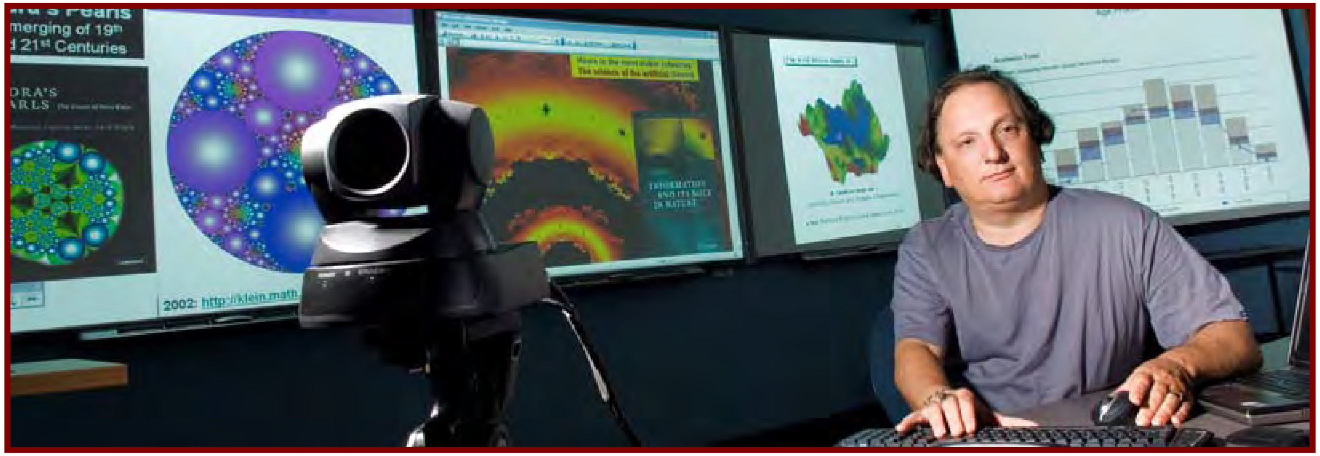


Newcastle
AMSI-AG Room

Some of my Favourite Convexity Results

Jon Borwein, FRSC www.cs.dal.ca/~jborwein

Canada Research Chair, Dalhousie

Laureate Professor, Newcastle

"Harald Bohr is reported to have remarked "Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove." (D.J.H. Garling)

Review of Michael Steele's *The Cauchy Schwarz Master Class in* the MAA Monthly, June-July 2005, 575-579.

Harald Bohr
1887-1951

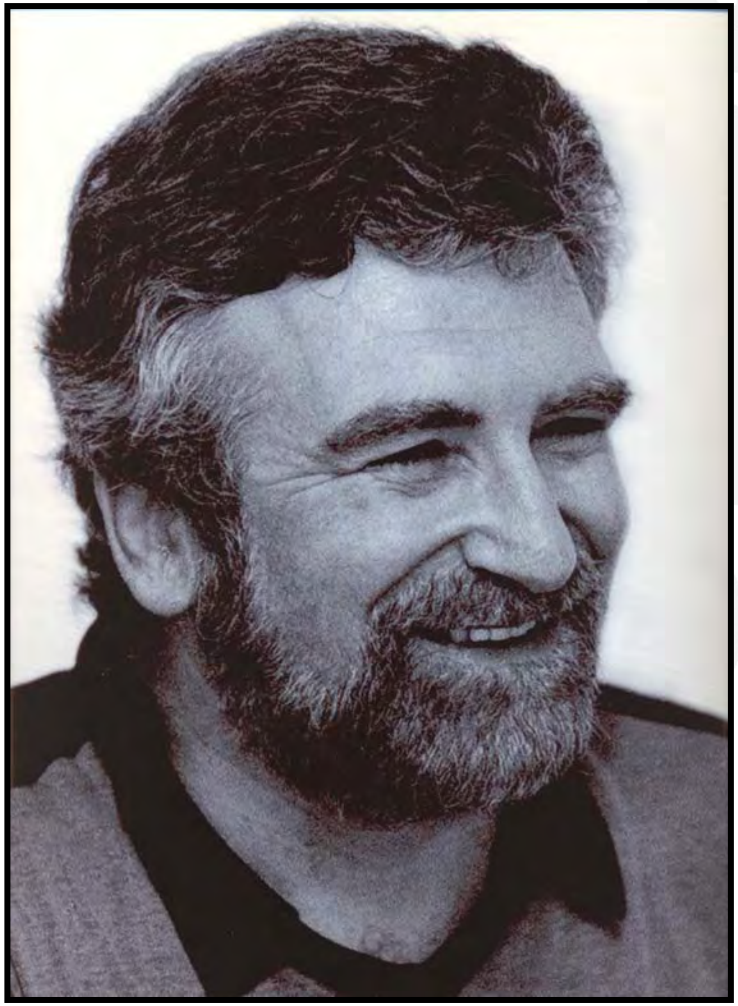
28/08/2008

The Bohrs

- One Nobel Prize
  - Nils (1885-1962)
  - Physics (1922)
- One Olympic Medal
  - Harald (1887-1951)
  - Soccer (1908)

Abstract of Convexity Talk I

In honour of my friend Boris Mordhukovich



We met in 1990. He said

"How old are you?"

I said *"39 and you?"*

He replied "48."

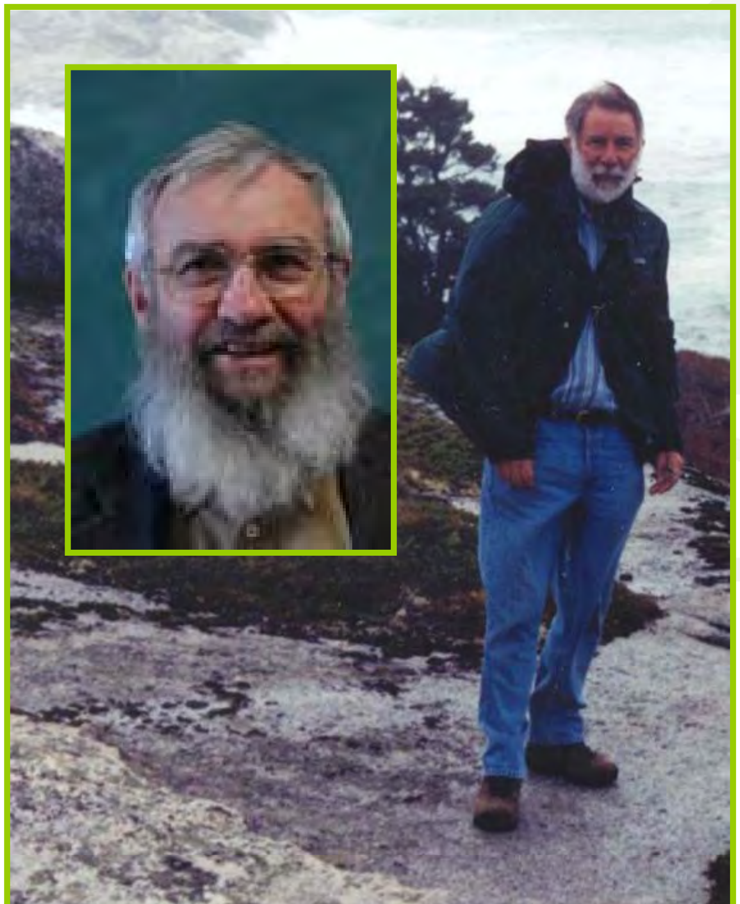
I left thinking he was 48 and he thinking I was 51.

Some years later Terry Rockafellar corrected our cultural misconnect.

What was it?

Abstract of Convexity Talk II

This is a revised version of a talk given in March and in June 2007 to celebrate Tony Thompson's 70^{th} birthday and Jon Thompson's 65^{th}



The three have a lot in common

- Substantial white beards
- Great energy and commitment
- Many contributions to the community
- I like them each enormously
- They are all older than me

.

G’day from Newcastle, Oz

Please pass on my best wishes to Boris. I have fond memories of his generous help as a reviewer to my first paper in JOTA many years ago. He was a great friend to Alex Rubinov and is of course a fine mathematician. Pass on my birthday greetings! Barney Glover

Abstract of Convexity Talk

I offer various examples of convexity appearing (often unexpectedly) over the years in my research.

Each example illustrates either the power of convexity, or of modern symbolic computation, or of both …

PUB: $$f_{\mathcal{A}}(x) := \sup_{A \in \mathcal{A}} \|A(x)\|$$

Proof. lsc and p.w. bounded is finite hence continuous and so the linear operators are uniformly bounded.

I start with a brief advert for computer-assisted mathematics and collaborative tools.

Drive

Dalhousie Distributed Research Institute and Virtual Environment

D-Drive's Nova Scotia location lends us unusual freedom when interacting globally. Many cities around the world are close enough in a chronological sense to comfortably accommodate real-time collaboration.

C2C Seminar: Example from SFU

- running biweekly since 2005, Ontario joined on 25/09/07. Chile this year

Local Presentation Remote Presentation

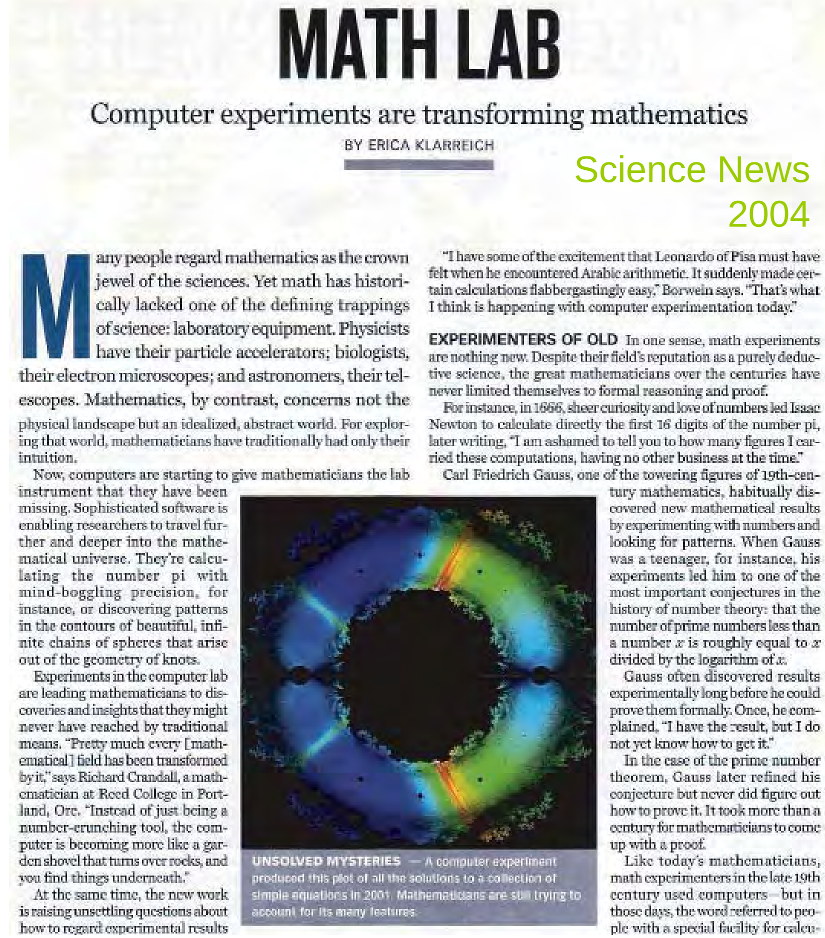
Speaker Remote Audience

Presentation Slides Local Camera Placement

Solving Checkers: one of top 10 Science breakthroughs of 2007

Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results



MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results

UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

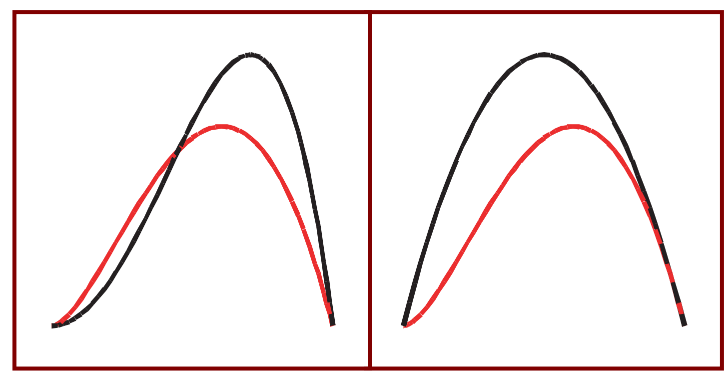
Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-

Science News 2004



Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

Mathematics by Experiment

SECOND EDITION

Jonathan Borwein
David Bailey

"[E]xperimental mathematics is here to stay. The reader who wants to get an introduction to this exciting approach to doing mathematics can do no better than [this book]."

—*Notices of the American Mathematical Society*

"Let me cut to the chase: every mathematics library requires a copy of this book. . . . Every supervisor of higher degree students requires a copy on their shelf. Welcome to the rich world of computer-supported mathematics!"

—*Mathematical Reviews*

Mathematicians have always used experiments and visualization to explore new ideas and ways to prove them.

Using examples that truly represent the experimental methodology, this book provides the historical context of, and rationale behind, experimental mathematics. It shows how today, the use advanced computing technology provides mathematicians with an amazing, previously unimaginable "laboratory," in which examples can be analyzed, new ideas tested, and patterns discovered.

This is a perfect introduction to the history and current state of research and technology in the growing field of experimental mathematics. In fact, it is an indication of that growth that this second edition contains over one hundred pages describing new research since the publication of the first edition.

For those interested in further examples and insights, the book *Experimentation in Mathematics: Computational Paths to Discovery* is highly recommended.

ISBN 1-56881-211-6
9 781568 812113

A K Peters, Ltd.

BORWEIN
BAILEY

Mathematics by Experiment

A K Peters

SECOND EDITION

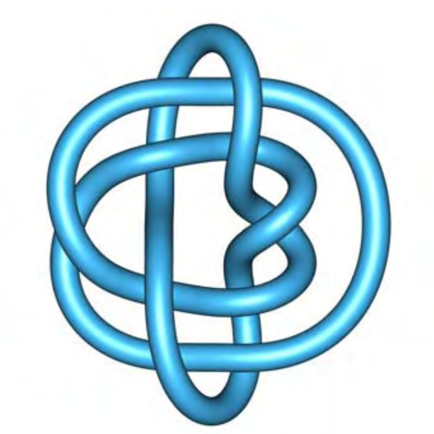
Mathematics by Experiment

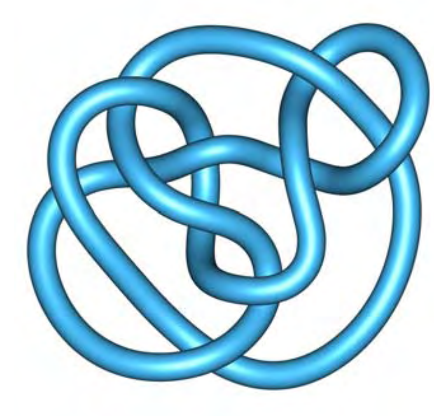
Plausible Reasoning in the 21st Century

Jonathan Borwein
David Bailey

The Perko Pair 10_{161} and 10_{162}

are two adjacent 10-crossing knots (1900)





- first shown to be the same by Ken Perko in 1974
- and beautifully made dynamic in KnotPlot (open source)

Outline of Convexity Talk

A. Generalized Convexity of Volumes (Bohr-Mollerup).

B. Coupon Collecting and Convexity.

C. Convexity of Spectral Functions.

D. Madelung's Constant for Salt.

The talk ends when I do

There are three bonus tracks!

Full details are in the four reference texts

Generalized Convexity of Volumes

A. Generalized Convexity of Gamma (Bohr-Mollerup).

Γ is usually defined for $\mathrm{Re}(x) > 0$ as

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\, dt. \qquad (1)$$

Theorem 1 (Bohr-Mollerup) Γ *is the unique function* $f : (0,\infty) \to (0,\infty)$ *such that:*
(a) $f(1) = 1$*; (b)* $f(x+1) = x f(x)$*;*
(c) f is log-convex ($x \to \log(f(x))$ is convex*).*

- Application is often *automatable* in a computer algebra system, as we now illustrate:

Generalized Convexity of Volumes

A. Generalized Convexity of Gamma (Beta function).

The β–*function* is defined by

$$\beta(x,y) \;:=\; \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad (1)$$

for $\mathrm{Re}(x), \mathrm{Re}(y) > 0$. As is often established using polar coordinates and double integrals

$$\beta(x,y) \;=\; \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \qquad (2)$$

Proof Use $f := x \to \beta(x,y)\,\Gamma(x+y)/\Gamma(y)$.
(a) and (b) are easy. For (c) we show f is log-convex via Hölder's inequality. Thus $f = \Gamma$ as required. **QED**

- Γ is *hyper-transcendental* as is ζ.

Generalized Convexity of Volumes

A. Convexity of Volumes (Blaschke-Santalo inequality).

For a convex body C in R^n its *polar* is

$$C^\circ \;:=\; \{y \in R^n : \langle y, x\rangle \leq 1 \text{ for all } x \in C\}.$$

Denoting n-dimensional Euclidean volume of $S \subseteq R^n$ by $V_n(S)$, **Blaschke-Santalo** says

$$V_n(C)\,V_n(C^\circ) \;\leq\; V_n(E)\,V_n(E^\circ) \;=\; V_n^2(B_n(2)) \tag{1}$$

where maximality holds (only) for *any* ellipsoid E and $B_n(2)$ is the Euclidean unit ball.

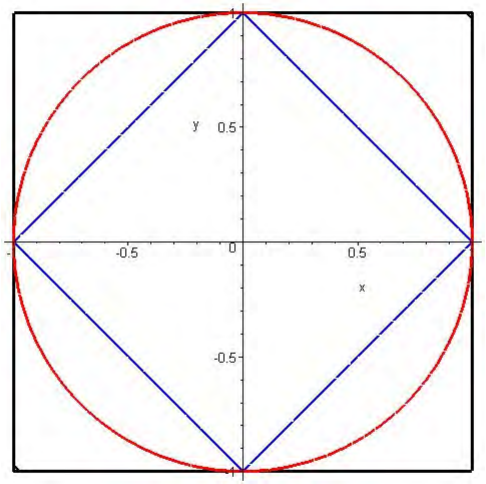
Question How to explain cases of this as convexity estimates?

Generalized Convexity of Volumes

A. Convexity of Volumes (Dirichlet Formulae).

The volume of the ball in the $\|\cdot\|_p$-norm, $V_n(p)$, was first determined by Dirichlet

$$V_n(p) = 2^n \frac{\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}.$$

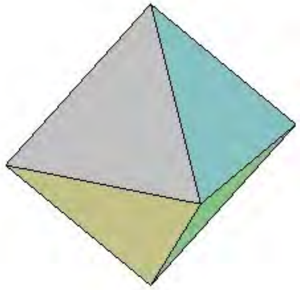


1, 2 and ∞-balls in R^2

When $p=2$,

$$V_n = 2^n \frac{\Gamma(\frac{3}{2})^n}{\Gamma(1+\frac{n}{2})} = \frac{\Gamma(\frac{1}{2})^n}{\Gamma(1+\frac{n}{2})},$$

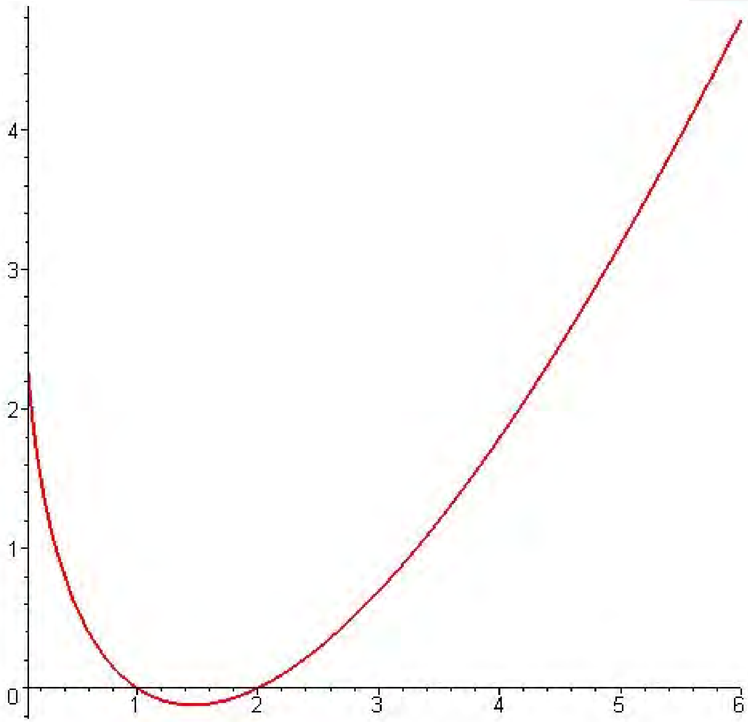
is more concise than that usually recorded. *Maple* code derives this formula as an iterated integral for arbitrary p and fixed n.



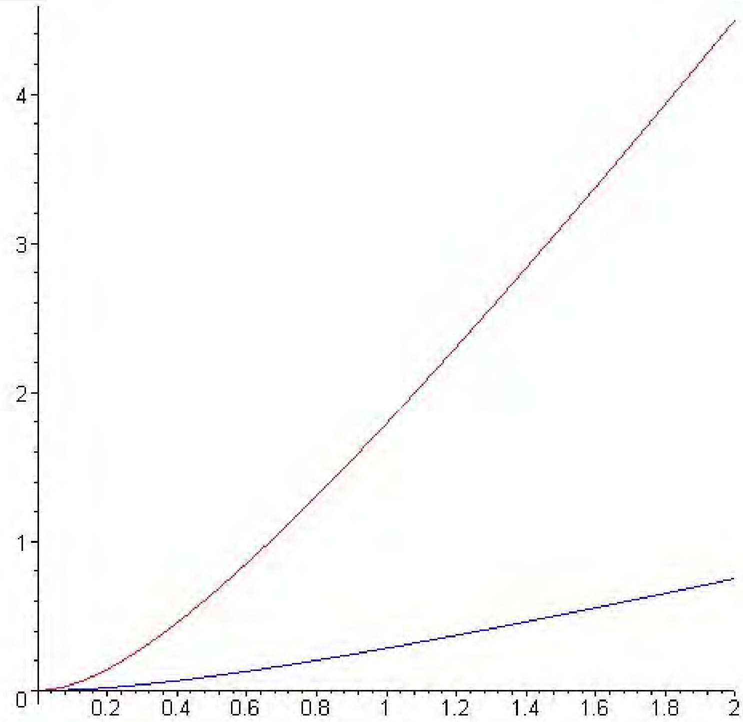
1-ball in R^3

Generalized Convexity of Volumes

A. Convexity of Volumes (Ease of Drawing Pictures).



$\log \Gamma(x)$



$\log V_a(1/x)$ for $a = 4/3, 3$

Discover the formula for $\sum_{n\geq 1} V_n(2)$

Generalized Convexity of Volumes

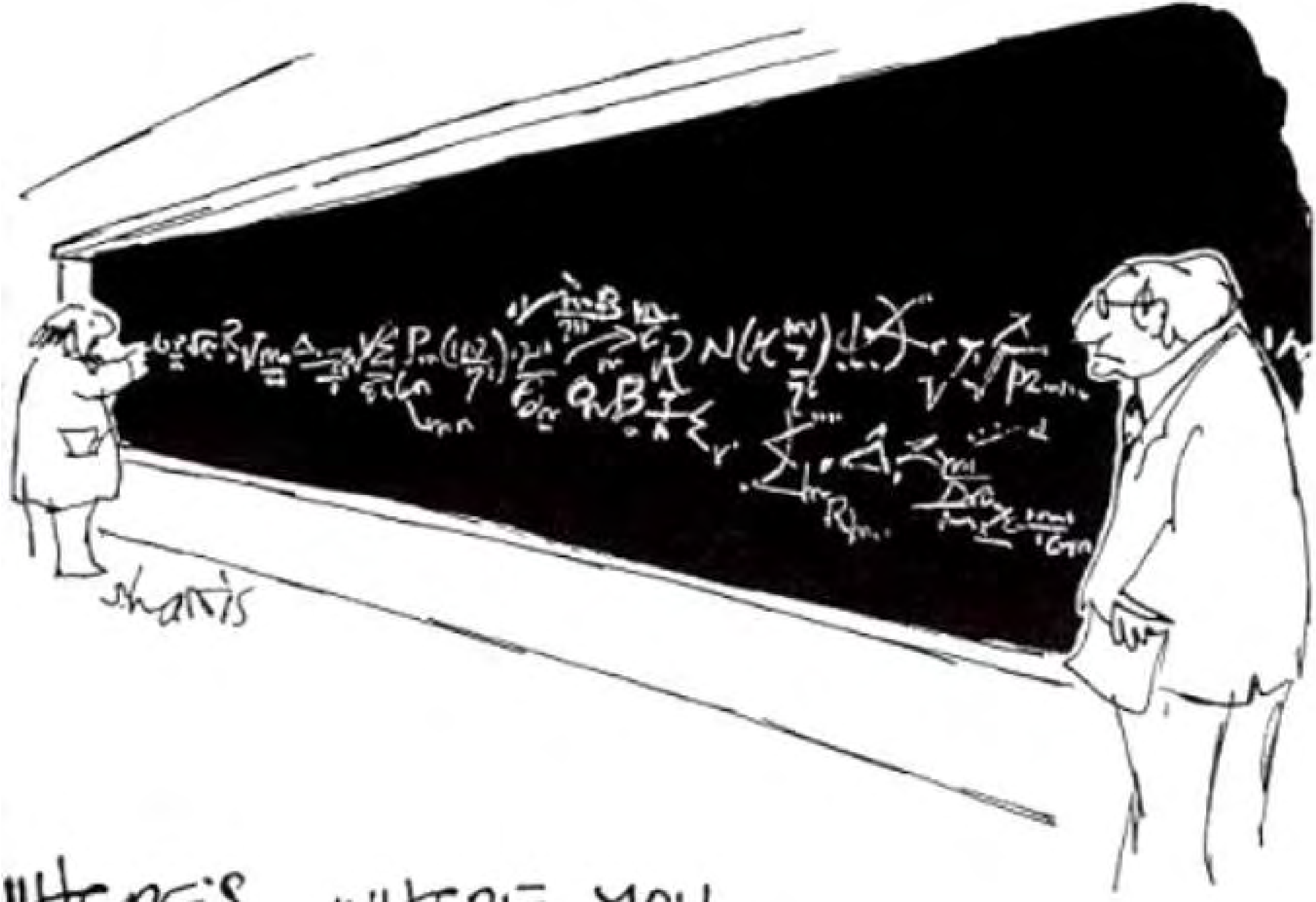
A. Convexity of Volumes ('mean' log-convexity).

Theorem 2 [(H,A) log-concavity] *The function* $V_\alpha(p) := 2^\alpha \Gamma(1+\frac{1}{p})^\alpha / \Gamma(1+\frac{\alpha}{p})$ *satisfies*

$$V_\alpha(p)^\lambda V_\alpha(q)^{1-\lambda} < V_\alpha\left(\frac{1}{\frac{\lambda}{p}+\frac{1-\lambda}{q}}\right), \qquad (1)$$

for all $\alpha > 1$, *if* $p, q > 1$, $p \neq q$, *and* $\lambda \in (0,1)$.

$\alpha = n$, $\frac{1}{p}+\frac{1}{q} = 1$ with $\lambda_1 = \lambda_2 = 1/2$ recovers the $p-$norm case of Blaschke-Santalo; and the *lower bound.* This extends to substitution norms. **Q.** How far can one take this?



"HERE'S WHERE YOU MADE YOUR MISTAKE."

Outline of Convexity Talk

A. Generalized Convexity of Volumes (Bohr-Mollerup).

B. Coupon Collecting and Convexity.

C. Convexity of Spectral Functions.

D. Madelung's Constant for Salt.

The talk ends when I do

Coupon Collecting and Convexity

B. The origin of the problem.

Consider a network *objective function* p_N:

$$p_N(q) := \sum_{\sigma \in S_N} \left(\prod_{i=1}^{N} \frac{q_{\sigma(i)}}{\sum_{j=i}^{N} q_{\sigma(j)}} \right) \left(\sum_{i=1}^{N} \frac{1}{\sum_{j=i}^{N} q_{\sigma(j)}} \right),$$

summed over *all* $N!$ permutations; so a typical term is

$$\left(\prod_{i=1}^{N} \frac{q_i}{\sum_{j=i}^{N} q_j} \right) \left(\sum_{i=1}^{N} \frac{1}{\sum_{j=i}^{n} q_j} \right).$$

For example, with $N = 3$ this is

$$q_1 q_2 q_3 \left(\frac{1}{q_1 + q_2 + q_3} \right) \left(\frac{1}{q_2 + q_3} \right) \left(\frac{1}{q_3} \right) \left(\frac{1}{q_1 + q_2 + q_3} + \frac{1}{q_2 + q_3} + \frac{1}{q_3} \right).$$

This arose as the objective function in a 1999 PhD on coupon collection. Ian Affleck wished to show p_N was convex on the positive orthant. **I hoped not!**

Coupon Collecting and Convexity

B. Doing What is Easy.

First, we try to simplify the expression for p_N. The *partial fraction decomposition* gives:

$$\begin{aligned}
p_1(x_1) &= \frac{1}{x_1}, \\
p_2(x_1, x_2) &= \frac{1}{x_1} + \frac{1}{x_2} - \frac{1}{x_1 + x_2}, \\
p_3(x_1, x_2, x_3) &= \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{1}{x_1 + x_2} - \frac{1}{x_2 + x_3} - \frac{1}{x_1 + x_3} \\
&+ \frac{1}{x_1 + x_2 + x_3}. \qquad (1)
\end{aligned}$$

Partial fractions are an arena in which computer algebra is hugely useful. Try performing the third case in (1) by hand. It is tempting to predict the "same" pattern will hold for $N = 4$. This is easy to confirm (by computer) and so we are led to:

Coupon Collecting and Convexity

B. A Very Convex Integrand. (Is there a direct proof?)

A year later, Omar Hijab suggested re-expressing p_N as the joint expectation of Poisson distributions. This leads to:

If $x = (x_1, \cdots, x_n)$ is a point in the positive orthant R^n_+, then

$$p_N(x) = \left(\prod_{i=1}^{n} x_i\right) \int_{R^n_+} e^{-\langle x,y\rangle} \max(y_1, \cdots, y_n)\, dy,$$

where $\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$ is the Euclidean inner product.

Now $y_i \to x_i\, y_i$ and standard techniques show $1/p_N$ is concave, as the integrand is. [We can now ignore probability if we wish!]

Q. "inclusion-exclusion" convexity: **OK** for $1/g(x) > 0$, g concave.

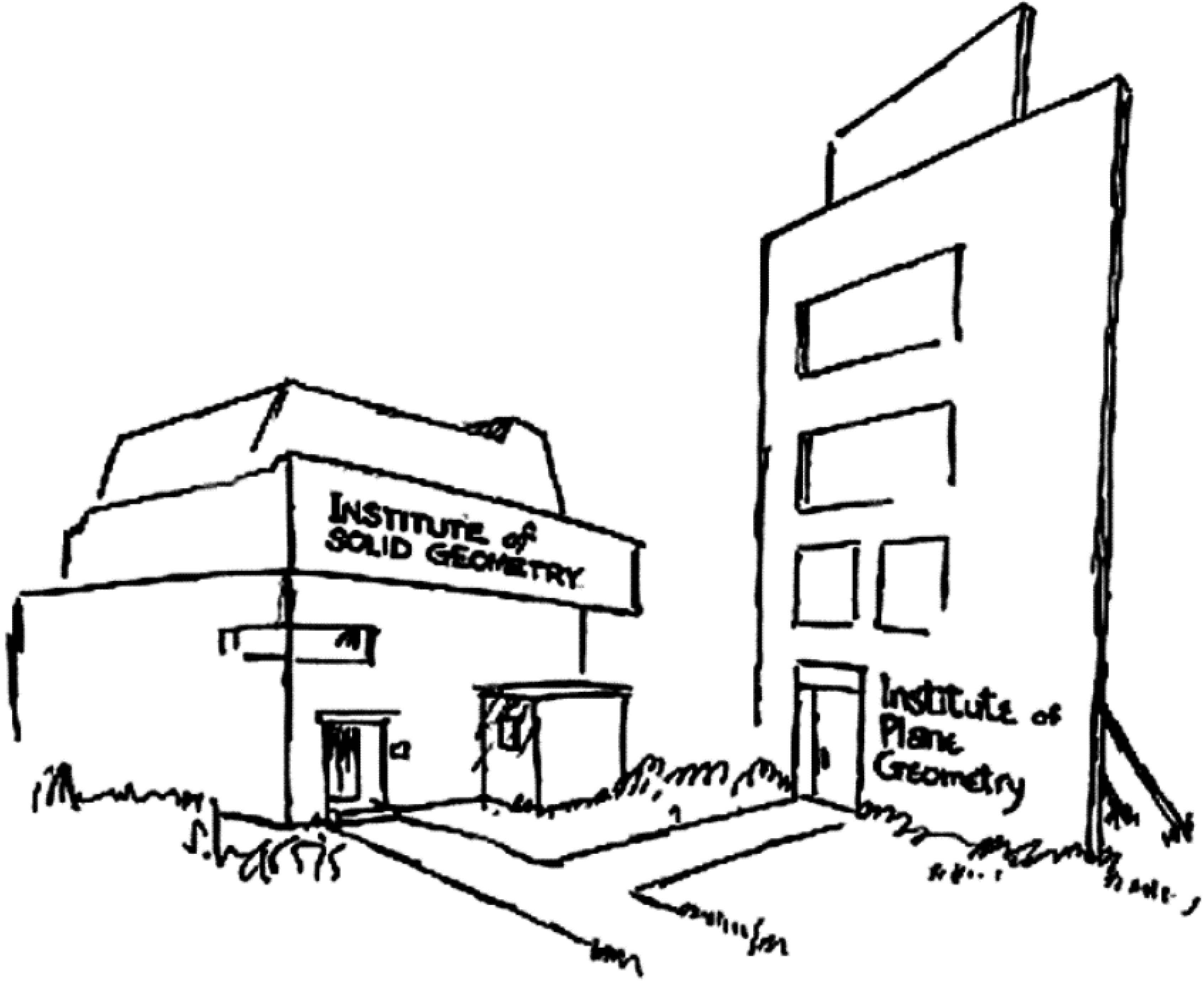
Goethe’s One Nice Comment About Us

“Mathematicians are a kind of Frenchmen:

whatever you say to them they translate into their own language, and right away it is something entirely different.”

(Johann Wolfgang von Goethe)

Maximen und Reflexionen, no. 1279


INSTITUTE of
SOLID GEOMETRY
Institute of
Plane
Geometry

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The talk ends when I do

Convexity of Spectral Functions

C. Eigenvalues of symmetric matrices (Lewis and Davis).

$\lambda(S)$ lists **decreasingly** the (real, resp. non-negative) eigenvalues of a (symmetric, resp. PSD) n-by-n matrix S. The **Fenchel conjugate** is the convex closed function given by

$$f^*(x) := \sup_y \langle y, x\rangle - f(y).$$

Theorem (Spectral conjugacy) If $f : R^n \mapsto [-\infty, \infty]$ is a symmetric function, it satisfies the formula $(f \circ \lambda)^* = f^* \circ \lambda$.

Corollary [Davis/Lewis] Suppose $f : R^n \mapsto [-\infty, \infty]$ is symmetric. Then the "spectral function" $f \circ \lambda$ is closed and convex (resp. differentiable) if and only if f is closed and convex (resp. differentiable).

Convexity of Spectral Functions

C. Three Amazing Examples (Lewis).

I. Log Determinant Let $\mathrm{lb}(x) := -\log(x_1 x_2 \cdots x_n)$ which is clearly symmetric and convex. The corresponding spectral function is $S \mapsto -\log\det(S)$.

II. Sum of Eigenvalues Ranging over permutations, let $f_k(x) := \max_\pi \{x_{\pi(1)} + x_{\pi(2)} + \cdots + x_{\pi(k)}\}$. This is clearly symmetric and convex. The corresponding spectral function is $\sigma_k(S) := \lambda_1(S) + \lambda_2(S) + \cdots \lambda_k(S)$.

In particular the largest eigenvalue, σ_1, is a continuous convex function of S and is differentiable if and only if the eigenvalue is simple.

Convexity of Spectral Functions

C. Three Amazing Examples (Lewis).

III. $k-$th Largest Eigenvalue The $k-$th largest eigenvalue may be written as

$$\mu_k(S) = \sigma_k(S) - \sigma_{k-1}(S).$$

In particular, this represents μ_k as the difference of two convex continuous, hence locally Lipschitz, functions of S and so we discover the very difficult result that for each k, $\mu_k(S)$ is a locally Lipschitz function of S.

- Hard analogues exist for *singular values*, *hyperbolic polynomials*, *Lie algebras*, etc.

Trace class operators

Convexity of Barrier Functions

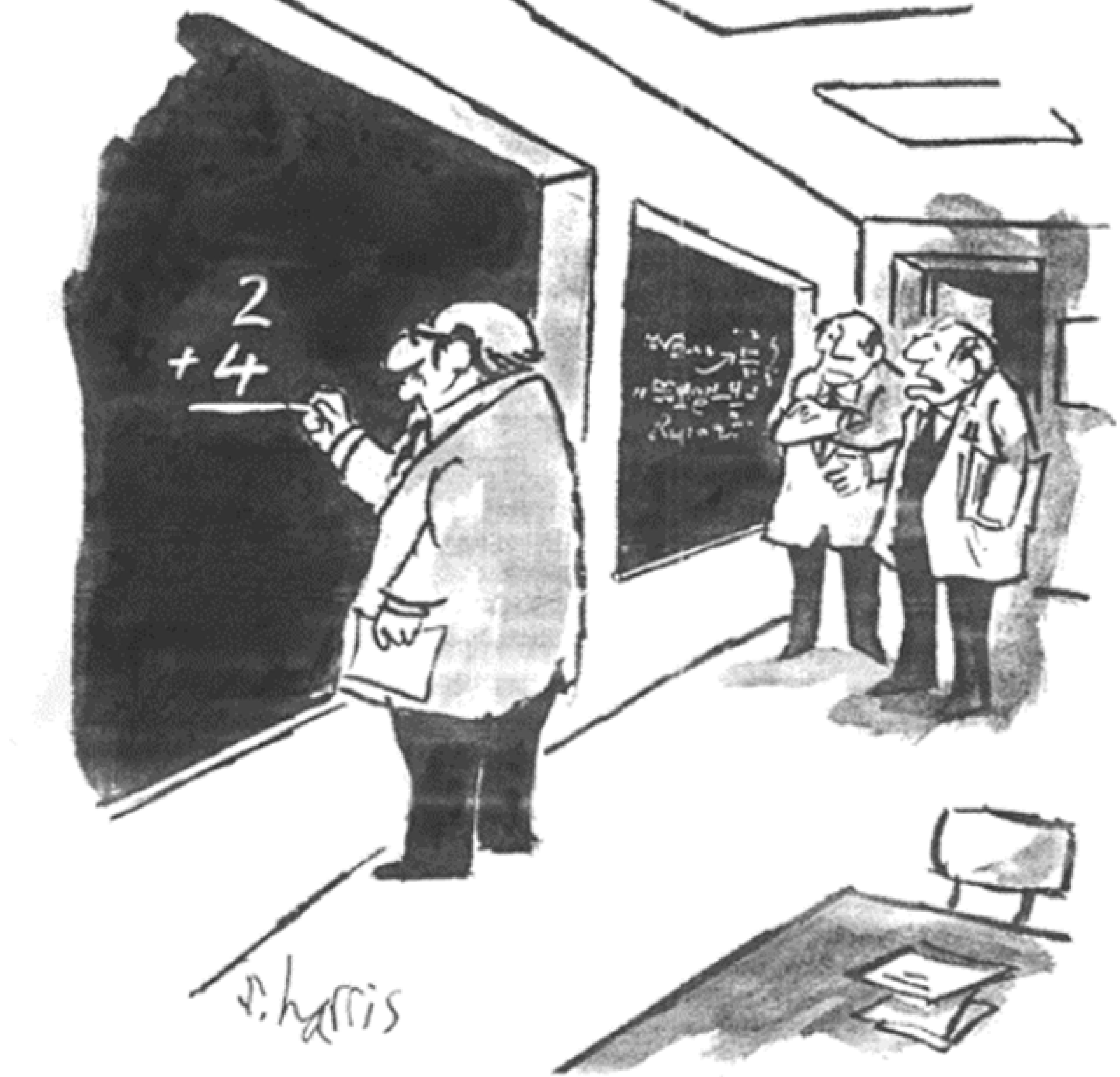
C. A Fourth Amazing Example (Nesterov & Nemirovskii).

IV Self-concordant Barrier Functions Let A be a nonempty open convex set in R^N. Define, for $x \in A$,

$$F(x) := \lambda_N((A - x)^o),$$

where λ_N is N-dimensional Lebesque measure and $(A - x)^o$ is the polar set. Then F is an essentially Fréchet smooth, log-convex, barrier function for A.

- Central to modern *interior point methods*.
- The orthant yields lb$(x) := -\sum_{k=1}^{N} \log x_k$.
- Hilbert space analog? (JB-JV, CUP, 2008)



"He was very big in Vienna."

Outline of Convexity Talk

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The talk ends when I do

Full details are in the three reference texts

D. Madelung's Constant:
David Borwein   CMS Career Award

$$= \sum_{n,m,p}{}' \frac{(-1)^{n+m+p}}{\sqrt{n^2+m^2+p^2}}$$

This polished solid silicon bronze sculpture is inspired by the work of David Borwein, his sons and colleagues, on the conditional series above for salt, Madelung's constant. This series can be summed to uncountably many constants; one is Madelung's constant for **electro-chemical stability of sodium chloride**. (Convexity is hidden here too!)

This constant is a period of an elliptic curve, a real surface in four dimensions. There are uncountably many ways to imagine that surface in three dimensions; one has negative gaussian curvature and is the tangible form of this sculpture. (As described by the artist.)

Peter Borwein
in front of
Helaman Ferguson's
work

CMS Meeting
December 2003
SFU Harbour Centre

Ferguson uses high tech tools and micro engineering at NIST to build monumental math sculptures

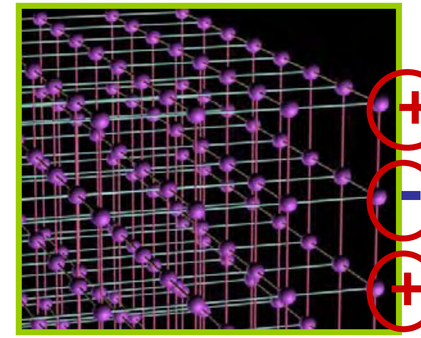
D. Madelung's Constant

perturbation

$$M_3(s) := \sum_{n,m,p}{}' \frac{(-1)^{n+m+p}}{\{n^2+m^2+p^2\}^s}$$

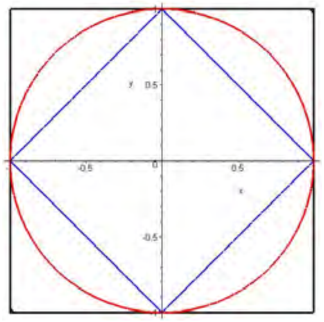
$$M_2(s) := \sum_{n,m}{}' \frac{(-1)^{n+m}}{\{n^2+m^2\}^s}$$

Alternating charges on a cubic lattice



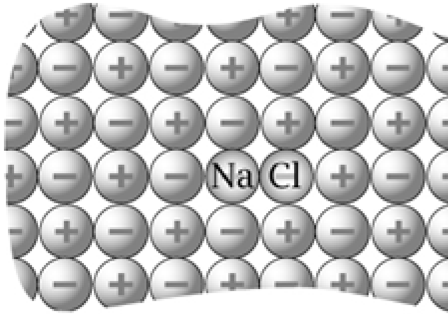
In many texts, the potential, $M_3(1/2)$, is 'added' over *increasing spheres*: $\boxed{\sum_{n=1}^{\infty}(-1)^n r_3(n)/\sqrt{n}}$ but $r_3(n)/\sqrt{n} \not\to 0$! [$r_3(n)$ is # of reps. of n as sum of 3 squares.]

The sum over *increasing cubes* does converge to the value chemists expect (by Mellin transform methods): $-1.74756459\ldots$ — needs a solar-system size crystal to be realistic!



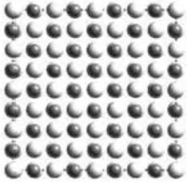
D. Madelung's Constant in 2D

$$M_2(s) \;=\; {\sum_{n,m}}' \frac{(-1)^{n+m}}{\left(n^2+m^2\right)^s}$$



$$\boxed{M_2(1/2) = \sum_{n=1}^{\infty} (-1)^n r_2(n)/\sqrt{n}}$$

Now if M_2 is `added` over spheres (ℓ^2 balls) the n-th term *tends to zero* and the sum *agrees* with that over increasing squares (ℓ^∞) but the sum over increasing diamonds (ℓ^1) diverges-Riemann sum!



✓ For C a closed convex symmetric body set

$$M_C(s) := \lim_{N\to\infty} {\sum_{n,m\in NC}}' \frac{(-1)^{n+m}}{\left(n^2+m^2\right)^s}$$

$$M_C(s) = \lim_{N\to\infty} \sum\nolimits'_{n,m \in NC} \frac{(-1)^{n+m}}{(n^2+m^2)^s}$$

Theorem (BBP, 1998) $M_C(s)$ *exists, is analytic and is* independent *of* C *for Re*$(s) > 1/2$. [In $\mathbf{R}^k$ this holds for Re$(s) > (k-1)/2$.]

1. Re$(s) > 1$ needed for absolute convergence.
2. $M_{\{\|\cdot\|_2 \le 1\}}(s) = -4\zeta(s)(1-2^{1-s})L_{-4}(s)$ converges precisely for Re$(s) > 1/4$. This relies on correctness ? of the wonderful exact determination of the average size of $r_2(n)$ [Cappell and Shaneson,2007]: *the number of lattice points in a circle of radius* $\sqrt{t}$ *is* $\pi t + O\left(t^{1/4+\varepsilon}\right)$ (best possible).


THEORIES
PROOFS
S. Harris

Three Bonus Track Follows

A. Generalized Convexity of Volumes (Bohr-Mollerup).

B. Coupon Collecting and Convexity.

C. Convexity of Spectral Functions.

D. Madelung's Constant for Salt.

References

E. Entropy and NMR.

F. Inequalities and the Maximum Principle.

G. Trefethen's 4th Digit Challenge Problem.

E. CONVEX CONJUGATES and NMR (MRI)

The *Hoch and Stern information measure* in complex N-space is $H(z) := \sum_{j=1}^{N} h(z_j/b)$ where h is convex and given (for scaling b) by

$$h(z) := |z| \ln\left(|z| + \sqrt{1+|z|^2}\right) - \sqrt{1+|z|^2}$$

for quantum theoretic (NMR) reasons. Recall the *Fenchel–Legendre conjugate*

$$f^*(y) = \sup_x \langle x, y\rangle - f(x).$$

Our symbolic convex analysis package produced

$$h^*(z) = \cosh(|z|).$$

Compare the *Shannon entropy* $z\ln(z) - z$ whose conjugate is $\exp(z)$.

I'd never have tried by hand! Effective dual algorithms are now possible!

Knowing `Closed Forms' Helps

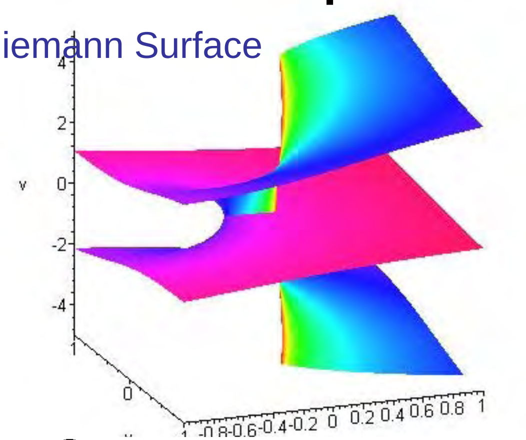
For example

$$(\exp\exp)^*(y) = y\ln(y) - y\{W(y) + W(y)^{-1}\}$$

where *Maple* or *Mathematica* recognize the complex *Lambert W function* given by

$W(x)e^{W(x)} = x$.

Riemann Surface



Thus, the conjugate's series is:

$$-1+(\ln(y)-1)\,y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{3}{8}y^4+\frac{8}{15}y^5+O\left(y^6\right).$$

The literature is all in the last decade since W got a name!

WHAT is ENTROPY?

Despite the narrative force that the concept of entropy appears to evoke in everyday writing, in scientific writing entropy remains a thermodynamic quantity and a mathematical formula that numerically quantifies disorder. When the American scientist Claude Shannon found that the mathematical formula of Boltzmann defined a useful quantity in information theory, he hesitated to name this newly discovered quantity entropy because of its philosophical baggage. The mathematician John Von Neumann encouraged Shannon to go ahead with the name entropy, however, since “no one knows what entropy is, so in a debate you will always have the advantage."

The American Heritage Book of English Usage, p. 158

Information Theoretic Characterizations Abound

Theorem. Up to a positive scalar multiple

$$H(\overrightarrow{p}) = -\sum_{k=1}^{N} p_k \log p_k$$

is the unique continuous function on finite probabilities such that [a.] **Uncertainly grows**:

$$H\left(\overbrace{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}}^{n}\right)$$

increases with n.

[b.] **Subordinate choices are respected**: for distributions $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$ and $0 < p < 1$,

$$H\left(p\,\overrightarrow{p_1}, (1-p)\,\overrightarrow{p_2}\right) = p\,H(\overrightarrow{p_1}) + (1-p)\,H(\overrightarrow{p_2}).$$


s.harris

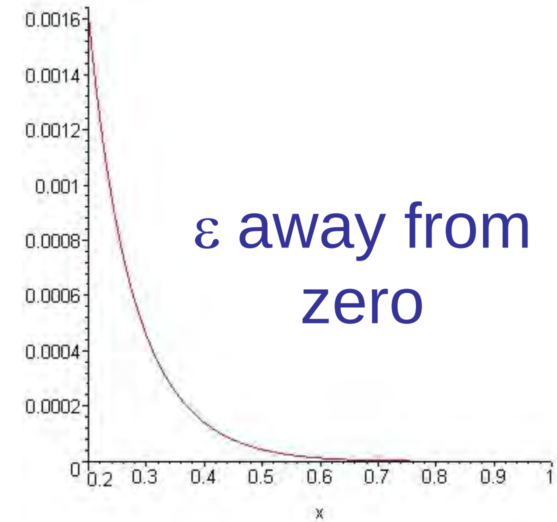
F. Inequalities and the Maximum Principle

- Consider the two *means*

$$\mathcal{L}^{-1}(x,y) := \frac{x-y}{\ln(x)-\ln(y)}$$

and

$$\mathcal{M}(x,y) := \sqrt[\frac{3}{2}]{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{2}}$$



A conformal function estimated reduced to

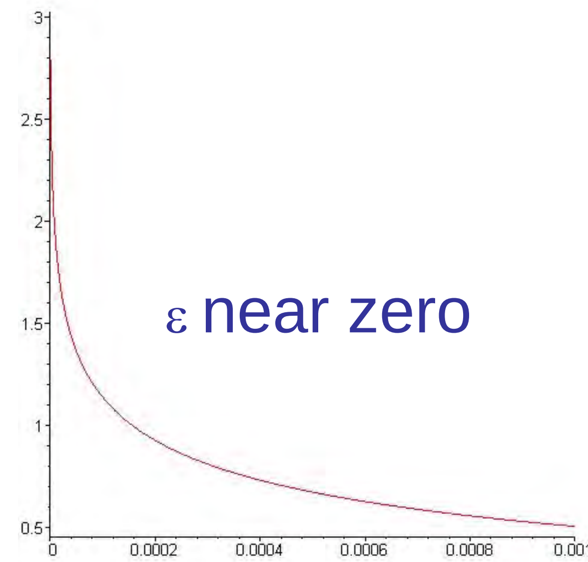
$$\mathcal{L}(\mathcal{M}(x,1),\sqrt{x}) > \mathcal{L}(x,1) > \mathcal{L}(\mathcal{M}(x,1),1)$$

for $0 < x < 1$.

tight

We first discuss showing

$$\mathcal{E}(x) := \mathcal{L}(\mathcal{M}(x,1),\sqrt{x}) - \mathcal{L}(x,1) > 0.$$



I. Numeric/Symbolic Methods

- $\lim_{x \to 0^+} \mathcal{E}(x) = \infty$.

- *Newton-like iteration* shows that $\mathcal{E}(x) > 0$ on $[0.0, 0.9]$.

When we make each step effective.
This is hardest for the integral.

- *Taylor series* shows $\mathcal{E}(x)$ has 4 zeroes at 1.

$$= \frac{7}{51840}(x-1)^4 - \frac{7}{20736}(x-1)^5 + O\left((x-1)^6\right)$$

- *Maximum Principle* shows there are no more zeroes inside $C := \{z : |z-1| = \frac{1}{4}\}$:

$$\frac{1}{2\pi i}\int_C \frac{\mathcal{E}'}{\mathcal{E}} = \#(\mathcal{E}^{-1}(0); C)$$

Drive

II. Graphic/Symbolic Methods

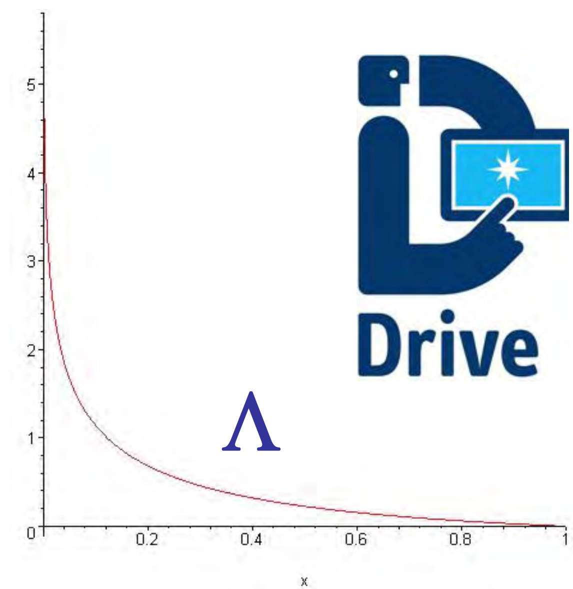
Consider the opposite (cruder) inequality

$$\Lambda := \mathcal{L}(x,1) - \mathcal{L}(\mathcal{M}(x,1),1) > 0.$$

We may observe that it holds since:

- $\mathcal{M}$ is a mean;

- $\mathcal{L}(x,1)$ decreases with x.



- There is an algorithm (Collins) for universal algebraic inequalities.

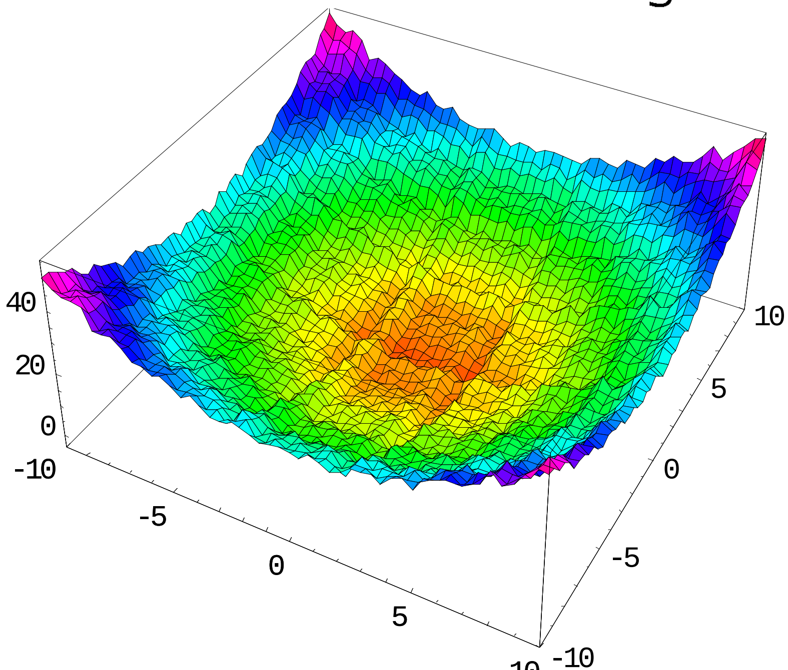
F. Nick Trefethen's 100 Digit/100 Dollar Challenge, Problem 4 (SIAM News, 2002)

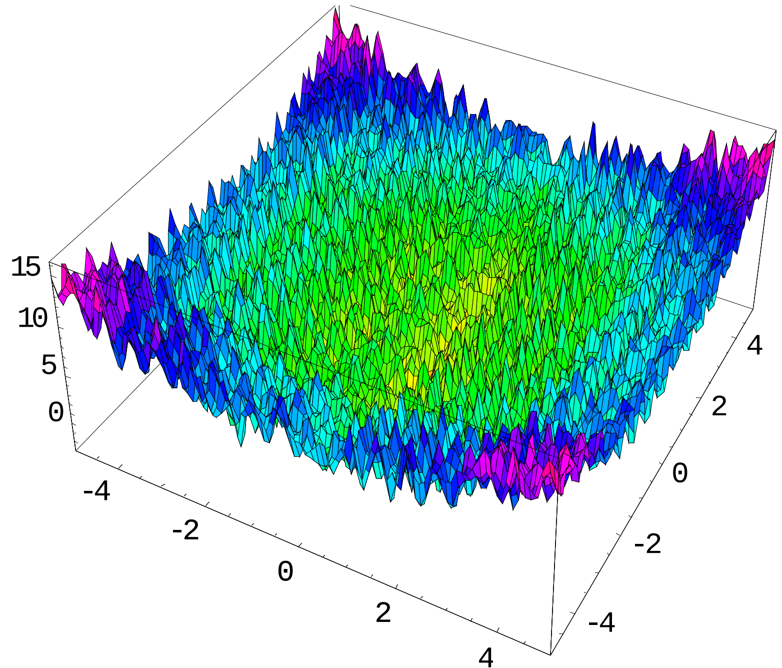
 # 4. What is the global minimum of the function

$$\exp(\sin(50x)) + \sin(60e^y) + \sin(70\sin x)$$

$$+ \sin(\sin(80y)) - \sin(10(x+y)) + (x^2+y^2)/4?$$

- no bounds are given.





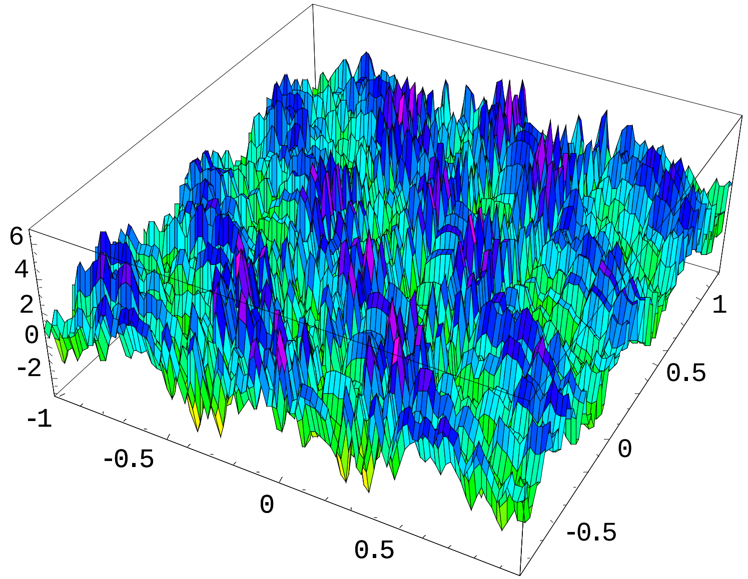
… HDHD Challenge, Problem 4

- This model has been numerically solved by LGO, MathOptimizer, MathOptimizer Pro, TOMLAB /LGO, and the *Maple GOT* (by Janos Pinter who provide the pictures).

- The solution found agrees to 10 places with the announced solution (the latter was originally based (**provably**) on a huge grid sampling effort, interval analyisis and local search).

x*~ (-0.024627…, 0.211789…)
f*~-3.30687…

Close-up picture near global solution: the problem still looks rather difficult … *Mathematica 6* can solve this by "zooming"!



See lovely SIAM solution book by Bornemann, Laurie, Wagon and Waldvogel and my Intelligencer Review at http://users.cs.dal.ca/~jborwein/digits.pdf

REFERENCES


Enigma

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J.M. Borwein and A.S. Lewis, *Convex Analysis and Nonlinear Optimization. Theory and Examples*, CMS-Springer, Second extended edition, 2005.

J.M. Borwein and J.D. Vanderwerff, *Convex functions, constructions, characterizations and counterexamples*, CUP, 2009.

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

- J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.