum_{i=1}^{n} m_2 f(x_i) + \sum_{i=1}^{n'} m_1 f(x'_i) \right),$$

where $\sum_{i=1}^{n} x_i = m_1 x$, $\sum_{i=1}^{n'} x'_i = m_2 y$. Define the following sequence in X:

$$z_j = \begin{cases} x_i & (i-1)m_2 + 1 \leq j \leq im_2, 1 \leq i \leq n, \\ x'_i & nm_2 + (i-1)m_1 + 1 \leq j \leq nm_2 + im_1 1 \leq i \leq n'. \end{cases}$$

Notice that we have $j \in \{1, 2, \ldots, nm_2 + n'm_1\}$. Notice also that we have

$$\sum_{j=1}^{nm_2+n'm_1} z_j = m_2 \sum_{i=1}^{n} x_i + m_1 \sum_{i=1}^{n'} x'_i = m_1 m_2 (x+y),$$

and

$$\sum_{j=1}^{nm_2+n'm_1} f(z_j) = m_2 \sum_{i=1}^{n} f(x_i) + m_1 \sum_{i=1}^{n'} f(x'_i).$$

As a result, we have

$$\frac{1}{m_1} \sum_{i=1}^{n} f(x_i) + \frac{1}{m_2} \sum_{i=1}^{n'} f(x_i') = \frac{1}{m_1 m_2} \sum_{j=1}^{nm_2+n'm_1} f(z_j) \geq po(x+y).$$

Taking infima over m_1, m_2 and $x_1, \ldots, x_n, x_1', \ldots, x_{n'}'$ satisfying $m_1 x = \sum_{i=1}^{n} x_i$ and $m_2 y = \sum_{i=1}^{n'} x_i'$ implies that po is also subadditive, and therefore sublinear. Finally, assume that $q \leq f$ is such that q is $\mathbb{N}$-sublinear. Then if $x_1, \ldots, x_n, x \in X$ and $m \in \mathbb{N}$ are such that $mx = \sum_{i=1}^{n} x_i$, then we have

$$q(x) \stackrel{(*)}{=} \frac{1}{m} q(mx) \stackrel{(**)}{\leq} \frac{1}{m} \sum_{i=1}^{n} q(x_i) \leq \frac{1}{m} \sum_{i=1}^{n} f(x_i),$$

where in $(*)$ we used the fact that q is homogeneous and in $(**)$ we used the fact that q is subadditive. In particular, taking the infimum over the right hand side gives $q \leq po$, which completes the proof. $\qquad\square$

Definition 23 ($\mathbb{N}$-Sublinear minorant of two functions) Assume that X is a monoid and $f, g : X \to (-\infty, \infty]$. Define

$$f \wedge g(x) = \inf \left\{ \frac{n_1 f(x_1) + n_2 g(x_1)}{n} \;\middle|\; n_1 x_1 + n_2 x_2 = nx \right\}.$$

It is straightforward to check that if f, g are $\mathbb{N}$-sublinear, so is $f \wedge g$.

3 Examples

Example 2 (Vector spaces) If X is a real vector space, then by definition, $x \in \mathrm{conv}(A)$ if for every $n \in \mathbb{N}$, every $\alpha_1, \ldots, \alpha_n \in (0, 1)$ and every $x_1, \ldots, x_n \in A$,

$$\left(\sum_{i=1}^{n} \alpha_i \right) x = \sum_{i=1}^{n} \alpha_i x_i.$$

Taking $\beta_i = \frac{\alpha_i}{\sum \alpha_i} > 0$, this is equivalent to

$$x = \sum_{i=1}^{n} \beta_i x_i, \quad \sum_{i=1}^{n} \beta_i = 1,$$

which is the standard definition of a convex hull in a vector space over $\mathbb{R}$. $\quad\diamond$

Example 3 ($\mathbb{R}$ as a $\mathbb{Q}$-module) Consider $X = \mathbb{R}$ as a vector space over $\mathbb{Q}$. In such case $x \in \mathrm{conv}(A)$ if for every $n \in \mathbb{N}$, every $q_1, \ldots, q_n \in \mathbb{Q}_+ \setminus \{0\}$ and every $x_1, \ldots, x_n \in A$,

$$qx = \sum_{i=1}^{n} q_i x_i, \quad q = \sum_{i=1}^{n} q_i,$$

which is equivalent to

$$x = \sum_{i=1}^{n} q_i' x_i, \quad \sum_{i=1}^{n} q_i' = 1, \quad q_i' \in [0,1] \cap \mathbb{Q},$$

i.e., we take only *rational* convex combinations.   ◇

We now present examples of monoids and of the behaviour of the hull operator.

Example 4 (The lattice $\mathbb{Z}^d$) Consider $X = \mathbb{Z}^d$ with the addition induced from $\mathbb{R}^d$. For every $A \subseteq X$, we have

$$\mathrm{conv}_{\mathbb{Z}^d}(A) = \mathrm{conv}_{\mathbb{R}^d}(A) \cap \mathbb{Z}^d, \qquad (7)$$

where $\mathrm{conv}_{\mathbb{R}^d}(A)$ is the standard convex hull of A in $\mathbb{R}^n$. To see this, first note that if $x \in \mathrm{conv}_{\mathbb{Z}^d}(A)$, then there exist $x_1, \ldots, x_n \in A$, and $m_1, \ldots, m_n, m \in \mathbb{N}$ such that $mx = \sum_{i=1}^{n} m_i x_i$, $m = \sum_{i=1}^{n} m_i$. This implies that

$$x = \sum_{i=1}^{n} \frac{m_i}{m} x_i, \quad \sum_{i=1}^{n} \frac{m_i}{m} = 1,$$

which means that $x \in \mathrm{conv}_{\mathbb{R}^d}(A)$, and so $\mathrm{conv}_{\mathbb{Z}^d}(A) \subseteq \mathrm{conv}_{\mathbb{R}^d}(A) \cap \mathbb{Z}^d$. To prove to other inclusion, use induction on the dimension. If $d = 1$, and $x \in \mathrm{conv}_{\mathbb{R}}(A) \cap \mathbb{Z}$, then x is an integer which is also a convex combination of two other integers x_1, x_2. Therefore, we can write $x = q_1 x_1 + q_2 x_2$ with $q_1, q_2 \in \mathbb{Q}$, and so there exist $m_1, m_2, m \in \mathbb{Z}$ such that $mx = m_1 x_1 + m_2 x_2$ and $m = m_1 + m_2$. To prove the general case, assume that $x \in \mathrm{conv}_{\mathbb{R}^d}(A) \cap \mathbb{Z}^d$. Then there exist $x_1, \ldots, x_n \in A$ and $\alpha_1, \ldots, \alpha_n \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$. By Carathéodory's Theorem [25], we can write $x = \sum_{i=1}^{n'} \alpha_j x_j$, where $n' \leq d + 1$ (we might have to rearrange the points $x_1, \ldots, x_n$). If $\dim\big(\mathrm{span}\{x_1, \ldots, x_{n'}\}\big) < d$, use the induction hypothesis to conclude that we can write $x = \sum_{i=1}^{n'} q_i x_i$, with $q_i \in \mathbb{Q}$. Otherwise, we have the following linear system.

$$\begin{bmatrix} x_1 & x_2 & \ldots & x_{d+1} \\ 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{d+1} \end{bmatrix} = x.$$

where $x_1, \ldots, x_{d+1}$ are written as column vectors. In this case, one can show that the system has a *unique* solution. Thus the matrix is invertible. Since the matrix has integer coefficients, it follows that the q_i's are rational. And so once again we can write $x = \sum_{1}^{n'} q_i x_i$ with $q_i \in \mathbb{Q}$, which implies that $x \in \mathrm{conv}_{\mathbb{Z}^d}(A)$.   ◇

Example 5 (General lattices in $\mathbb{R}^d$) We say that $v_1, \ldots, v_k \in \mathbb{R}^d$ are independent over $\mathbb{Z}$ if

$$\sum_{i=1}^{k} m_i v_i = 0, \quad m_i \in \mathbb{Z} \Longrightarrow m_i = 0.$$

Assume that $\Gamma = \mathrm{span}_{\mathbb{Z}}\{v_1, \ldots, v_k\}$, where $v_1, \ldots, v_k \in \mathbb{R}^d$ are independent over $\mathbb{Z}$. Let $T : \mathbb{R}^k \to \mathbb{R}^d$ be defined as

$$T(\alpha_1, \ldots, \alpha_k) = \sum_{i=1}^{k} \alpha_i v_i.$$

T is linear and $T(\mathbb{Z}^k) = \Gamma$. Also, since $v_1, \ldots, v_k$ are independent over $\mathbb{Z}$, it follows that $T\big|_{\mathbb{Z}^k}$ is invertible. Finally, since Γ is a $\mathbb{Z}$-module, it follows from Proposition 1 that

$$\mathrm{conv}_\Gamma(A) = \left\{ \sum_{i=1}^{n} q_i a_i \ \middle| \ a_i \in A, q_i \in \mathbb{Q} \cap [0,1], \sum_{i=1}^{n} q_i = 1 \right\}.$$

Hence,

$$
\begin{aligned}
\mathrm{conv}_\Gamma(A) &= T\left(\mathrm{conv}_{\mathbb{Z}^k}\left(T^{-1}A\right)\right) \\
&\overset{(*)}{=} T\left(\mathrm{conv}_{\mathbb{R}^k}\left(T^{-1}A\right) \cap \mathbb{Z}^k\right) \\
&\overset{(**)}{=} T\left(\mathrm{conv}_{\mathbb{R}^k}\left(T^{-1}A\right)\right) \cap T\mathbb{Z}^k \\
&\overset{(***)}{=} \mathrm{conv}_{\mathbb{R}^d}(A) \cap \Gamma,
\end{aligned}
$$

where in $(*)$ we used Example 4, in $(**)$ we used the invertibility of T over $\mathbb{Z}^k$, and in $(***)$ we used the linearity of T. $\diamond$

Example 6 (Dyadic rationals) Let X be the rational numbers of the form $\frac{m}{2^n}$, where $m, n \in \mathbb{Z}$. We have that X is 2-semidivisible as $X = 2X$, since $\frac{m}{2^n} = 2\frac{m}{2^{n-1}}$, but for any odd number k we do not have $1 = k \cdot \frac{m}{2^n}$. Thus, X is not divisible. $\diamond$

Example 7 (Arctan semigroup) Let $X = ([0, \infty), \oplus)$ with addition defined by

$$a \oplus b = \frac{a+b}{1+ab}.$$

Note that if $a, b \neq 0$ then $a \oplus b = \frac{1}{a} \oplus \frac{1}{b}$. The unit is 0 as $a \oplus 0 = a$. Also, for all $a \geq 0$, $a \oplus 1 = 1$. Hence, $\mathrm{conv}(\{0\}) = \{0\}$ and $\mathrm{conv}(\{1\}) = \{1\}$. For every $a > 0$ we have $a \oplus a = \frac{1}{a} \oplus \frac{1}{a}$. Thus, if $a \neq 1$ then $\frac{1}{a} \in \mathrm{conv}(\{a\})$. This means that $\{0\}$ and $\{1\}$ are the only convex singletons. Also, since $a \oplus 1 = 1$ for every $a \in X$, then for every $A \subseteq X$, we have

$$\mathrm{conv}(A \cup \{1\}) = \mathrm{conv}(A) \cup \{1\}.$$

Finally, note that for every $a \geq 0$, we have

$$3a = a \oplus a \oplus a = \frac{3a + a^3}{1 + 3a^2},$$

and the function $a \mapsto \frac{3a+a^3}{1+3a^2}$ is onto $[0, \infty)$. Thus, X is 3-semidivisible. On the other hand, $a \oplus a = \frac{2a}{1+a^2} \leq 1$, and so X is not divisible. In fact is is divisible precisely for all odd numbers. $\diamond$

The next example illustrates that finding convex or affine functions on a group is solving potentially subtle functional equations and inequalities

Example 8 (Hyperbolic group) Let X_p be the collection of all 2×2 symmetric matrices of the form $e^{\frac{2\pi i l}{p}} M(\theta)$, where $M(\theta) = \begin{bmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{bmatrix}$, $\theta \in \mathbb{R}$ and $0 \leq l \leq p - 1$. Then X_p is a group under the standard matrix multiplication, as we have

$$\left(e^{\frac{2\pi i l_1}{p}} M(\theta_1) \right) \cdot \left(e^{\frac{2\pi i l_2}{p}} M(\theta_2) \right) = e^{\frac{2\pi i (l_1 + l_2)}{p}} M(\theta_1 + \theta_2).$$

In particular, the group is commutative. Also, if $p|n$, we have that $M(\theta)^n = \left(e^{\frac{2\pi i l}{p}} M(\theta) \right)^n = M(n\theta)$ for all $0 \leq l \leq p - 1$. Thus, in this case we have $nX_p \subsetneq X_p$. Otherwise, if $p \nmid n$, then we have $\left(e^{\frac{2\pi l}{p}} M(\theta) \right)^n = e^{\frac{2\pi n l}{p}} M(n\theta)$. Since $\theta \mapsto n\theta$ and $e^{\frac{2\pi l}{p}} \mapsto e^{\frac{2\pi n l}{p}}$ is one-to-one and onto (the second since $p \nmid n$), it follows that in this case $nX_p = X_p$. Altogether, we conclude that X_p is n-divisible if and only if $p \nmid n$.

Next, we would like to show that it is easy to produce convex functions on the group X_p. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is a convex function then defining $F\left(e^{\frac{2\pi i l}{p}} M(\theta) \right) = f(\theta)$ is also convex. To see this, for $m_1, \ldots, m_n \in \mathbb{N}$ and $x_1, \ldots, x_n, x \in X$ satisfying $mx = \sum_{i=1}^{n} m_i x_i$, $m = \sum_{i=1}^{n} m_i$, assume that $x = e^{\frac{2\pi i l}{p}} M(\theta), x_1 = e^{\frac{2\pi i l_1}{p}} M(\theta_1), \ldots, x_n = e^{\frac{2\pi i l_n}{p}} M(\theta_n)$. Thus, we have

$$e^{\frac{2\pi i m l}{p}} M(m\theta) = e^{\frac{2\pi i}{p} \sum_{j=1}^{n} m_j l_j} M(m_1\theta_1 + \ldots m_n\theta_n). \tag{8}$$

Note that if $e^{\frac{2\pi i l}{p}} M(\theta)$ is the identity matrix, then $l = \theta = 0$. Therefore, if $e^{\frac{2\pi i l_1}{p}} M(\theta_1) = e^{\frac{2\pi i l_1}{p}} M(\theta_1)$, then $l_1 = l_2$ and $\theta_1 = \theta_2$. In particular, (8) implies that $m\theta = \sum_{i=1}^{n} m_i \theta_i$. Hence, we have

$$mF(x) = mf(\theta) \leq \sum_{i=1}^{n} m_i f(\theta_i) = \sum_{i=1}^{n} m_i F(x_i).$$

Note that restriction to $M(\theta)$ (determinant one) is a divisible subgroup. Also, consider the group

$$X_{\mathbb{R}} = \left\{ e^{it} M(\theta) \ \middle| \ t, \theta \in \mathbb{R} \right\},$$

again with the standard multiplication. Then $X_{\mathbb{R}}$ is a divisible group, since for every $t, \theta \in \mathbb{R}$ and every $n \in \mathbb{N}$, we have

$$e^{it} M(\theta) = \left(e^{i\frac{t}{n}} M(\theta/n) \right)^n.$$

Note that for every p, X_p is a semidivisible subgroup of $X_{\mathbb{R}}$. Finally, note that if we consider $X_{\mathbb{R}}$ as a topological space, equipped with the topology induced from $\mathbb{R}^4$, then $X_{\mathbb{R}}$ is connected since we can write $X_{\mathbb{R}} = \Phi(\mathbb{R}^2)$, where $\Phi : (t, \theta) \mapsto e^{it} M(\theta)$ is continuous. See [5] for a more detailed discussion on convexity in topological groups. ◇

Example 9 (Circle group) Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the standard coset addition. In this case, if $x = [m/n]$ $m, n \in \mathbb{N}$ then $nx = [0]$. Thus,

$$\mathrm{conv}(\{0\}) = \big\{ x \in \mathbb{T} \mid x \text{ has finite order} \big\}.$$

Also, for every $x \in X$, $x + y \in \mathrm{conv}(\{x\})$ for every $y \in X$ which is of finite order. Thus, there are no convex singletons in X. ◇

Example 10 (Prüfer group) This is a subgroup of the circle group $\mathbb{T}$, which is given by

$$\mathbb{Z}(p^\infty) = \big\{ \exp\big(2\pi i m/p^n\big) \mid m, n \in \mathbb{N} \cup \{0\} \big\},$$

i.e., all p^n-th roots of unity. Every element in this group has a finite order and so by the previous example (and also by example 1), the only two convex sets are $\emptyset$ and the entire group. It is also known that $\mathbb{Z}(p^\infty)$ is divisible. To see this, note that it is enough to show that $X = qX$ for every prime q. Let $x = \exp\big(2\pi i m/p^n\big)$. If $n = 0$ then $x = 1 = 1^q$. Assume then that $n > 0$. If $q = p$ then $x = y^q$ where $y = \exp\big(2\pi i m/p^{n+1}\big)$. If $q \neq p$ then since the greatest common divisor of p^n and q is 1, there exist $a, b \in \mathbb{Z}$ such that $ap^n + bq = 1$. So $x = x^{ap^n + bq} = x^{ap^n} x^{bq} = x^{bq}$. Choosing $y = x^b$, then $x = y^q$, as needed. ◇

Example 11 (Extensions of $\mathbb{Q}$) Consider $X = \mathbb{Q} + \theta \mathbb{Q}$, where θ is irrational, with the addition operation then the mapping $\Phi : a + \theta b \mapsto (a, b)$ is a group homomorphism from X to $\mathbb{Q}^2$. Thus

$$\mathrm{conv}_X(A) = \Phi^{-1}\big(\mathrm{conv}_{\mathbb{Q}^2}(\Phi(A))\big).$$

Similarly, we can consider extensions of $\mathbb{Q}$ be any number of algebraically independent numbers. ◇

Example 12 (Half line with multiplication) If $X = ((0, \infty), \cdot)$, this semigroup is isomorphic to $(\mathbb{R}, +)$ via $x \mapsto \log(x)$. Thus,

$$\mathrm{conv}_X(A) = \exp\big(\mathrm{conv}_{(\mathbb{R}, +)}(\log(A))\big). \tag{9}$$

If instead we choose $X = ([0, \infty), \cdot)$, then if $0 \in A$, we have

$$\mathrm{conv}_X(A) = \{0\} \cup \exp\big(\mathrm{conv}_{(\mathbb{R}, +)}(\log(A))\big),$$
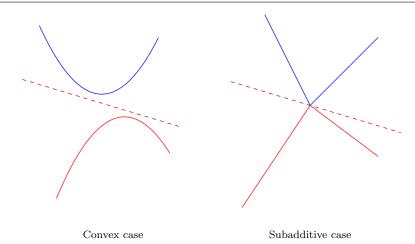
if $0 \notin A$ then (9) still holds. ◇

Convex case   Subadditive case

Fig. 1 Separation in groups

Example 13 (σ-algebras with symmetric differences) Given a set S, let X be a
σ-algebra of subsets of S. For $A, B \in X$, let $A + B = A \triangle B = (A \cup B) \backslash (A \cap B)$.
Clearly $A \triangle B = B \triangle A$. Also, note that for every $A \in \mathcal{F}$, $A \triangle \emptyset = A$, and
$A \triangle A = \emptyset$. Thus, $\emptyset$ is the additive unit and $A = -A$. It also follows that
from every $A \in X$ and $n \in \mathbb{N}$, $2nA = \emptyset$ and $(2n-1)A = (2nA) \triangle A =
\emptyset \triangle A = A$. Thus, $2nX = \{\emptyset\} \subsetneq X$ and $(2n-1)X = X$, and so X is $(2n-1)$-
semidivisible but not $2n$-semidivisible. Next, assume that $A_1, \ldots, A_n, A \in X$
and $m_1, \ldots, m_n, m \in \mathbb{N}$ are such that $mA = \sum_{i=1}^{n} m_i A_i$ and $m = \sum_{i=1}^{n} m_i$.
Then by the above arguments we have in fact

$$mA = \sum_{i:2\nmid m_i} m_i A_i = \sum_{i:2\nmid m_i} A_i.$$

Thus, if $\mathcal{A} \subseteq X$, then we can write

$$\mathrm{conv}(\mathcal{A}) = \left\{ A \subseteq X \;\middle|\; A = \sum_{i=1}^{n} A_i, \;\; A_i \in \mathcal{A}, \;\; n \in \mathbb{N} \right\}.$$

Note that we always have $\emptyset \in \mathrm{conv}(\mathcal{A})$ since $A + A = \emptyset = 2\emptyset$. This group can
also be studied as a topological group. See [5]. $\diamond$

4 Interpolation of scalar-valued functions

We begin with a slight extension of a seminal result.

Theorem 1 (Kaufman [19]**)** *Let X be a monoid and $f, g : X \to [-\infty, \infty)$
satisfying $g \leq f$, where f and $-g$ are subadditive. Then there exists a function
$a : X \to \mathbb{R}$ which is additive and satisfies $g \leq a \leq f$.*

Theorem 1 is a generalization of Kaufman's Hahn-Banach result which itself extends the seminal result by Mazur and Orlicz [26]. Under the assumption that X is semidivisible, the following holds.

Theorem 2 (Interpolation of convex functions) *Assume that X is a semidivisible monoid, and $f : X \to [-\infty, \infty]$ and $-g : X \to [-\infty, \infty]$ are convex. Then there exists a function $a : X \to [-\infty, \infty]$ which is generalised affine and satisfies $g \leq a \leq f$.*

We illustrate the two results in Figure 1.

Proof First, since $f, -g$ are convex and $g \leq f$, we have

$$mf(x) \geq \sum_{i=1}^{n} m_i g(x_i),$$

$$mx = \sum_{i=1}^{n} m_i x_i, \quad m = \sum_{i=1}^{n} m_i. \tag{10}$$

If $f = g$, then f is generalised affine and the proof is complete. Assume then that there exists $x_0 \in X$ and $r \in \mathbb{R}$ such that $f(x_0) > r > g(x_0)$. In such case, either we have

$$mf(x) \geq m_0 r + \sum_{i=1}^{n} m_i g(x_i), \tag{11}$$

whenever we have

$$mx = m_0 x_0 + \sum_{i=1}^{n} m_i x_i, \quad m = m_0 + \sum_{i=1}^{n} m_i,$$

or else

$$(m' - m_0') f(y) + m_0' r \geq \sum_{i=1}^{n'} m_i' g(y_i), \tag{12}$$

whenever we have

$$m_0' x_0 + (m' - m_0') y = \sum_{i=1}^{n'} m_i' y_i, \quad m' = \sum_{i=1}^{n} m_i', \quad m_0' \leq m'.$$

To see this, assume that neither (11) nor (12) hold. Multiplying (11) by m' and (12) by m, we can find integers $m_0, \ldots, m_n, m_0', \ldots, m_{n'}' \in \mathbb{N}$ and elements $x_1, \ldots, x_n, y_1, \ldots, y_{n'} \in X$ satisfying

$$m = m_0 + \sum_{i=1}^{n} m_i, \quad mx = m_0 x_0 + \sum_{i=1}^{n} m_i x_i, \tag{13}$$

$$m' = \sum_{i=1}^{n'} m_i', \quad m_0' \leq m', \quad m_0' x_0 + (m' - m_0') y = \sum_{i=1}^{n'} m_i' y_i, \tag{14}$$

such that

$$m_0' \sum_{i=1}^{n} m_i g(x_i) + m_0 \sum_{i=1}^{n'} m_i' g(y_i) > m_0' m f(x) + m_0(m' - m_0') f(y)$$

$$\geq \big(m_0' m + m_0(m' - m_0')\big) f(z),$$

where z satisfies

$$
\begin{aligned}
\big(m_0' m + m_0(m' - m_0')\big) z &= m_0' m x + m_0(m' - m_0') y \\
&\overset{(13)}{=} m_0' m_0 x_0 + m_0' \sum_{i=1}^{n} m_i x_i + m_0(m' - m_0') y \\
&\overset{(14)}{=} m_0 \sum_{i=1}^{n'} m_i' y_i + m_0' \sum_{i=1}^{n} m_i x_i. \qquad (15)
\end{aligned}
$$

Such z always exists since X is semidivisible, i.e., $X = p^l X$ for some prime p and $l \in \mathbb{N}$ and by Proposition 6 we may assume that $m_0' m + m_0(m' - m_0') = p^l$. Now, we have

$$
\begin{aligned}
m_0' \sum_{i=1}^{n} m_i + m_0 \sum_{i=1}^{n'} m_i' \; \overset{(13)\wedge(14)}{=} \; & m_0'(m - m_0) + m_0 m' \\
= \; & m_0' m - m_0' m_0 + m_0 m' \\
= \; & m_0' m + m_0(m' - m_0').
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\big(m_0' m + m_0(m' - m_0')\big) f(z) \; \overset{(*)}{\geq} \; & \big(m_0' m + m_0(m' - m_0')\big) g(z) \\
\overset{(**)}{\geq} \; & m_0' \sum_{i=1}^{n} m_i g(x_i) + m_0 \sum_{i=1}^{n'} m_i' g(y_i), \quad (16)
\end{aligned}
$$

where in $(*)$ we used the fact that $g \leq f$ and in $(**)$ we used the fact that g is concave. Now, (16) is a contradiction to (10). Thus, we must have that either (11) or (12) hold. Assume first that (11) holds. Define

$$h(x) = \sup \left[\frac{1}{k} \left(k_0 r + \sum_{i=1}^{n} k_i g(x_i) \right) \right], \qquad (17)$$

where the supremum is taken over all $k, k_0, k_1, \ldots, k_n \in \mathbb{N}$ and $y_1, \ldots, y_n \in X$ such that $kx = k_0 x_0 + \sum_{i=1}^{n} k_i y_i$ and $k = k_0 + \sum_{i=1}^{n} k_i$. By choosing $k_1 = \cdots = k_n = 0$, we have $h(x_0) \geq r > g(x_0)$. Since g is concave we also have that $h \geq g$, and by (11) it follows that $h \leq f$. Next, we would like to show that h is concave, and that (10) holds for h instead of g. To show the concavity, let $m_1, \ldots, m_n \in \mathbb{N}$, and $x_1, \ldots, x_n, x \in X$ such that $mx = \sum_{i=1}^{n} m_i x_i$ and $m = \sum_{i=1}^{n} m_i$. Let $\epsilon > 0$, and for each $1 \leq i \leq n$, choose $k_i, k_{i,0}, \ldots, k_{i,n_i} \in \mathbb{N}$ and

$y_{i,1}, \ldots, y_{i,n'} \in X$ such that $k_i x = k_{i,0} x_0 + \sum_{j=1}^{n_i} k_{i,j} y_{i,j}$, $k_i = k_{i,0} + \sum_{j=1}^{n_i} k_{i,j}$ such that

$$k_i h(x_i) - \frac{k_i \epsilon}{m} \le k_{i,0} r + \sum_{j=1}^{n_i} k_{i,j} g(y_{i,j}). \tag{18}$$

Now, we have

$$
\left(m \prod_{i=1}^{n} k_i \right) x = \sum_{i=1}^{n} \left(m_i \prod_{j \ne i} k_j \right) k_i x_i
$$

$$
= \sum_{i=1}^{n} \left(m_i \prod_{j \ne i} k_j \right) \left(k_{i,0} x_0 + \sum_{j=1}^{n_i} k_{i,j} y_{i,j} \right)
$$

$$
= \sum_{i=1}^{n} \left(m_i k_{i,0} \prod_{j \ne i} k_j \right) x_0 + \sum_{i=1}^{n} \left(m_i \prod_{j \ne i} k_j \right) \left(\sum_{j=1}^{n_i} k_{i,j} y_{i,j} \right).
$$

Also, we have

$$
m \prod_{i=1}^{n} k_i = \sum_{i=1}^{n} \left(m_i k_{i,0} \prod_{j \ne i} k_j \right) + \sum_{i=1}^{n} \left(m_i \prod_{j \ne i} k_j \right) \sum_{j=1}^{n_i} k_{i,j}.
$$

Thus, by the definition of h (17), we have

$$
mh(x) \ge \frac{1}{\prod_{i=1}^{n} k_i} \left[\sum_{i=1}^{n} \left(m_i k_{i,0} \prod_{j \ne i} k_j \right) r + \sum_{i=1}^{n} \left(m_i \prod_{j \ne i} k_j \right) \sum_{j=1}^{n_i} k_{i,j} g(y_{i,j}) \right]
$$

$$
= \frac{1}{\prod_{i=1}^{n} k_i} \sum_{i=1}^{n} m_i \prod_{j \ne i} k_j \left[k_{i,0} r + \sum_{j=1}^{n_i} k_{i,j} g(y_{i,j}) \right]
$$

$$
\overset{(18)}{\ge} \frac{1}{\prod_{i=1}^{n} k_i} \sum_{i=1}^{n} \left(m_i \prod_{j \ne i} k_j \right) \left(k_i h(x_i) - \frac{k_i \epsilon}{m} \right)
$$

$$
= \sum_{i=1}^{n} m_i h(x_i) - \epsilon.
$$

Since ϵ is arbitrary, it follows that h is concave. Finally, we would like to show that if $mx = \sum_{i=1}^{n} m_i x_i$, $m = \sum_{i=1}^{n}$, then $\sum_{i=1}^{n} m_i h(x_i) \le mf(x)$. This follows from the fact that h is concave together with the fact that $h \le f$. The existence and the properties of h show that g is *not* the maximal element in the class of all concave functions that satisfy (10). Analogously, if (12) holds, define

$$h'(x) = \inf \left[\frac{1}{k'} \left(k_0' r + (k' - k_0') f(y) \right) \right], \tag{19}$$
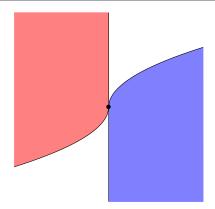
Fig. 2 Failure of finite affine separation

where the infimum is taken over all $k_0' \geq 0$, $k' \in \mathbb{N}$, and $y \in X$ such that $k'x = k_0'x_0 + (k' - k_0')y$. If $k' = k_0'$ we define the right side of (19) to be r. Choosing $k' = k_0'$ gives $h'(x_0) \leq r < f(x_0)$ and choosing $k_0 = 0$ gives $h'(x) \leq f(x)$ for all $x \in X$. Since (12) holds and g is concave, we also have that $g \leq h'$ and (10) holds with h' instead of f. Also, in an analogous way to the previous case, one can show that h' is convex. To conclude the proof, define the following ordered set $\mathcal{D}$ of all pairs of the form (h, h'), where h is concave, h' is convex, and (10) holds if we replace g by h or f by h'. Define the partial order on $\mathcal{D}$ to be $(h, h') \leq (w, w') \iff h \leq w$ and $w' \leq h'$. Since $(g, f) \in \mathcal{D}$, this chain is non-empty and therefore has a maximal element. By the above consideration we conclude the maximal element is generalised affine.   □

Remark 11 Note that we used the semidivisibility only to show that either (11) or (12) must hold. We did not use this fact again in the proof.   ◇

Remark 12 Similarly, the results hold if we work in a semimodule.   ◇

Remark 13 In general we cannot expect the affine function a to be better than generalised affine in Theorem 2, even if X is a vector space. This is illustrated by the example of $f(x) = -\sqrt{x}$ if $x \geq 0$ and $f(x) = -\infty$ if $x < 0$ and $g(x) = -f(-x)$, where $X = \mathbb{R}$. The only separator comes from letting a to be $+\infty$ when $x > 0$, $-\infty$ when $x < 0$ and 0 when $x = 0$. See Figure 2.   ◇

On the other hand, using Proposition 4, we have the following.

Corollary 3 *Assume that X is a group. If either f or g is everywhere finite and the other function is somewhere finite, then a is finite and affine.*

The vector space version of the following result is used in [17] as the basis for Hahn-Banach theory. Once established, one imposes additional core conditions on A, B to show cl $C \cap$ cl D is a separating half-space. Here one uses the algebraic closure. We take a different (more modern) approach in the next section.

Corollary 4 (Stone's lemma for monoids) *Assume that X is a semidivisible monoid and $A, B \subseteq X$ are disjoint convex sets. Then there exist $C, D \subseteq X$ disjoint and convex such that $A \subseteq C$, $B \subseteq D$ and $C \cup D = X$.*

Proof Let $f = \iota_A$, $g = -\iota_B$, where

$$\iota_A(x) = \begin{cases} 0 & x \in A \\ \infty & x \notin A \end{cases},$$

and similarly for ι_B. Then $f, -g : X \to [-\infty, \infty]$ are convex. Use Theorem 2 to deduce the existence of a generalised affine function $a : X \to [-\infty, \infty]$ with $-\iota_B \leq a \leq \iota_A$. Choosing

$$C = \{x \in X \mid a(x) < 0\}, \quad D = \{x \in X \mid a(x) \geq 0\},$$

concludes the proof.   □

   Theorem 2 also implies the following.

Corollary 5 *Assume that X is semidivisible monoid and $f : X \to [-\infty, \infty]$ is convex. Then f is the supremum over its generalised affine minorants.*

Proof Clearly we have

$$f(x) \geq \sup\{a(x) \mid a \leq f, \ a \text{ is affine}\}. \tag{20}$$

To show that equality holds, assume to the contrary that we have a strict inequality in (20). The function g which equals the supremum at x and $-\infty$ everywhere else is concave. By Theorem 2, there exists an affine function a such that

$$\sup\{a(x) \mid a \leq f, \ a \text{ is affine}\} < a(x) < f(x),$$

which is a contradiction.   □

Example 14 (Non separation) In the non-divisible setting, Theorem 2 fails even for everywhere finite functions. Take for example $X = \mathbb{Z}^2$. Let $A = \text{conv}_{\mathbb{R}^2}(\{(0,2), (1,0)\})$ and $B = \text{conv}_{\mathbb{R}^2}(\{(0,1), (2,0)\})$, and
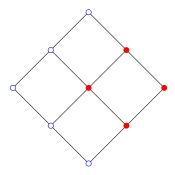
$$f(x) = 2\sqrt{5}d_A(x) - 1,$$
$$g(x) = -2\sqrt{5}d_B(x) + 1.$$

where $d_A(x) = \inf_{a \in A} \|x - a\|_{\mathbb{R}^2}$. Note that for every $x \in \mathbb{Z}^2$ such that $x \notin A$, we have $d_A(x) \geq \frac{1}{\sqrt{5}}$. Similarly, if $x \in \mathbb{Z}^2$ and $x \notin B$, we have $d_B(x) \geq \frac{1}{\sqrt{5}}$. For every $x \in \mathbb{Z}^2$, either $x \notin A$ or $x \notin B$ and so $d_A(x) + d_B(x) \geq \frac{1}{\sqrt{5}}$. Hence,

$$f(x) - g(x) = 2\sqrt{5}(d_A(x) + d_B(x)) - 2 \geq 0,$$

and so $g \leq f$ on $\mathbb{Z}^2$. Also, f and $-g$ are convex, since they are convex on all of $\mathbb{R}^2$ (the distance to a convex set in a vector space is a convex function). Assume

Fig. 3 Convex separation in a lattice

that a is affine and satisfies $g \le a \le f$. By the choice of f and g, a has to be finite everywhere. Since a is affine, we can write $a(m_1, m_2) = c + \alpha_1 m_1 + \alpha_2 m_2$, where $c, \alpha_1, \alpha_2 \in \mathbb{R}$. Since $a \le f$, we can choose $x = (0, 2)$ and $x = (1, 0)$ and obtain

$$c + 2\alpha_2 \le -1, \quad c + \alpha_1 \le -1.$$

Similarly, since $a \ge g$ we get

$$c + 2\alpha_1 \ge 1, \quad c + \alpha_2 \ge 1.$$

Altogether, we get both $c \le -3$ and $c \ge 3$.   ◇

Example 15 Let $(X, \wedge)$ be a semimodule induced by a semilattice X. This is divisible since $x \wedge x = x$. Thus, $\mathrm{conv}(S)$ is the sub semilattice generated by S. In this case convex and subadditive functions coincide, and so Theorems 1 and 2 both assert the un-obvious result that disjoint sub meet-lattices lie in partitioning sublattices. See Figure 3. Note that since X contains nontrivial idempotent elements, it cannot be embedded in a group (see [16]). See also [28] for a study of convexity in semilattices.   ◇

Part II: Convex operators on groups

5 Analysis of convex operators on groups

We turn now to results for operators on groups. By Example 3 and Remark 4, we could derive many of these results using $\mathbb{Q}$-modules but we prefer to highlight the use of only monoidal structure.

5.1 Subdifferential calculus of operators

Here we assume that X, Y are groups, and $f : X \to (Y \cup \{\infty\}, \leq)$, where ∞ is a maximal element with respect to the partial order $\leq$ on Y. Assume also that $\leq$ is compatible with the group operation, i.e., if $x \geq y$ iff $x - y \geq 0$. We also assume that the order is at least *inductive*, i.e., that every countable chain has an upper bound. In Subsection 5.3, we will need to further assume that $\leq$ is a *complete* order, i.e., that every order bounded set has an infimum and supremum. Of course Y may be $\mathbb{R}$ as before.

Remark 14 A partial order in a Banach space is order complete if and only it is latticial. Moreover, order completeness of the range characterises the Hahn-Banach extension theorem holding. By contrast if the cone has a bounded complete base, the order is inductive. Thus, in Euclidean space all pointed closed convex cones induce inductive orders. (See [4, 7–9] for much more on these technicalities in the vector space setting.) ◇

As in Definition 19, f is said to be *subadditive* if $f(x + y) \leq f(x) + f(y)$. We can similarly define $\mathbb{N}$-sublinear and convex functions.

Definition 24 (Domain of convex function) Let X, Y be groups and $f : X \to Y \cup \{\infty\}$ be convex. Define the domain of f to be the set

$$\mathrm{dom}(f) = \big\{ x \in X \mid f(x) < \infty \big\}.$$

It is easily shown that the domain of a convex function of a convex subset of X. The *core* of the domain is then:

Definition 25 (Core of domain) Let X, Y be groups and let $f : X \to Y \cup \{\infty\}$ be a convex function. Define the core of the domain of f to be

$$\mathrm{core}(\mathrm{dom}(f)) = \big\{ x \in X \mid \forall h \in X, \exists n \in \mathbb{N}, \exists g \in X, ng = h, f(x + g) < \infty \big\}.$$

By choosing $h = 0$, it follows that $\mathrm{core}(\mathrm{dom}(f)) \subseteq \mathrm{dom}(f)$. More generally, we can define the core of a convex function.

Definition 26 (Core of convex set) Let X be a group and $C \subseteq X$ a convex set. Define the core of C to be the set

$$\mathrm{core}(C) = \big\{ x \in X \mid \forall h \in X, \exists n \in \mathbb{N}, \exists g \in X, ng = h, x + g \in C \big\}.$$

Again, we have $\mathrm{core}(C) \subseteq C$. Now we define the directional derivative.

Definition 27 (Directional derivative) Let X be a group, $(Y \cup \{\infty\})$ a group with an inductive order, and $f : X \to Y \cup \{\infty\}$ a convex function. For $x \in \mathrm{core}(\mathrm{dom}(f))$, define

$$f_x(h) = \inf \big\{ n\big(f(x + g) - f(x) \big) \mid ng = h, \ f(x + g) < \infty \big\}.$$

Before proceed to the study of directional derivatives, we need the following technical proposition.

Proposition 13 *Assume that $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are two decreasing sequences in an inductive and compatible cone. Then*

$$\inf_{n\in\mathbb{N}}\left\{a_n + b_n\right\} = \inf_{n\in\mathbb{N}} a_n + \inf_{n\in\mathbb{N}} b_b.$$

Proof Let $n, m \in N$ with $n > m$. Then since $\{b_n\}_{n\in\mathbb{N}}$ is decreasing, we have $a_n + b_n \leq a_n + b_m$. Thus, we have

$$\inf_{n\in\mathbb{N}}\left\{a_n + b_n\right\} \leq \left[\inf_{n\in\mathbb{N}} a_n\right] + b_m.$$

Taking the infimum over m gives $\inf_{n\in\mathbb{N}}\left\{a_n + b_n\right\} \leq \inf_{n\in\mathbb{N}} a_n + \inf_{m\in\mathbb{N}} b_m$. The converse inequality is clear. This completes the proof.   □

We have the following.

Proposition 14 (One-sided derivatives, I) *Assume that a group X is a p-semidivisible group, and $(Y, \leq)$ is a group with an inductive order. Assume also that $f : X \to Y \cup \{\infty\}$ is convex and $x \in \mathrm{core}(\mathrm{dom}(f))$. Then f_x is an everywhere finite, $\mathbb{N}$-sublinear function.*

Proof For arbitrarily large $n, n' \in \mathbb{N}$ with $n < n'$ we can find $g, g' \in X$ such that $ng = n'g' = h$ and $f(x + g) < \infty$, $f(x + g') < \infty$. We have $n'(x + g') = n(x+g)+(n'-n)x$, and so by convexity $n'f(x+g') \leq nf(x+g)+(n'-n)f(x)$. Therefore, we have

$$n'(f(x + g') - f(x)) \leq n(f(x + g) - f(x)).$$

Also, if $g, g' \in X$ are such that $ng = n'g' = h$, then $(n + n')x = n(x - g) + n'(x + g)$ and so again by convexity, we have

$$n(f(x) - f(x - g)) \leq n'(f(x + g') - f(x)).$$

Thus, the sequence $\left\{n\big(f(x+g) - f(x)\big) \mid ng = h,\ f(x+g) < \infty\right\}$ is decreasing and bounded from below. Since $\leq$ is an inductive order on Y, $f_x(h)$ exists and is finite. To show that $f_x(0) \leq 0$, note that we can choose $g = 0$ in Definition 27 and obtain $f_x(0) \leq 0$. To prove the positive homogeneity of f_x, choose, $g, g' \in X$ such that $p^l g = ph$ and $p^l g' = h$. Then we have $p^{l+1}(x+g') = p^{l+1}x + ph = p^{l+1}x + p^l g = (p^{l+1} - p^l)x + p^l(x + g)$. Thus, since f is convex, we have

$$p^{l+1}f(x + g') \leq (p^{l+1} - p^l)f(x) + p^l f(x + g),$$

or in other words,

$$p^{l+1}\big(f(x + g') - f(x)\big) \leq p^l\big(f(x + g) - f(x)\big).$$

Taking the limit as $l \to \infty$ and using the fact that the sequence in Definition 27 is decreasing, we get $pf_x(h) \leq f_x(ph)$. On the other hand, we have,

$$
\begin{aligned}
pf_x(h) &= \inf\left\{pn\big(f(x+g) - f(x)\big) \mid ng = h\right\} \\
&\overset{(*)}{\geq} \inf\left\{pn\big(f(x+g) - f(x)\big) \mid png = ph\right\} \\
&\overset{(**)}{=} f_x(ph).
\end{aligned}
$$

In $(*)$ we used the fact that if $pg = h$ then $png = ph$ (but we might have a bigger set on which we take the infimum). In $(**)$ we used the fact in Definition 27 the infimum is taken over a decreasing sequence. This shows that $p_x(ph) = pf_x(h)$. Finally, to show subadditivity, note that $p(x + g_1 + \cdots + g_p) = (x + pg_1) + \cdots + (x + pg_p)$, and so by convexity of f,

$$
p\big(f(x + g_1 + \cdots + g_p) - f(x)\big)
$$
$$
\leq \big(f(x + pg_1) - f(x)\big) + \cdots + \big(f(x + pg_p) - f(x)\big). \quad (21)
$$

Multiply (21) by n and then choose $g_1, \ldots, g_p$ such that $ng_1 = h_1, \ldots, ng_p = h_p$. This is possible since we may assume without loss of generality that $n = p^l$ for some $l \in \mathbb{N}$, and this is because the sequence $\left\{n\big(f(x+g) - f(x)\big) \mid ng = h, \ f(x+g) < \infty\right\}$ is decreasing. We get

$$
p\big(nf(x + g_1 + \cdots + g_p) - f(x))\big) \leq \sum_{j=1}^{p} n\big(f(x + g_j) - f(x)\big) \quad (22)
$$

By Definition 27, we have

$$
p\big(nf(x + g_1 + \cdots + g_p) - f(x))\big) \geq pf_x(h_1 + \cdots + h_p). \quad (23)
$$

To evaluate the right side of (22), note that for each $1 \leq j \leq p$, the sequence

$$
\left\{n\big(f(x + g_j) - f(x)\big) \mid ng_j = h_j, \ f(x + g_j) < \infty\right\}
$$

is decreasing. Thus, using Proposition 13 and taking the infimum over the right side of (22), we get,

$$
\begin{aligned}
&\inf\left\{\sum_{j=1}^{p} n\big(f(x + g_j) - f(x)\big) \ \middle| \ ng_j = ph_j, \ f(x + g_j) < \infty, \ 1 \leq j \leq p\right\} \\
&= \sum_{j=1}^{p} \inf\left\{n\big(f(x + g_j) - f(x)\big) \ \middle| \ ng_j = ph_j, \ f(x + g_j) < \infty, \ 1 \leq j \leq p\right\} \\
&= \sum_{j=1}^{p} f_x(ph_j). \quad (24)
\end{aligned}
$$

Combining (23) and (24), we get

$$
pf_x(h_1 + \ldots h_p) \leq f_x(ph_1) + \cdots + f_x(ph_p),
$$

and so, since $f_x(ph_j) = pf_x(h_j)$, $1 \leq j \leq p$, we get

$$f_x(h_1 + \ldots h_p) \leq f_x(h_1) + \cdots + f_x(h_p).$$

Note that here we used the fact that $\leq$ is compatible with the group operations on Y, and therefore we have $py_1 \leq py_2 \implies y_1 \leq y_2$. Next, note that since is assumed to be p prime, $p \geq 2$. Choosing $h_3 = \cdots = h_p = 0$, we get

$$
\begin{aligned}
f(h_1 + h_2) &\leq f_x(h_1) + f_x(h_2) + f_x(h_3) + \cdots + f_x(h_p) \\
&\overset{(*)}{\leq} f_x(h_1) + f_x(h_2) + 0 \\
&= f_x(h_1) + f_x(h_2).
\end{aligned}
$$

where in $(*)$ we used the fact that $f_x(0) \leq 0$. Altogether we have that f_x is subadditive and $f(px) = pf(x)$. Now apply Proposition 7 to deduce that f is $\mathbb{N}$-sublinear, and the proof is complete. $\square$

Remark 15 The proof of Proposition 14 shows that the sequence $n\big(f(x+g) - f(x)\big)$, $ng = h$, $f(x+g) < \infty$, is decreasing. If we assume that we have both $pX = X$ and $qX = X$, then in (27), we can choose $n = p^l$ or $n = q^l$ for every $l \in \mathbb{N}$ and the infimum would be the same in both cases. $\diamond$

In the case when f is not only convex, but actually $\mathbb{N}$-sublinear, we have the following stronger result.

Proposition 15 (One-sided derivatives, II) *Assume that X is a group, $(Y, \leq)$ is a group with an inductive order, and $f : X \to Y \cup \{\infty\}$ is $\mathbb{N}$-sublinear map, and $x \in \mathrm{core}(\mathrm{dom}(f))$. Then f_x is an everywhere finite $\mathbb{N}$-sublinear map, that satisfies in addition $f_x(0) = 0$, $f_x(x) = -f_x(-x) = f(x)$.*

Proof When f is $\mathbb{N}$-sublinear, (27) becomes

$$f_x(h) = \inf \big\{ f(nx + h) - nf(x) \; \big| \; f(nx + h) < \infty \big\}.$$

Since f is positively homogeneous, it is easy to see that $f_x(x) = -f_x(-x) = f(x)$ and $f_x(0) = 0$. To show the positive homogeneity of f_x, use the fact that, as in the proof of Proposition 14, the sequence $\big\{ f(nx + h) - nf(x) \big\}$ is decreasing, and so we have for all $m \in \mathbb{N}$,

$$
\begin{aligned}
f_x(mh) &= \inf \big\{ f(nx + mh) - nf(x) \; \big| \; f(nx + mh) < \infty \big\} \\
&= \inf \big\{ f(mkx + mh) - mkf(x) \; \big| \; f(mkx + mh) < \infty \big\} \\
&= m \inf \big\{ f(kx + h) - kf(x) \; \big| \; f(kx + h) < \infty \big\} \\
&= mf_x(h).
\end{aligned}
$$

To show the subadditivity, take $n_1, n_2 \in \mathbb{N}$. Since f is subadditive, we have,

$$
\begin{aligned}
f_x(h_1 + h_2) &\leq f((n_1 + n_2)x + h_1 + h_2) - (n_1 + n_2)f(x) \\
&\leq \big(f(n_1 x + h_1) - n_1 f(x)\big) + \big(f(n_2 x + h_2) - n_2 f(x)\big).
\end{aligned}
$$

Taking the infimum over all $n_1, n_2 \in \mathbb{N}$ such that $f(n_1 x + h_1) < \infty$, $f(n_2 x + h_2) < \infty$, the subadditivity follows. $\square$

Given two monoids X and Y, let $\mathcal{L}(X, Y)$ be the collection of all additive maps between X and Y. As in the vector space setting, define the following:

$$\partial f(x_0) = \left\{ a \in \mathcal{L}(X, Y) \;\middle|\; f(x_0) + a(h) \leq f(x_0 + h) \right\}.$$

In the vector space setting it is usually required that $a(x - x_0) \leq f(x) - f(x_0)$. However, in order to avoid taking differences, we use the above definition. Let $\mathcal{L}(X, Y)$ be the space of all additive maps between X and Y. Then it follows that $\partial f(x_0) \subseteq \mathcal{L}(X, Y)$.

Proposition 16 *Assume that X is a p-semidivisible group, $(Y, \leq)$ is a group with an inductive order, and $f : X \to Y \cup \{\infty\}$ is subadditive and satisfies $f(px) = pf(x)$ for all $x \in X$. If $x \in \mathrm{core}(\mathrm{dom}(f))$, then $f_x \leq f$ and*

$$f_x(x) + f_x(-x) \leq 0.$$

Proof To prove the first assertion, note that

$$
\begin{aligned}
f_x(h) &\overset{(*)}{=} \inf \left\{ n\big(f(x + g) - f(x)\big) \;\middle|\; ng = h, \; f(x + g) < \infty \right\} \\
&= \inf \left\{ p^l\big(f(x + g) - f(x)\big) \;\middle|\; p^l g = h, \; f(x + g) < \infty \right\} \\
&\leq \inf \left\{ p^l f(g) \;\middle|\; p^l g = h, \; f(x + g) < \infty \right\} \\
&\overset{(**)}{=} f(h),
\end{aligned}
$$

where in $(*)$ we used the fact that $\left\{ n\big(f(x+g) - f(x)\big) \;\middle|\; ng = h, \; f(x+g) < \infty \right\}$ is a decreasing sequence and in $(**)$ we used the fact that $f(px) = pf(x)$. To prove the second assertion, choose g such that $pg = x$ and note that

$$
\begin{aligned}
f_x(x) + f_x(-x) &\leq p\big(f(x + g) - f(x)\big) + m\big(f(x - g) - f(x)\big) \\
&= f((p + 1)x) + f((p - 1)x) - 2pf(x) \\
&\leq 0,
\end{aligned}
$$

where in the last inequality we used the subadditivity of f. □

Proposition 17 *If $(Y, \leq)$ satisfies that for every $m \in \mathbb{N}$ $my_1 \leq my_2 \implies y_1 \leq y_2$ then $\partial p(x_0)$ is convex in $\mathcal{L}(X, Y)$.*

Proof For $a_1, \dots, a_n, a \in \mathcal{L}(X, Y)$, assume that $ma = \sum_{i=1}^n m_i a_i$, $m = \sum_{i=1}^n m_i$. Then we have

$$m\big(f(x_0) + a(x)\big) = \sum_{i=1}^n (f(x_0) + a_i(x)) \leq \sum_{i=1}^n m_i f(x) = mf(x).$$

By the assumption on Y, it follows that $f(x_0) + a(x) \leq f(x)$. □

5.2 The maximum or max formula

We show that the well known max formula [6, 9] holds in this generality.

Theorem 3 (Max formula) *Assume that X is a p-semidivisible group, that $(Y, \leq)$ is an additive group with an inductive order, and $f : X \to Y \cup \{\infty\}$ is convex. Assume also that for some $x_0 \in \mathrm{core}(\mathrm{dom}(f))$, we have*

$$f_{x_0}(x_0) + f_{x_0}(-x_0) \leq 0. \tag{25}$$

Then we have

$$f_{x_0}(h) = \max \big\{ a(h) \;\big|\; a \in \partial f(x_0) \big\}. \tag{26}$$

In particular, f admits additive minorants, and $\partial f(x_0) \neq \emptyset$. The maximal element in (26) is bounded.

Proof Define $\mathcal{C}$ to be the set of all pairs (φ, S), where $S \subseteq X$, and $\varphi : X \to Y \cup \{\infty\}$ is $\mathbb{N}$-sublinear and satisfies $\varphi \leq f_{x_0}$, and $\sup_{s \in S} \big(\varphi(s) + \varphi(-s)\big) \leq 0$. Define a partial order on $\mathcal{C}$ by

$$(\varphi_1, S_1) \leq (\varphi_2, S_2) \iff \varphi_1 \geq \varphi_2, \; S_1 \subseteq S_2.$$

$(\mathcal{C}, \leq)$ is inductive, as both $\leq$ and $\subseteq$ are inductive orders. By Proposition 14, we have $f_{x_0}(0) = 0$, implying that $(f_{x_0}, \{0\}) \in \mathcal{C}$ and so $\mathcal{C} \neq \emptyset$. Therefore, $\mathcal{C}$ has a maximal element $(\bar{\varphi}, \bar{S})$. We claim that we must have $\bar{S} = X$. Otherwise, choose $y \in X \setminus \bar{S}$. Since $(\bar{\varphi}, \bar{S}) \in \mathcal{C}$, in particular it follows that the function $\bar{\varphi}$ satisfies the hypotheses of Proposition 16. Also, $y \in \mathrm{core}(\mathrm{dom}(f))$ since $\bar{\varphi} \leq f_{x_0}$ and f_{x_0} is everywhere finite (by Proposition 14). Therefore, Proposition 16 implies that $\bar{\varphi}_y \leq \bar{\varphi}$ and $\bar{\varphi}_y(y) + \bar{\varphi}_y(-y) \leq 0$. This means that $(\bar{\varphi}_y, \bar{S} \cup \{y\}) \in \mathcal{C}$, which is a contradiction to the maximality of $(\bar{\varphi}, \bar{S})$. Thus, we have $\bar{S} = X$. Next, we claim that $\bar{\varphi}$ is additive on X. If not, then since $\bar{\varphi}$ is subadditive, there must exist $x, h \in X$ such that $\bar{\varphi}(x+h) - \bar{\varphi}(h) < \bar{\varphi}(x)$. But then $\bar{\varphi}_x \leq \bar{\varphi}$ which is again a contradiction to the maximality of $(\bar{\varphi}, \bar{S})$. Since $\bar{\varphi} \leq f_{x_0}$ and by (25) we have $\bar{\varphi}(-x_0) \leq f_{x_0}(-x_0) \leq -f_{x_0}(x_0)$, it follows that $\bar{\varphi}(x_0) = f_{x_0}(x_0)$ and $\bar{\varphi}$ is bounded. Choosing $a = \bar{\varphi}$ proves (26). Since $x \in \mathrm{core}(\mathrm{dom}(f))$, Definition 27 implies that the maximal element in (26) is indeed bounded. This completes the proof. $\qquad \square$

An instructive setting is when Y is the symmetric matrices endowed with the (non-lattical) semidefinite order.

Remark 16 (Well posedness) If f is $\mathbb{N}$-sublinear and $ng = x$, then by positive homogeneity, we have $n\big(f(x-g) - f(x)\big) = f((n-1)x) - f(nx) = -f(x)$ and $n\big(f(x+g) - f(x)\big) = f((n+1)x) - f(nx) = f(x)$. In particular, $f_x(x) + f_x(-x) \leq 0$ for every $x \in \mathrm{core}(\mathrm{dom}(f))$. Thus, every $\mathbb{N}$-sublinear function satisfies the assumptions of Theorem 3. $\qquad \diamond$

Remark 17 Using Proposition 15, we have that Theorem 3 holds if f is $\mathbb{N}$-sublinear, even if we omit the subdivisibility assumption. $\qquad \diamond$

5.3 Fenchel-Rockafellar duality

In this section we will consider functions $f : X \to Z \cup \{\infty\}$, where Z is a group. Since the range is always $Z \cup \{\infty\}$, as in vector spaces, it is natural to define the *additive dual group* of a group X to be

$$X^* = \mathcal{L}(X, Z) = \big\{ \varphi : X \to Z \mid \varphi \text{ is additive} \big\}.$$

Then X^* is an additive group with the addition being point-wise addition of elements in Z. We emphasise that X^* is *not* the group of homomorphisms of X. How rich a notion this is depends on the given groups X and Z.

Consider now $(Z, \leq)$ which is order complete. We still require that $\leq$ is compatible with the group operation. Define the *conjugate function* $f^\star : X^* \to Z \cup \{\infty\}$ to be

$$f^\star(\varphi) = \sup_{x \in X} \big\{ \varphi(x) - f(x) \big\}. \tag{27}$$

The conjugate function has been studied extensively in the vector space setting. See for example [6, 9, 30]. Note that $f^*(\varphi) = +\infty$ will happen if (27) has no upper bound. Before proving the *Fenchel duality theorem* for groups, we need the following proposition.

Proposition 18 *Assume that X_1, X_2, Z are groups, where X_1 is semidivisible and $(Z, \leq)$ is an order complete group. Let $T : X_1 \to X_2$ be additive, and assume that $f : X_1 \to Z \cup \{\infty\}$ and $g : X_2 \to Z \cup \{\infty\}$ are convex. If we define $h : X_2 \to Z \cup \{\infty\}$ by*

$$h(u) = \inf_{x \in X_1} \big[f(x) + g(Tx + u) \big],$$

then h is convex, and it domain is given by

$$\mathrm{dom}(h) = \mathrm{dom}(g) - T\mathrm{dom}(g). \tag{28}$$

Proof First, note that since g is convex and T is additive, it follows that $g \circ T : X_1 \to Z \cup \{\infty\}$ is convex. Next, to show the convexity of h, let $m_1, \ldots, m_n \in \mathbb{N}$, $u_1, \ldots, u_n, u \in X_2$ such that $mu = \sum_{i=1}^n m_i u_i$, $m = \sum_{i=1}^n m_i$. Let $x_1, \ldots, x_n \in X_1$. By Proposition 6, we may assume that $m = p^l$, where p is a prime satisfying $pX = X$. Hence, there exists $x \in X_1$ such that $mx = \sum_{i=1}^n m_i x_i$. We have

$$mh(u) \leq m\big(f(x) + g(Tx + u) \big)$$

$$\leq \sum_{i=1}^n m_i \big(f(x_i) + g(Tx_i + u_i) \big).$$

Taking the infimum over $x_1, \ldots, x_n \in X$, we get

$$mh(u) \leq \sum_{i=1}^n m_i h(u_i).$$

The proof of (28) is immediate. This completes the proof. $\square$

Theorem 4 (Fenchel-Young inequality for groups) *Suppose that X, Z, are groups, Z is order complete, and $f : X \to Z \cup \{\infty\}$. Then for every $x \in X$ and every $\varphi \in X^*$,*

$$f(x) + f^\star(\varphi) \geq \varphi(x).$$

Equality holds if and only if $\varphi \in \partial f(x)$.

Proof By definition (27), $\varphi(x) - f(x) \leq f^\star(\varphi)$ which implies $f(x) + f^\star(\varphi) \geq \varphi(x)$. If $\varphi \in \partial f(x)$, then $f(x) + \varphi(y - x) \leq f(y)$ and so $f(x) - \varphi(x) \leq f(y) - \varphi(y)$. Taking the infimum over the right side gives $f(x) - \varphi(x) \leq -f^\star(\varphi)$ which then gives $f(x) + f^\star(\varphi) = \varphi(x)$. Conversely, by the definition of $f^\star$, if $f(x) + f^\star(\varphi) = \varphi(x)$ then $\varphi(y - x) \leq f(y) - f(x)$, and so $\varphi \in \partial f(x)$ as required. □

Example 16 If X is a meet lattice then additive functions are identically 0, since for every $m \in \mathbb{N}$ we have

$$f(x) = f(\overbrace{x \wedge \cdots \wedge x}^{m \text{ times}}) = mf(x).$$

Hence $X^* = \{0\}$ and Theorem 4 simply gives $f(x) \geq \inf_{x \in X} f(x)$. ◇

For an additive map $T : X_1 \to X_2$ define the *adjoint* $T^* : X_2^* \to X_1^*$ in the usual way

$$(T^* x_2^*)(x_1) = x_2^*(Tx_1), \quad x_1 \in X_1, \ x_2^* \in X_2^*.$$

We are now in a position to state and prove the Fenchel duality theorem.

Theorem 5 (Weak and strong Fenchel duality) *Let X_1, X_2, Z, be groups, and $(Z, \leq)$ an order complete group. Given $f : X_1 \to Z \cup \{\infty\}$, $g : X_2 \to Z \cup \{\infty\}$ and an additive map $T : X_1 \to X_2$, define*

$$P = \inf_{x \in X_1} \big\{ f(x) + g(Tx) \big\},$$
$$D = \sup_{\varphi \in X_2^*} \big\{ -f^\star(T^*\varphi) - g^\star(\varphi) \big\}.$$

Then $P \geq D$ (weak duality). In particular, if $P = -\infty$ then $D = -\infty$. If, in addition, X_1 is semidivisible, f and g are convex and we assume

$$0 \in \mathrm{core}\big(\mathrm{dom}(g) - T\,\mathrm{dom}(f)\big),$$

then $P = D$ (strong duality) and D is attained when finite.

Proof To prove weak duality, note that $P \geq D$ is equivalent to

$$\inf_{\substack{x \in X_1 \\ \varphi \in X_2^*}} \Big[f(x) + f^\star(T^*\varphi) + g(Tx) + g^\star(-\varphi) \Big] \geq 0.$$

By Theorem 4, we have $f(x) + f^\star(T^*\varphi) \geq (T^*\varphi)(x)$ and $g(Tx) + g^\star(-\varphi) \geq -\varphi(Tx)$. Then by the definition of T^* we have $(T^*\varphi)(x) - \varphi(Tx) = 0$.

To prove strong duality, define $h : X_2 \to Z \cup \{\infty\}$,

$$h(u) = \inf_{x \in X_1} \big\{ f(x) + g(Tx + u) \big\}.$$

By Proposition 18, h is convex and $\mathrm{dom}(h) = \mathrm{dom}(g) - T\,\mathrm{dom}(f)$ is a convex set. Since we assume that $0 \in \mathrm{core}\big(\mathrm{dom}(g) - T\,\mathrm{dom}(f)\big)$, applying Theorem 3 for h and $x_0 = 0$ implies that there exists $\varphi : X_2 \to Z$ additive such that

$$-\varphi(u) \leq h(u) - h(0).$$

Note that since we choose $x_0 = 0$ in Theorem 3, the condition $h_{x_0}(x_0) + h_{x_0}(-x_0) \leq 0$ holds, as $h_x(0) = 0$ always. Note also that Theorem 3 says that the maximal element is bounded and therefore we can choose φ whose range is Z. Altogether, we get

$$\begin{aligned} h(0) \leq h(u) + \varphi(u) &\leq f(x) + g(Tx + u) + \varphi(u) \\ &= \big[f(x) - (T^*\varphi)(x) \big] + \big[g(Tx + u) - (-\varphi(Tx + u)) \big]. \end{aligned}$$

Taking the infimum over $x \in X_1$, $u \in X_2$ implies

$$h(0) \leq -f^\star(T^*\varphi) - g^\star(-\varphi) \leq D.$$

Since $h(0) = P$, strong duality follows. Again the dual supremum is attained when finite. $\square$

Example 17 If X_2 is a meet lattice, then $X_2^* = \{0\}$ and

$$D = -f^\star(0) - g^\star(0) = \inf_{x \in X_1} f(x) + \inf_{x \in X_2} g(x)$$

which is clearly smaller than P. $\diamond$

Remark 18 Assume that in Theorem 5 we have $\mathbb{N}$-sublinear functions rather than convex functions. Then if we use Proposition 15, Theorem 5 still holds even if we omit the subdivisibility assumption. $\diamond$

Next we discuss applications of Theorem 5. One of the classical applications, is a representation for the subdifferential of a sum of convex functions. We show that such a result holds for groups as well.

Theorem 6 (Sum rule for subdifferentials) *Suppose* $f : X_1 \to Z \cup \{\infty\}$, $g : X_2 \to Z \cup \{\infty\}$, *for* $(Z, \leq)$ *an order complete group and* $T : X_1 \to X_2$ *is additive. Then*

$$\partial\big(f + g \circ T\big)(x_0) \supseteq \partial f(x_0) + T^* \partial g(x_0).$$

If, in addition, X_1 is semidivisible, $0 \in \mathrm{core}\big(\mathrm{dom}(g) - T\,\mathrm{dom}(f)\big)$, while f and g are convex, then equality holds.

Proof The first inclusion follows immediately. To prove the equality case, let $\phi \in \partial\big(f + g \circ T\big)(x_0)$. Then the function $(f - \phi) + g \circ T$ is minimised at x_0. Assume without loss of generality that the minimum is 0. By the strong Fenchel duality result with $P = D = 0$, there exists $\varphi \in X_2^*$ such that

$$0 = -(f - \phi)^\star(T^*\varphi) - g^\star(-\varphi) = -f^\star(T^*\varphi + \phi) - g^\star(-\varphi).$$

Hence, for every $x_1 \in X_1$ and $x_2 \in X_2$, we have

$$0 \le (f - \phi)(x_1) - T^*\varphi(x_1) + g(x_2) + \varphi(x_2). \tag{29}$$

In particular, choosing $x_1 = x_0$, we have for all $x_2 \in X_2$,

$$-\varphi(x - Tx_0) \le (f - \phi)(x_0) + g(x_2) = g(x_2) - g(Tx_0),$$

where in the last equality we used our assumption that $(f - \phi)(x_0) + g(Tx_0) = 0$. Thus, we have $-\varphi \in \partial g(Tx_0)$. Also, by (29), we have

$$\sup_{x_1 \in X_1} \big(- g(Tx_1) - T^*\varphi(x_1)\big) \le \inf_{x_1 \in X_1} \big((f - \phi)(x_1) - T^*\varphi(x_1)\big).$$

Thus there exists $z_0 \in Z$ such that for all $x_1 \in X_1$,

$$-g(Tx_1) \le (T^*\varphi)(x_1) + z_0 \le (f - \phi)(x_1)$$

and equality holds when $x_1 = x_0$. Hence $z_0 = 0$ and $T^*\varphi + \phi \in \partial f(x_0)$, which completes the proof. $\qquad\square$

Another application of Theorem 5 is a Hahn-Banach theorem for groups.

Theorem 7 (Hahn-Banach theorem for groups) *Let X be a group, $X' \subseteq X$ a subgroup, and $(Z, \le)$ an order complete group. Assume that $f : X \to Z$ is $\mathbb{N}$-sublinear and $h : X' \to Z$ is additive such that $h \le f$ on X'. Then there exists $\bar{h} : X \to Z$ additive such that $\bar{h} \le f$ and $\bar{h} = h$ on X'.*

Proof Choose $X_1 = X_2 = X$ and let $T : X \to X$ be the identity map. Choose $g : X' \to Z \cup \{\infty\}$ to be $g = -h + \iota_{X'}$, where

$$\iota_{X'}(x) = \begin{cases} 0 & x \in X', \\ \infty & x \notin X'. \end{cases}$$

Since $f : X \to Z$, $\operatorname{dom}(f) = X$. Also, $\operatorname{dom}(g) = X'$. Thus $0 \in \operatorname{core}(\operatorname{dom}(f) - T\operatorname{dom}(g))$ and we can thus use Theorem 5. Note that by Remark 18 we do not need to assume subdivisibility as we are dealing with $\mathbb{N}$-sublinear functions. Now, by Theorem 5, we have

$$\begin{aligned}
0 &\le \inf_{x \in X} \big\{ f(x) - h(x) + \iota_{X'}(x) \big\} \\
&= \inf_{x \in X} \big\{ f(x) + g(x) \big\} \\
&= \sup_{\varphi \in X^*} \big\{ - f^\star(\varphi) - g^\star(-\varphi) \big\}. \tag{30}
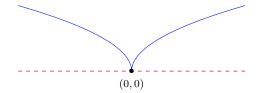\end{aligned}$$

Fig. 4 Single minorant in $\mathbb{R}$

Thus, there exists $\varphi \in X^*$ such that for all $x \in X'$, $f^\star(\varphi) \leq \varphi(x) - h(x)$. Since f is sublinear, $f(0) = 0$ and so it follows that $f^\star(\varphi) \geq 0$ or in other words $h(x) \leq \varphi(x)$, $x \in X'$. Since X' is a subgroup and φ is additive, we have $h(x) = \varphi(x)$ on X' and $g^\star(-\varphi) = 0$. Now (30) implies that $f^\star(\varphi) = 0$, which implies that $\varphi(x) \leq f(x)$ for all $x \in X$. □

Remark 19 If X and Z are groups and $f, g : X \to Z \cup \{\infty\}$ are additive with $g \leq f$, then $f = g$. However, if X is only a semigroup, this is no longer always true. As a result, we cannot expect strong Hahn-Banach type theorems on arbitrary semigroups. ◇

Theorem 8 (Sandwich theorem for groups) *Assume that X_1 is a semidivisible group, X_2 a group, and $(Z \cup \{\infty\}, \leq)$ a group with complete order. Let $f : X_1 \to Z \cup \{\infty\}$, $-g : X_2 \to Z \cup \{\infty\}$ be convex and $T : X_1 \to X_2$ be additive, such that $g \circ T \leq f$. Assume that $0 \in \text{core}(\text{dom}(g) - T \, \text{dom}(f))$. Then there exists an additive function $a : X \to Z$ such that $g \circ T \leq a \leq f$.*

Proof Using Theorem 5, we have $P \leq 0$ and so there exists $\varphi \in X_2^*$ such that $(-g)^\star(-\varphi) \leq -f^\star(T^*\varphi)$. This implies that

$$\sup_{x \in X_1} \big(-g(Tx) - T^*\varphi(x) \big) \leq \inf_{x \in X_1} \big(f(x) - T^*\varphi(x) \big). \tag{31}$$

$T^*\varphi$ is the required additive function. If $P = -\infty$ in Theorem 5, then $P < -\alpha < 0$ for every $\alpha > 0$ and so inequality (31) still holds. □

Remark 20 By Proposition 15, Theorem 8 holds if we replace convex functions by $\mathbb{N}$-sublinear, even if we omit the subdivisibility assumption. ◇

Remark 21 Even for $X = \mathbb{R}$, the only additive minorant may be $a = 0$. Consider the subadditive (non-convex) function $f(x) = \sqrt{|x|}$. See Figure 4. ◇

6 Subadditive optimisation

Let $f, g_1, \ldots, g_k : X \to [-\infty, \infty]$ and $b \in \mathbb{R}$. Define $v : \mathbb{R}^k \to [-\infty, \infty]$ by

$$v(b) = v(b_1, \ldots, b_k) = \inf \big\{ f(x) \mid x \in X, \ g_1(x) \leq b_1, \ldots, g_k(x) \leq b_k \big\}. \tag{32}$$

v is also known as the *value function*. We have the following.

Proposition 19 (Subadditive and sublinear value functions) *Assume that X is a monoid and $f, g_1, \ldots, g_k : X \to [-\infty, \infty]$ are subadditive. Then the function $v : \mathbb{R}^k \to [-\infty, \infty]$ defined by (32) is subadditive. If, in addition, X is assumed to be p-semidivisible and $f, g_1, \ldots, g_k$ satisfy $f(px) = pf(x)$, $g_i(px) = pg_i(x)$, $1 \le i \le k$ then v satisfies $v(px) = pv(x)$. In particular, by Proposition 7, v is convex.*

Proof Let $x_1, x_2 \in X$ be such that $g_i(x_1) \le b_i$, $g(x_2) \le c_i$, $1 \le i \le k$. Since $g_1, \ldots, g_k$ are subadditive, $g_i(x_1 + x_2) \le g_i(x_1) + g_i(x_2) \le b_i + c_i$, $1 \le i \le k$. Thus, of $b = (b_1, \ldots, b_k)$, $c = (c_1, \ldots, c_k)$, then

$$v(b + c) \le f(x_1 + x_2) \le f(x_1) + f(x_2),$$

where we used the subadditivity of f. Taking the infimum over the right side, the first assertion follows. To prove the second assertion, we only need to prove positive homogeneity. Indeed, for every $x \in X$, since X is p-semidivisible, there exists $y \in X$ satisfying $x = py$. As a result,

$$
\begin{aligned}
v(pb) &= \inf \big\{ f(x) \ \big| \ x \in X, \ g_1, (x) \le pb_1, \ldots, g_k(x) \le pb_k \big\} \\
&= \inf \big\{ f(py) \ \big| \ y \in X, \ g_1(py) \le pb_1, \ldots, g_k(py) \le pb_k \big\} \\
&= \inf \big\{ pf(y) \ \big| \ y \in X, \ pg_1(y) \le pb_1, \ldots, pg_k(y) \le pb_k \big\} \\
&= p \inf \big\{ f(y) \ \big| \ y \in X, \ g_1(y) \le b_1, \ldots, g_k(y) \le b_k \big\} \\
&= pv(b),
\end{aligned}
$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 22 The result holds if the module is over a semidivisible semiring R and f and g are subadditive functions. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\diamond$

In the sublinear case, we may now apply Theorem 3 to the function h of Proposition 19 to describe h in terms of additive minorants.

Example 18 Let $b \in \mathbb{R}$, and let

$$\inf \big\{ -x \ \big| \ 2x \le b, \ x \in \mathbb{Z} \big\} = - \left\lceil \frac{b}{2} \right\rceil.$$

Thus, in the nondivisible setting, even if $k = 1$ and f and g_1 are additive, v need not be homogeneous. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\diamond$

In general integer programming [1, 34] adding the sub additive, but not $\mathbb{N}$-homogeneous, *ceiling* function $\lceil \cdot \rceil$ allows one to reconstruct integer value functions but the additive minorants do not suffice. This is discussed in [2, 32]. It is interesting to ask what class of groups allows an analogue of the ceiling?

We note also that methods that were originally developed to study linear programming results in vector spaces, such as the cutting-plane method [20], can also be used to study *integer* linear programming problems. See also [1, 23] and the survey [10] for more information on the cutting-planes method, and [2, 14, 15, 23] for more information on integer programming.

6.1 Lagrange multipliers in action

Suppose now that we have an optimisation problem with m constraints:

$$\inf \big\{ f(x) \mid g_1(x) \le 0, \ldots, g_k(x) \le 0 \big\}.$$

Let $g(x) = (g_1(x), \ldots, g_k(x)) \in \mathbb{R}^m$. Define the *Lagrangian function* $L : X \times \mathbb{R}^k \to (-\infty, \infty]$ to be

$$L(x, \lambda) = f(x) + \lambda \cdot g(x).$$

Here, $\lambda \cdot g(x)$ is the standard inner product in $\mathbb{R}^k$. We say that $\bar{\lambda} \in \mathbb{R}^k$ is a *Lagrange multiplier* if the Lagrangian function $L(\,\cdot\,, \bar{\lambda})$ has the same infimum as f on X. We will now show that Lagrange multipliers can be used to compute the subdifferential of the maximum of convex function. In the vector space case, this fact has several different proofs. We chose this particular version to show the use of Lagrange multipliers in the group setting.

Theorem 9 *Let X be a semidivisible group and $f_i : X \to (-\infty, \infty]$ be convex functions, where $i \in I$, I being a finite index set. Let $f = \max_{1 \le i \le k} f_i$. For $x_0 \in \bigcap_{i \in I(x_0)} \mathrm{core}(\mathrm{dom}(f_i))$, where $I(x_0) = \{1 \le i \le k \mid f_i(x_0) = f(x_0)\}$. Then we have*

$$\partial f(x_0) = \mathrm{conv}\left(\bigcup_{i \in I(x_0)} \partial f_i(x_0) \right).$$

Proof The inclusion $\supseteq$ follows immediately from the fact the subdifferential is convex (Proposition 17 with $Y = \mathbb{R}$). To prove the other inclusion, consider the constrained minimisation problem

$$\inf \big\{ t \mid t \in \mathbb{R}, \ x \in X, \ f_1(x) \le t, \ldots, f_k(x) \le t \big\}. \tag{33}$$

Note that this infimum equals $\inf_{x \in X} f(x)$. Assume first that $0 \in \partial f(x_0)$, which means that the infimum in (33) is attained at x_0. Define the following auxiliary value function $v : \mathbb{R}^{I(x_0)} \to [-\infty, \infty]$,

$$v(b) = \inf \big\{ t \mid f_i(x) - t \le b_i, \ \ i \in I(x_0) \big\}.$$

We have $v(b) \ge f(x_0) - \max_{i \in I(x_0)} |b_i| > -\infty$. Also, since we assumed that $x_0 \in \bigcap_{i \in I(x_0)} \mathrm{core}(\mathrm{dom}(f_i))$, it follows that $0 \in \mathrm{core}(\mathrm{dom}(v))$. By Proposition 19, v is convex. Thus, by Theorem 3, there exists $\bar{\lambda} \in \partial v(0)$ (again we are allowed to use the max formula because we are at $x_0 = 0$). We note also that if $b \in \mathbb{R}_+^{I(x_0)}$ then we also have $v(b) \le f(x_0)$ (infimum over a larger set) and also $v(0) = f(x_0)$. Thus, we have

$$f(x_0) = v(0) \le v(b) + \bar{\lambda} \cdot b \le f(x_0) + \bar{\lambda} \cdot b,$$

which means that $\bar{\lambda} \in \mathbb{R}_{+}^{I(x_0)}$. Hence,

$$
\begin{aligned}
t &\geq v((f_i(x) - t)_{i \in I(x_0)}) \\
&\geq v(0) - \bar{\lambda} \cdot (f_i(x) - t)_{i \in I(x_0)} \\
&= f(x_0) - \bar{\lambda} \cdot (f_i(x) - t)_{i \in I(x_0)},
\end{aligned}
$$

and so

$$
t + \bar{\lambda} \cdot (f_i(x) - t)_{i \in I(x_0)} \geq f(x_0),
$$

which means that $\bar{\lambda}$ is a minimiser for the Lagrangian function. In other words, we can find $\bar{\lambda} \in \mathbb{R}^{I(x_0)}$ that minimises

$$
t + \sum_{i \in I(x_0)} \lambda_i (f_i(x) - t) = t \left(1 - \sum_{i \in I(x_0)} \lambda_i \right) + \sum_{i \in I(x_0)} \lambda_i f_i(x). \qquad (34)
$$

We must have $\sum_{i \in I(x_0)} \bar{\lambda}_i = 1$. If not, then we can choose t that would make (34) go to $-\infty$. Thus, we have

$$
\sum_{i \in I(x_0)} \bar{\lambda}_i f_i(x_0) \leq \sum_{i \in I(x_0)} \bar{\lambda}_i f_i(x),
$$

and so $0 \in \partial \left(\sum_{i \in I(x_0)} \bar{\lambda}_i f_i \right) (x_0)$. If, in general, we have that $\phi \in \partial f(x_0)$, then $0 \in \partial (f - \phi)(x_0)$ and then we repeat the same argument to conclude that $\phi \in \partial \left(\sum_{i \in I(x_0)} \bar{\lambda}_i f_i \right) (x_0)$. Altogether, we get

$$
\partial f(x_0) \subseteq \bigcup \left\{ \partial \Big(\sum_{i \in I(x_0)} \lambda_i f_i \Big)(x_0) \, \middle| \, \lambda_i \geq 0, \sum_{i \in I(x_0)} \lambda_i = 1 \right\}.
$$

Now, Theorem 6 implies that the right side is equal to

$$
\operatorname{conv} \left(\bigcup_{i \in I(x_0)} \partial f_i(x_0) \right),
$$

and so we have

$$
\partial f(x_0) \subseteq \operatorname{conv} \left(\bigcup_{i \in I(x_0)} \partial f_i(x_0) \right),
$$

which proves the other inclusion and concludes the proof. $\qquad \square$

Remark 23 Combining Theorem 9 with Proposition 5 allows us to consider subadditive optimisation problems with finitely many constraints.

7 Conclusion

This paper grew out of a lecture that the first author gave in 1983 and then put aside until 2015 when the second author joined him in recreating and extending the original results. One original intention was to better understand the difficulty of integer programming as that of programming over a non-divisible group. See also [3, 12]. In so doing we have uncovered many interesting connections but as of now made little progress directly for integer programming.

Surely there are many other classical results for which one can find elegant and even useful generalisations. Hopefully this paper will serve as an invitation to others to join the pursuit.

References

[1] D. S. Atkinson and P. M. Vaidya, *A cutting plane algorithm for convex programming that uses analytic centers*, Math. Programming **69** (1995), no. 1, Ser. B, 1–43. Nondifferentiable and large-scale optimization (Geneva, 1992).

[2] C. E. Blair and R. G. Jeroslow, *The value function of an integer program*, Math. Programming **23** (1982), no. 3, 237–273.

[3] N. L. Boland, A. C. Eberhard, F. G. Engineer, M. Fischetti, M. W. P. Savelsbergh, and A. Tsoukalas, *Boosting the feasibility pump*, Math. Program. Comput. **6** (2014), no. 3, 255–279.

[4] J. M. Borwein, *Continuity and differentiability properties of convex operators*, Proc. London Math. Soc. (3) **44** (1982), no. 3, 420–444.

[5] J. M. Borwein and O. Giladi, *Some remarks on convex analysis in topological groups*, to appear in J. Convex. Anal. (2015). Preprint available at `https://www.carma.newcastle.edu.au/jon/LCGroups.pdf`.

[6] J. M. Borwein and A. S. Lewis, *Convex analysis and nonlinear optimization*, 2nd ed., CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 3, Springer, New York, 2006. Theory and examples.

[7] J. M. Borwein, J. P. Penot, and M. Théra, *Conjugate convex operators*, J. Math. Anal. Appl. **102** (1984), no. 2, 399–414.

[8] J. M. Borwein and M. Théra, *Sandwich theorems for semicontinuous operators*, Canad. Math. Bull. **35** (1992), no. 4, 463–474 (English, with English and French summaries).

[9] J. M. Borwein and J. D. Vanderwerff, *Convex functions: constructions, characterizations and counterexamples*, Encyclopedia of Mathematics and its Applications, vol. 109, Cambridge University Press, Cambridge, 2010.

[10] S. Boyd and L. Vandenberghe, *Localization and Cutting-plane Methods*. Available at `http://web.stanford.edu/class/ee392o/localization-methods.pdf`.

[11] R. E. Burkard, H. Hamacher, and J. Tind, *On abstract duality in mathematical programming*, Z. Oper. Res. Ser. A-B **26** (1982), no. 7, A197–A209 (English, with German summary).

[12] M. Fischetti, F. Glover, and A. Lodi, *The feasibility pump*, Math. Program. **104** (2005), no. 1, Ser. A, 91–104.

[13] L. Fuchs, *Infinite abelian groups. Vol. I*, Pure and Applied Mathematics, Vol. 36, Academic Press, New York-London, 1970.

[14] R. E. Gomory, *Outline of an algorithm for integer solutions to linear programs*, Bull. Amer. Math. Soc. **64** (1958), 275–278.

[15] R. E. Gomory and W. J. Baumol, *Integer programming and pricing*, Econometrica **28** (1960), 521–550. MR0115822 (22 #6620)

[16] A. Hamel, *Variational Principles on Metric and Uniform Spaces*, 2005. Habilitation Thesis. Available at `http://sundoc.bibliothek.uni-halle.de/habil-online/05/05H167/index.htm`.

[17] R. B. Holmes, *Geometric functional analysis and its applications*, Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 24.

[18] P. Juutinen, G. Lu, J. J. Manfredi, and B. Stroffolini, *Convex functions on Carnot groups*, Rev. Mat. Iberoam. **23** (2007), no. 1, 191–200.

[19] R. Kaufman, *Interpolation of additive functionals*, Studia Math. **27** (1966), 269–272.

[20] J. E. Kelley Jr., *The cutting-plane method for solving convex programs*, J. Soc. Indust. Appl. Math. **8** (1960), 703–712.

[21] Y. Kemprasit, N. Triphop, and A. Wasanawichit, *Some divisible matrix groups*, Ital. J. Pure Appl. Math. **28** (2011), 225–230.

[22] J. Lawson, *Clifford algebras, Möbius transformations, Vahlen matrices, and B-loops*, Comment. Math. Univ. Carolin. **51** (2010), no. 2, 319–331.

[23] A. N. Letchford and A. Lodi, *Primal cutting plane algorithms revisited*, Math. Methods Oper. Res. **56** (2002), no. 1, 67–81. Special issue on combinatorial and integer programming.

[24] G. Lu, J. J. Manfredi, and B. Stroffolini, *Convex functions on the Heisenberg group*, Calc. Var. Partial Differential Equations **19** (2004), no. 1, 1–22.

[25] J. Matoušek, *Lectures on discrete geometry*, Graduate Texts in Mathematics, vol. 212, Springer-Verlag, New York, 2002.

[26] S. Mazur and W. Orlicz, *Sur les espaces métriques linéaires. II*, Studia Math. **13** (1953), 137–179 (French).

[27] K. Murota, *Discrete convex analysis*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.

[28] P. Poncet, *Convexities on ordered structures have their Krein-Milman theorem*, J. Convex Anal. **21** (2014), no. 1, 89–120.

[29] D. J. S. Robinson, *A course in the theory of groups*, 2nd ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996.

[30] R. T. Rockafellar, *Convex analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original; Princeton Paperbacks.

[31] C. Rosendal, *Automatic continuity of group homomorphisms*, Bull. Symbolic Logic **15** (2009), no. 2, 184–214.

[32] J. Tind and L. A. Wolsey, *An elementary survey of general duality theory in mathematical programming*, Math. Programming **21** (1981), no. 3, 241–261.

[33] M. L. J. van de Vel, *Theory of convex structures*, North-Holland Mathematical Library, vol. 50, North-Holland Publishing Co., Amsterdam, 1993.

[34] H. P. Williams, *Integer programming and pricing revisited*, IMA J. Math. Appl. Bus. Indust. **8** (1997), no. 3, 203–213. Duality in practice.

[35] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.