An explicit non-expansive function whose subdifferential is the entire dual ball

Jonathan M. Borwein and Scott Sciffer

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ABSTRACT. We provide an explicit recipe for constructing a function on an arbitrary real Banach space whose Clarke and approximate subdifferentials are identically equal to the dual unit ball.

1 Introduction

Our aim, motivated by ideas in [6], is to provide on any Banach space X an explicit Lipschitz function (think of a 'dimpled' golf ball) whose Clarke and approximate subdifferential is identically the ball. In [3, 2] the existence of such a function was established by Baire category techniques as part of more general results, but no direct construction was provided. Indeed, this construction appears new even in two dimensions. The history of the subject is described in some detail in [3, 2]. It is possible to be much more precise about the prevalence of such "maximal" subdifferentials [4]. (See also [12, 13].)

For a real-valued function $f : A \to \mathbf{R}$ we say that f is K-*Lipschitz* on A if $K > 0$ and $|f(x) - f(y)| \le K\|x - y\|$ for all $x, y \in A$. When $K = 1$, f is called *non-expansive*.

The right-hand lower *Dini derivative* of f at a point x in the direction v is given by

$$f^-(x; v) := \liminf_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t},$$

while the—possibly empty—*Dini subdifferential* $\partial_- f$ is given by

$$\partial_- f(x) := \{x^* \in X^* \mid \langle x^*, v \rangle \le f^-(x; v) \text{ for all } v \in X\}.$$

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The *Clarke derivative* of f at a point x in the direction v is given by

$$f^\circ(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

while the *Clarke subdifferential* $\partial_c f$ is given by

$$\partial_c f(x) := \{x^* \in X^* |\ \langle x^*, v \rangle \leq f^\circ(x; v) \text{ for all } v \in X\}.$$

Note that $f^\circ(x; v)$ is upper semicontinuous as a function of (x, v). Being nonempty and weak*-compact (convex) valued, the multifunction $\partial_c f : A \to 2^{X^*}$ is norm-to-weak* upper semicontinuous. Detailed properties about Dini and Clarke subdifferentials can be found in [5], which is a bible of sorts for nonsmooth analysts.

Let us recall that a Lipschitz function is Clarke *regular* at a point x if its classical right-hand *Dini derivative* at the point x given by

$$f'(x; v) := \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t},$$

exists and equals the Clarke derivative at the point. In this case we write $\partial := \partial_c = \partial_-$.

2 Construction

We state and prove our core result:

Theorem 1 (Maximal Clarke subdifferential [3, 2]) *Let X be an arbitrary real Banach space. There exists a non-expansive function $f \colon X \to \mathbf{R}$ such that*

$$\partial_c f(x) = B_{X^*} \ \text{for all} \ x \in X, \tag{1}$$

where B_{X^*} is the closed unit ball in the dual space.

Proof. We start the proof with a 'seeding' of open sets in X.

Base sets Let $\{U_n \colon n \in \mathbf{N}\}$ be a collection of dense open subsets of X such that:

(i) Each U_n is a disjoint union of open norm balls with radius $r_\gamma < 1/n$; say $U_n = \bigcup_{\gamma \in \Gamma_n} B(x_\gamma; r_\gamma)$.

(ii) The collection is nested, that is, $U_{n+1} \subset U_n$ for each $n \in \mathbf{N}$.

(iii) The centres of the balls at each level, Γ_n, are not contained at the next level: for $n \in \mathbf{N}$ and $\gamma \in \Gamma_n$, $x_\gamma \notin U_{n+1}$.

A standard maximality argument shows these requirements are easily fulfilled—if hard to prescribe more concretely except in a few polyhedral norms—at level n, we replace each $B(x_\gamma; r_\gamma)$ by a maximal disjoint union of open balls of radius less than $1/(n + 1)$ lying in $B(x_\gamma; r_\gamma) \setminus \{x_\gamma\}$. $\qquad\square$
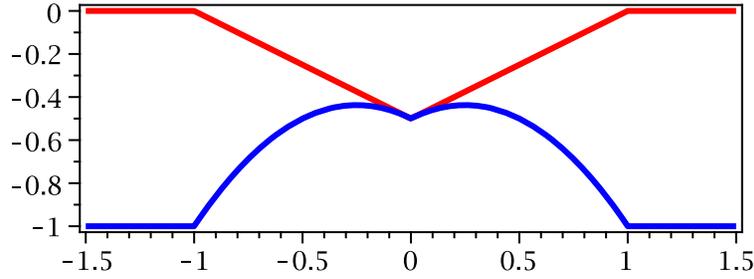
Figure 1: $f_1 \geq g_1$ on $B(0; 1)$.

Remark 1 Since separable metric spaces are Lindelöf, the collection is countable when X is separable.

The 'seeding' is in the spirit of a simpler construction used among others by Katriel [10] to answer a question in [7], as a way of constructing surprising or pathological Lipschitz functions from familiar benign ones (see [1] and the references therein). □

From this point on the process is entirely constructive, but it seems difficult to make the 'seeding' equally so—even in two-dimensional space.

Initial steps. Begin with f_0, g_0 defined by $f_0(x) := 0$ and $g_0(x) := -1$. We now give f_0 some 'dimples' by defining

$$ f_1(x) := \begin{cases} f_0(x) + (1 - \frac{1}{2})(\|x - x_\gamma\| - r_\gamma), & \text{if } x \in B(x_\gamma; r_\gamma), \gamma \in \Gamma_1; \\ f_0(x), & \text{if } x \notin U_1. \end{cases} $$

Also, setting $\delta_\gamma := r_\gamma$ at this first iteration, define

$$ g_1(x) := \begin{cases} f_1(x) - \|x - x_\gamma\|^2, & \text{if } x \in B(x_\gamma; \delta_\gamma), \gamma \in \Gamma_1; \\ g_0(x), & \text{if } x \notin U_1. \end{cases} $$

Note that f_1 is Lipschitz with Lipschitz constant $1/2$ while g_1 is Lipschitz on U_1 with constant not exceeding $3/2$ as in Figure 1. Also note that

$$ g_1(x) \leq f_1(x), $$

with equality occurring if and only if $x = x_\gamma$ for some $\gamma \in \Gamma_1$.

Observe that f_1 and g_1 are both regular at each $x_\gamma, \gamma \in \Gamma_1$. Further, if h_1 is any function such that $g_1(x) \leq h_1(x) \leq f_1(x)$ in a neighbourhood of $x_\gamma, \gamma \in \Gamma_1$, then the lower right-hand derivative of h_1 exists at x_γ and agrees with those of f_1 and g_1. All of this leads to the conclusion that

$$ g_1(x) \leq h_1(x) \leq f_1(x) \text{ for all } x \in U_1 \Rightarrow \partial_- h_1(x_\gamma) = \frac{1}{2} B_{X^*}. $$

when $\gamma \in \Gamma_1$.

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Induction steps. Suppose f_k, g_k have been constructed so that f_k is Lipschitz with Lipschitz constant $(1 - 1/2^k)$ and g_k is locally Lipschitz on U_k. Moreover, we have

$$-1 \leq g_{k-1}(x) \leq g_k(x) \leq f_k(x) \leq f_{k-1}(x) \leq 0 \tag{2}$$

with equality of f_k and g_k occurring if and only if $x = x_\gamma$ for some $\gamma \in \Gamma_i, i \leq k$.

We proceed to add more (smaller but steeper) 'dimples' to f_k. Define f_{k+1} and g_{k+1} as follows. Given any $\gamma \in \Gamma_{k+1}$, since $\gamma \notin \bigcup_{i \leq k} \Gamma_i$, by the $(1 - 1/2^k)$-Lipschitzness of f_k we can choose an $s_\gamma > 0$ such that

$$f_k(x_\gamma) + \left(1 - \frac{1}{2^{k+1}}\right)(\|x - x_\gamma\| - s_\gamma) > f_k(x) \quad \text{for all } \|x - x_\gamma\| > r_\gamma, \tag{3}$$

Then by the local continuity of f_k, g_k and the norm we may select $\delta_\gamma > 0$ such that

$$f_k(x) > f_k(x_\gamma) + \left(1 - \frac{1}{2^{k+1}}\right)(\|x - x_\gamma\| - s_\gamma) > g_k(x) \quad \text{for all } \|x - x_\gamma\| \leq \delta_\gamma. \tag{4}$$

Indeed, any positive constant s_γ less than $\min\{f_k(x_\gamma) - g_k(x_\gamma), r_\gamma/2^{k+1}\}$ will work. The first term is positive because the centres of balls at each level are excluded at the next level. The second is chosen with the knowledge that the 'dimple' is $1/2^{k+1}$ steeper than the function it is dimpling and ensures (3) holds.

Set

$$f_{k+1}(x) := \begin{cases} \min\{f_k(x), f_k(x_\gamma) + \left(1 - \frac{1}{2^{k+1}}\right)(\|x - x_\gamma\| - s_\gamma)\}, & x \in B(x_\gamma; r_\gamma), \gamma \in \Gamma_{k+1}; \\ f_k(x), & \text{if } x \notin U_{k+1} . \end{cases}$$

We also create

$$g_{k+1}(x) := \begin{cases} \max\{g_k(x), f_{k+1}(x) - \|x - x_\gamma\|^2\}, & x \in B(x_\gamma; \delta_\gamma), \gamma \in \Gamma_{k+1}; \\ \max\{g_k(x), f_{k+1}(x) - \delta_\gamma^2 + \delta_\gamma - \|x - x_\gamma\|\}, & \delta_\gamma \leq \|x - x_\gamma\| < r_\gamma, \gamma \in \Gamma_{k+1}; \\ g_k(x), & \text{if } x \notin U_{k+1}. \end{cases}$$

Observe that

$$-1 \leq g_k(x) \leq g_{k+1}(x) \leq f_{k+1}(x) \leq f_k(x) \leq 0 \tag{5}$$

with $g_{k+1}(x) = f_{k+1}(x)$ if and only if $x = x_\gamma$ for some $\gamma \in \Gamma_i, i \leq k + 1$.

Note also that f_{k+1} and g_{k+1} are regular at each $x_\gamma, \gamma \in \Gamma_{k+1}$. Further, if h_{k+1} is any function such that $g_{k+1}(x) \leq h_{k+1}(x) \leq f_{k+1}(x)$ in a neighbourhood of $x_\gamma, \gamma \in \Gamma_{k+1}$, then the right-hand Dini derivative of h_{k+1} exists at x_γ and agrees with those of f_{k+1} and g_{k+1}. All of this leads to the conclusion that

$$g_{k+1}(x) \leq h_{k+1}(x) \leq f_{k+1}(x) \text{ for all } x \in U_{k+1} \Rightarrow \partial_- h_{k+1}(x_\gamma) = \left(1 - \frac{1}{2^{k+1}}\right) B_{X^*}$$

when $\gamma \in \Gamma_{k+1}$, where again B_{X^*} is the closed unit ball in the dual space.

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The limiting step. Let f be the pointwise limit of the f_k, which limit exists by (5) since $f_k(x)$ is decreasing and bounded below. Clearly f is Lipschitz with Lipschitz constant no greater than one, as each f_k is. Also, (5) implies that $g_k \leq f \leq f_k$ for each k. Hence if $\gamma_k \in \Gamma_k$ we conclude that

$$\left(1 - \frac{1}{2^k}\right) B_{X^*} \subseteq \partial_- f(x_{\gamma_k}) \subseteq B_{X^*}. \tag{6}$$

Since $U := \bigcap_{n \in \mathbf{N}} U_n$ is a dense G_δ set in X, for every $x \in X$ there is a sequence $x_{\gamma_k} \to x$, $\gamma_k \in \Gamma_k$. But then by the norm-to-weak* upper semicontinuity property of $\partial_c f$ [5, Prop 2.1.5(b)] we conclude that $\partial_c f(x) = B_{X^*}$ for all $x \in X$, as required. $\qquad\square$

Remark 2 We note that the subdifferential is maximal since, by the Clarke Mean-value theorem [5], the Clarke subdifferential of any non-expansive function must lie in the unit ball. Conversely, any function satisfying (8) is necessarily non-expansive.

We also note that in any separable Banach space Rademacher's theorem implies that, for any function satisfying (8) and any *Haar null* set Ω [3], one must have

$$\overline{\mathrm{conv}}^*\{\nabla f(x_n) \xrightarrow{w^*} x^* : x_n \to x, x_n \notin \Omega\} = B_{X^*} \text{ for all } x \in X, \tag{7}$$

where ∇f denotes the Gâteaux derivative of f.

In particular, when the dual ball is strictly convex this means that the points where the gradient exists and w*-cluster at any prescribed point of the dual sphere are non-null in every neighbourhood of every point in the space. $\qquad\square$

An inspection of the proof of equation (6) of Theorem 1 shows that we have actually proven more. Recall that the *approximate subdifferential* [7, 8, 9, 11] of a Lipschitz function can be defined by

$$\partial_a f(x) := \bigcap_{\varepsilon > 0} \overline{\partial_- f(B(x, \varepsilon))}^*,$$

and satisfies $\overline{\mathrm{conv}}^* \partial_a f(x) = \partial_c f(x)$.

Theorem 2 (Maximal approximate subdifferential [3, 2]) *Let X be an arbitrary real Banach space. There exists a non-expansive function $f : X \to \mathbf{R}$ such that*

$$\partial_a f(x) = B_{X^*} \text{ for all } x \in X, \tag{8}$$

where B_{X^*} is the closed unit ball in the dual space.

Remark 3 Theorem 2 had previously only been proven with restrictions on the class of Banach spaces or on the rotundity properties of the norm [3, 2].

We emphasize that when $\partial_a f(x) \equiv B_{X^*}$, the approximate subdifferential encodes no positive information other than the fact that f is non-expansive.

It should be apparent that the use of the norm could be varied and extended to allow for the limiting approximate subdifferential to take on more exotic forms as in [1]. $\qquad\square$

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Jonathan Borwein, School of Mathematical and Physical Sciences, University of Newcastle, Callaghan NSW 2308, Australia. Email: jonathan.borwein@newcastle.edu.au
and
Faculty of Computer Science and Department of Mathematics, Dalhousie University, Halifax NS B3H 2W5, Canada.

Scott Sciffer, School of Mathematical and Physical Sciences, University of Newcastle, Callaghan NSW 2308, Australia. Email: scott.sciffer@newcastle.edu.au