Moments of Ramanujan's Generalized Elliptic Integrals and Extensions of Catalan's Constant

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Abstract

We undertake a thorough investigation of the moments of Ramanujan's alternative elliptic integrals and of related hypergeometric functions. Along the way we are able to give some surprising closed forms for Catalan-related constants and various new hypergeometric identities.

Key words: elliptic integrals, hypergeometric functions, moments, Catalan's constant.

1 Introduction and background

As in [7, pp. 178-179], for $0 \le s < 1/2$ and $0 \le k \le 1$, let

$$K^s(k) \;:=\; \frac{\pi}{2}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}-s,\frac{1}{2}+s\\ 1\end{matrix}\right|k^2\right) \tag{1}$$

and

$$E^s(k) \;:=\; \frac{\pi}{2}\,{}_2F_1\left(\left.\begin{matrix}-\frac{1}{2}-s,\frac{1}{2}+s\\ 1\end{matrix}\right|k^2\right). \tag{2}$$

We use the standard notation for hypergeometric functions, namely

$${}_2F_1\left(\left.\begin{matrix}a,b\\ c\end{matrix}\right|z\right) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\,\frac{z^n}{n!},$$

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and its analytic continuation, where $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ is the rising factorial or *Pochhammer symbol*; likewise,

$$ {}_3F_2\left(\left.{a,b,c \atop d,e}\right| z\right) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\,\frac{z^n}{n!}. $$

One of the key early results, due to Gauss (1812), is the closed form

$$ {}_2F_1\left(\left.{a,b \atop c}\right| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{3} $$

when $\mathrm{Re}(c-a-b) > 0$.

We are interested in the moments given by

$$ K_n = K_{n,s} := \int_0^1 k^n K^s(k)\,\mathrm{d}k, \qquad E_n = E_{n,s} := \int_0^1 k^n E^s(k)\;\mathrm{d}k. \tag{4} $$

for both integer and real values of n. We immediately note that $K^s = K^{(-s)}$. Also, Euler's transform [3, Eqn. (2.2.7)] and a contiguous relation yield

$$ E^{(-s)} = \frac{4s\,(1-k^2)}{2s-1}\,K^s + \frac{2s+1}{2s-1}\,E^s. $$

The corresponding integral form of K^s may be obtained by expanding $(1-k^2t)^{s-1/2}$ and using the identity $\Gamma(1/2-s)\Gamma(1/2+s) = \pi/\cos(\pi s)$:

$$ K^s(k) = \frac{\cos \pi s}{2}\int_0^1 \frac{t^{s-1/2}}{(1-t)^{1/2+s}(1-k^2\,t)^{1/2-s}}\,\mathrm{d}t \tag{5} $$

$$ = \cos(\pi\, s)\int_0^{\pi/2} \frac{\tan^{2s}(\theta)}{\left(1-k^2\,\sin^2\theta\right)^{1/2-s}}\,\mathrm{d}\theta. \tag{6} $$

The latter has the nice feature of looking like the cleanest classical definition when $s = 0$. These and many more forms for K^s, E^s can be obtained from http://dlmf.nist.gov/15.6. There are four values for which these integrals are truly special:

$$ s \in \Omega := \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}\right\}, $$

that is, when $\cos^2(\pi s)$ is rational.

These are Ramanujan's alternative elliptic integrals as displayed in [13] and first decoded in [7]. A comprehensive study is given in [5] (see also [11] and [2]). These four cases are all produce modular functions [7, §5.5] and study is currently experiencing a renewal of interest, especially regarding related elliptic series for $1/\pi$ ([6], [7, §5.5] and [8]).

1.1 Reciprocal series for π

Truly novel series for $1/\pi$, based on elliptic integrals, were discovered by Ramanujan around 1910 [6, 7]. The most famous, with $s = 1/4$ is:

$$\frac{1}{\pi} \;=\; \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\,(1103+26390k)}{(k!)^4 396^{4k}}. \tag{7}$$

Each term of (7) adds eight correct digits. Gosper used (7) for the computation of a then-record 17 million digits of π in 1985 — thereby completing the first proof of (7) [7, Ch. 3]. Shortly thereafter, David and Gregory Chudnovsky found the following variant, which uses $s = 1/3$ and lies in the quadratic number field $\mathbb{Q}(\sqrt{-163})$ rather than $\mathbb{Q}(\sqrt{58})$:

$$\frac{1}{\pi} \;=\; 12\sum_{k=0}^{\infty}\frac{(-1)^k\,(6k)!\,(13591409+545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \tag{8}$$

Each term of (8) adds 14 correct digits. The brothers used this formula several times, culminating in a 1994 calculation of π to over four billion decimal digits. Their extraordinary story was told in a prizewinning *New Yorker* article by Richard Preston. Remarkably, (8) was used again in late 2009 for the then-record computation of π to 2.7 trillion places. In consequence, Fabrice Bellard has provided access to two trillion-digit integers whose ratio is bizarrely close to π. A striking recent series due to Yao, see [16], is

$$\frac{1}{\pi} = \frac{\sqrt{15}}{18}\sum_{n=0}^{\infty}\frac{\sum_{k=0}^{n}\binom{n}{k}^4(4\,n+1)}{36^n}. \tag{9}$$

1.2 Classical results

The coupling equation between E^s and K^s is given in [7, p. 178] and can be derived from the generalized hypergeometric differential equation (see http://dlmf.nist.gov/15.10). It is

$$E^s = (1-k^2)\,K^s + \frac{k(1-k^2)}{1+2s}\frac{\mathrm{d}}{\mathrm{d}k}K^s. \tag{10}$$

Integrating this by parts leads to

$$K_{2,s} = \frac{(1+2s)\,E_{0,s} - 2s\,K_{0,s}}{2-2s}. \tag{11}$$

In the same fashion, multiplying by k^n before integrating the coupling provides a recursion for $K_{n+2,s}$:

$$K_{n+2,s} = \frac{(n-2\,s)\,K_{n,s} + (1+2\,s)\,E_{n,s}}{n+2\,(1-s)}. \tag{12}$$

We also consider the *complementary* integrals:

$$K'^s(k) := K^s(\sqrt{1-k^2}) \quad\text{and}\quad E'^s(k) := E^s(\sqrt{1-k^2}).$$

The four integrals then satisfy a version of *Legendre's identity*,

$$E^s K'^s + K^s E'^s - K^s K'^s = \frac{\pi}{2}\,\frac{\cos \pi s}{1+2s} \tag{13}$$

for all $0 \le k \le 1$.

In [7, pp. 198-99] the moments are determined for the classical case of $s = 0$ which give the original complete elliptic integrals K and E. These are linked by the equations (see [7, p. 9])

$$E = (1-k^2)\,K + k(1-k^2)\frac{\mathrm{d}K}{\mathrm{d}k}, \tag{14}$$

which is (10) with $s = 0$ and

$$E = K + k\frac{\mathrm{d}E}{\mathrm{d}k}, \tag{15}$$

from which we derive the following recursions:

Theorem 1 **(**$s = 0$**)** *For* $n = 0, 1, 2, \ldots$

$$(a)\ K_{n+2} = \frac{nK_n + E_n}{n+2} \quad \textit{and} \quad (b)\ E_n = \frac{K_n + 1}{n+2}. \tag{16}$$

The recursion holds for real n*. Moreover,*

$$K_0 = 2\,G, \qquad K_1 = 1, \tag{17}$$

$$E_0 = G + \frac{1}{2}, \qquad E_1 = \frac{2}{3}. \tag{18}$$

Here

$$G := \sum_{n\ge 0} \frac{(-1)^n}{(2n+1)^2} = L_{-4}(2)$$

is *Catalan's* constant whose irrationality is still not proven. This ignorance is part of our motivation for the current study. Indeed [1] uses this moment as a definition of G!

The current record for computation of G is 31.026 billion decimal digits in 2009. Computations often use the following central binomial formula due to Ramanujan [7, last formula] or its recent generalizations [10]:

$$\frac{3}{8}\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} + \frac{\pi}{8}\,\log(2+\sqrt{3}) = G. \tag{19}$$

Early in 2011, a string of base-4096 digits of Catalan's constant beginning at position 10 trillion was computed on an IBM *Blue Gene/ P* machine as part of a suite of similar computations [4]. The resulting confirmed base-8 digit string is

3470505377477705112261337162012525732721732452 2

(each quadruplet of base-8 digits corresponds to one base-4096 digit).

There are various ways to obtain the initial values, and one may also profitably study fractional moments, see below and [1].

2 Basic results

We commence in this section with various fundamental representations and evaluations. Then in section three we provide a generalization of Catalan's constant arising as the expectation of K^s. In section four we consider related contour integrals. Finally, in section five we look at negative and fractional moments.

2.1 Hypergeometric closed forms

A concise closed form for the moments is

Theorem 2 (Hypergeometric forms) *For* $0 \le s < \frac{1}{2}$ *we have*

$$K_{n,s} = \frac{\pi}{2(n+1)}\, {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}-s, \frac{1}{2}+s, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{matrix}\right| 1\right), \tag{20}$$

$$E_{n,s} = \frac{\pi}{2(n+1)}\, {}_3F_2\left(\left.\begin{matrix} -\frac{1}{2}-s, \frac{1}{2}+s, \frac{n+1}{2} \\ 1, \frac{n+3}{2} \end{matrix}\right| 1\right). \tag{21}$$

These hold in the limit for $s = \frac{1}{2}$.

Proof. To establish (20) and (21), we begin with

$$\begin{aligned} \int_0^1 x^{u-1}(1-x)^{v-1}\, {}_2F_1\left(\left.\begin{matrix} a, 1-a \\ b \end{matrix}\right| x\right) \mathrm{d}x &= \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(b)_n n!} \int_0^1 x^{n+u-1}(1-x)^{v-1}\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n(u)_n}{(b)_n(u+v)_n n!} \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \\ &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\, {}_3F_2\left(\left.\begin{matrix} a, 1-a, u \\ b, u+v \end{matrix}\right| 1\right). \end{aligned} \tag{22}$$

Similarly,

$$\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\, {}_3F_2\left(\left.\begin{matrix} a, -a, u \\ b, u+v \end{matrix}\right| 1\right) = \int_0^1 x^{u-1}(1-x)^{v-1}\, {}_2F_1\left(\left.\begin{matrix} a, -a \\ b \end{matrix}\right| x\right) \mathrm{d}x.$$

By applying these to (1) and (2) we immediately get (20) and (21). □

As long as $0 < s < 1/2$, the first series (20) is *Saalschützian* [14]. That is, the denominator parameters add to one more than those in the numerator, but is not well poised, and can be reduced to Gamma functions only for $n = \pm 1$ (with $n = -1$ a pole) since then it reduces to a ${}_2F_1$. The second (21) is not even Saalschützian, although it is nearly well poised (whose definition [14] we do not need) and also can be reduced to Gamma functions for $n = \pm 1$. Thus, for $|s| < 1/2$ we find

$$K_{1,s} = \frac{\cos \pi s}{1-4s^2}, \qquad E_{1,s} = \frac{2}{2s+3}\frac{\cos \pi s}{1-4s^2}. \tag{23}$$

In general we obtain:

Theorem 3 (Odd moments of K^s) *For odd integers $2m+1$ and $m = 0, 1, 2, \ldots$,*

$$K_{2m+1,s} = \frac{\cos \pi s\, m!^2}{4\,\Gamma\left(\frac{3}{2}-s+m\right)\Gamma\left(\frac{3}{2}+s+m\right)} \sum_{k=0}^{m} \frac{\Gamma\left(\frac{1}{2}-s+k\right)\Gamma\left(\frac{1}{2}+s+k\right)}{k!^2}. \tag{24}$$

Proof. In terms of the *Legendre function*,

$${}_2F_1\left(\left.\begin{matrix} a, 1-a \\ 1 \end{matrix}\right| z\right) =: P_{-a}(1-2z),$$

where

$$y = P_\nu(x) = {}_2F_1\left(\left.\begin{matrix} -\nu, \nu+1 \\ 1 \end{matrix}\right| \frac{1-x}{2}\right)$$

is a solution of the differential equation

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \nu(\nu+1)y = 0.$$

In consequence we may deduce that

$$\begin{aligned} {}_2F_1\left(\left.\begin{matrix} a, 1-a \\ 1 \end{matrix}\right| z\right) &= \frac{\sin \pi a}{\pi} \sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{k!^2} (1-z)^k \times \\ &\quad \left\{2\Psi(1+k) - \Psi(a+k) - \Psi(1-a+k) - \log(1-z)\right\}, \end{aligned} \tag{25}$$

where

$$\Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) \mathrm{d}t$$

is the *digamma* function, using [12, p. 44, first formula ($b = 1-a$)].

Now, by integrating the series (25) term-by-term and applying representation (22), we have

$$\begin{aligned} {}_3F_2\left(\left.\begin{matrix} a, 1-a, n \\ 1, n+1 \end{matrix}\right| 1\right) &= n \int_0^1 z^{n-1}\, {}_2F_1\left(\left.\begin{matrix} a, 1-a \\ 1 \end{matrix}\right| z\right) \mathrm{d}z \\ &= \frac{n! \sin \pi a}{\pi} \sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{k!(k+n)!} \times \\ &\quad \left\{\Psi(1+k) + \Psi(n+1+k) - \Psi(a+k) - \Psi(1-a+k)\right\}. \end{aligned}$$

We note in passing that this offers an apparently new approach for summing this class of hypergeometric series; we exploit (22) again in section 5.4.

Thence, for example, by creative telescoping, one finds for any positive integer n that

$${}_3F_2\left(\left.\begin{matrix} a, 1-a, n \\ 1, n+1 \end{matrix}\right| 1\right) = \frac{\Gamma(n)\,\Gamma(1+n)}{\Gamma(a+n)\,\Gamma(1-a+n)} \sum_{k=0}^{n-1} \frac{(a)_k (1-a)_k}{k!^2}. \tag{26}$$

Now, with $n = m+1$ in (26) we conclude the proof of Theorem 3. □

Similarly,

$$
{}_2F_1\left(\left.\begin{matrix} a, -a \\ 1 \end{matrix}\right| z\right) = \frac{\sin(\pi a)}{\pi a}\Big\{1 - a^2 \sum_{k=0}^{\infty} \frac{(a+1)_k(1-a)_k}{k!(k+1)!}(1-z)^{k+1} \times
$$
$$
[\Psi(a+1+k) + \Psi(1-a+k) - \Psi(k+1) - \Psi(k+2) + \ln(1-z)]\Big\}.
$$

For $m = 0$, Theorem 3 reduces to the evaluation given in (23). In general, it gives $\cos(\pi s)$ times a rational function. An equivalent, rather pretty, partial fraction decomposition is

$$
K_{2m+1,s} = \frac{\cos \pi s}{2} \sum_{k=0}^{m} \frac{m!^2}{(m-k)!(m+k+1)!}\left(\frac{1}{2k+1-2s} + \frac{1}{2k+1+2s}\right). \tag{27}
$$

This can easily be confirmed inductively, using say (76).

For $s = 0$ this result originates with Ramanujan. Adamchik [1] reprises its substantial history and extensions which include a formula due independently to Bailey and Hodgkinson in 1931 and which subsumes (26). A special case of Bailey's formula is

$$
{}_3F_2\left(\left.\begin{matrix} a, b, c+1 \\ a+b+n \end{matrix}\right| 1\right) = \frac{\Gamma(n)\Gamma(a+b+n)}{\Gamma(a+n)\Gamma(b+n)} \sum_{k}^{n-1} \frac{(a)_k(b)_k}{(c)_k(1)_k}. \tag{28}
$$

Example 1 (Digamma consequences) For $0 < a < 1/2$, consequences are neatly given using:

$$
\gamma(\nu) := \frac{1}{2}\left[\Psi\left(\frac{\nu+1}{2}\right) - \Psi\left(\frac{\nu}{2}\right)\right],
$$

for which

$$
\gamma\left(\frac{1}{2}\right) = \frac{\pi}{2}, \qquad \gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sqrt{2}} - \sqrt{2}\log(\sqrt{2}-1),
$$
$$
\gamma\left(\frac{1}{3}\right) = \frac{\pi}{\sqrt{3}} + \log 2, \qquad \gamma\left(\frac{1}{6}\right) = \pi + \sqrt{3}\log(2+\sqrt{3}).
$$

More generally,

$$
\sum_{k=0}^{\infty} \frac{(a)_k(1-a)_k}{\left(\frac{3}{2}\right)_k k!}\left[\Psi(k+1) + \Psi\left(k+\frac{3}{2}\right) - \Psi(k+a) - \Psi(k+1-a)\right] = \frac{2\gamma(a) - \pi\csc(\pi a)}{1-2a}.
$$

This in turn gives

$$
{}_3F_2\left(\left.\begin{matrix} a, 1-a, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix}\right| 1\right) = \frac{2\sin(\pi a)}{\pi(1-2a)}\gamma(a) - \frac{1}{1-2a}. \tag{29}
$$

Taking the limit as $a \to 1/2$ in (29) gives two useful specializations:

$$
(a) \qquad {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix}\right| 1\right) = \frac{4G}{\pi} \tag{30}
$$

$$
(b) \qquad \Psi'\left(\frac{1}{4}\right) = \pi^2 + 8\,G, \tag{31}
$$

with (30) being known but far from obvious. ◇

Example 2 (Odd moments of E^s) The corresponding form for $E_{2m+1,s}$ is:

$$\begin{aligned} E_{2m+1,s} &= \frac{\pi}{4\,(m+1)}\frac{1}{\Gamma(\frac{3}{2}+s)\Gamma(\frac{1}{2}-s)} + \frac{\pi}{4}\frac{m!}{\Gamma(\frac{1}{2}+s)\Gamma(-\frac{1}{2}-s)} \times \\ & \sum_{k=0}^{\infty}\frac{(\frac{3}{2}+s)_k(\frac{1}{2}-s)_k}{k!(k+m+2)!}\left\{\Psi\left(\frac{3}{2}+s+k\right)+\Psi\left(\frac{1}{2}-s+k\right)-\Psi\left(k+1\right)-\Psi\left(3+m+k\right)\right\}. \end{aligned}$$

This, however, can be replaced by

$$E_{2m-1,s} = \frac{\cos\pi s}{2(s+m)+1}\left\{\frac{1}{2s+1}+(2s+1)\sum_{k=0}^{m-1}\frac{(m-1)!^2}{(m-1-k)!(m+k)!}\frac{2k+1}{(2k+1)^2-4s^2}\right\}, \tag{32}$$

on combining (24) with (78) below. ◇

Example 3 (Other special values) For each $s \neq 0$ there are also two special values of r for which $K_{r,s}$ also reduce to a ${}_2F_1$. They are obtained by solving $r+3/2 = 1/2 \pm s$. This and similar calculations for $E_{n,s}$ yield

$$K_{(-2\pm 2s),s} = -\frac{\cos\pi s}{(1\mp 2s)^2}, \tag{33}$$

$$E_{(-2-2s),s} = -\frac{2}{(1+2\,s)}\frac{\cos(\pi\,s)}{(1-2\,s)^2}, \tag{34}$$

$$E_{(-4-2s),s} = -\frac{2}{(1+2\,s)}\frac{\cos(\pi\,s)}{(3+2\,s)^2}. \tag{35}$$

The r-recursions given above in (12) for $K_{r,s}$ and below in equation (78) for $E_{r,s}$ extend this to values of $r+2n$, for n integral. ◇

Example 4 (Alternative moment expansions) We also obtain

$$\begin{aligned} K_{0,s} &= \frac{\cos(\pi\,s)}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}+s\right)_n\left(\frac{1}{2}-s\right)_n}{n!\left(\frac{3}{2}\right)_n} \times \\ & \left\{\Psi\left(n+1\right)+\Psi\left(\frac{3}{2}+n\right)-\Psi\left(\frac{1}{2}+n+s\right)-\Psi\left(\frac{1}{2}+n-s\right)\right\}, \\ E_{0,s} &= \frac{\cos\pi s}{2s+1}+\cos\pi s\frac{2s+1}{6}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}+s\right)_n\left(\frac{1}{2}-s\right)_n}{n!\left(\frac{5}{2}\right)_n} \times \\ & \left\{\Psi\left(n+1\right)+\Psi\left(\frac{5}{2}+n\right)-\Psi\left(\frac{3}{2}+n+s\right)-\Psi\left(\frac{1}{2}+n-s\right)\right\}. \end{aligned}$$

◇

2.1.1 Half-integer values of s

For $s = m + 1/2$, and $m, n = 0, 1, 2\ldots$ we can obtain a terminating representation

$$\begin{aligned} K_{n,m+1/2} &= \frac{\pi}{2(n+1)}\, {}_3F_2\left(\left.{-m, m+1, \frac{n+1}{2} \atop 1, \frac{n+3}{2}}\right| 1\right) \\ &= \frac{(-1)^m \pi}{4} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2} - m\right)\Gamma\left(\frac{n+3}{2} + m\right)}, \end{aligned} \tag{36}$$

and likewise

$$E_{n,m+1/2} = \frac{\pi}{2} \sum_{k=0}^{m+1} \frac{(-m-1)_k\,(m+1)_k}{(n+1+2k)\,k!^2}. \tag{37}$$

2.2 The complementary integrals

By contrast, the complementary integral moments are somewhat less recondite.

Theorem 4 (Complementary moments) *For $n = 0, 1, 2, \ldots$ and $0 \le s < \frac{1}{2}$ we have*

$$K'_{n,s} = \frac{\pi}{4} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2-2s}{2}\right)\Gamma\left(\frac{n+2+2s}{2}\right)} \tag{38}$$

$$E'_{n,s} = \frac{\pi}{2(n+1)} \frac{\Gamma^2\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2-2s}{2}\right)\Gamma\left(\frac{n+4+2s}{2}\right)}. \tag{39}$$

These hold in the limit for $s = \frac{1}{2}$.

In particular, recursively we obtain for all real *n:*

$$\text{(a)}\quad K'_{n+2,s} = \frac{(n+1)^2}{(n+2)^2 - 4s^2} K'_{n,s}, \qquad \text{(b)}\quad E'_{n,s} = \frac{n+1}{n+2+2\,s} K'_{n,s}, \tag{40}$$

$$\text{where (c)}\quad K'_{0,s} = \frac{\pi}{4}\frac{\sin(\pi\, s)}{s}, \qquad \text{(d)}\quad K'_{1,s} = \frac{\cos \pi s}{1 - 4s^2}.$$

Proof. To establish (38) we recall that

$$K^{s'} = \frac{\pi}{2}\, {}_2F_1\left(\left.{\frac{1}{2} - s, \frac{1}{2} + s \atop 1}\right| 1 - k^2\right), \tag{41}$$

and so

$$\begin{aligned}
K'_{n,s} &= \frac{\pi}{2}\int_0^1 x^n \,{}_2F_1\left(\left.{\frac12 - s, \frac12 + s \atop 1}\right| 1-x^2\right) \mathrm{d}x \\
&= \frac{\pi}{4}\int_0^1 x^{\frac{n+1}{2}-1} {}_2F_1\left(\left.{\frac12 - s, \frac12 + s \atop 1}\right| 1-x\right) \mathrm{d}x \\
&= \frac{\pi}{4}\int_0^1 (1-x)^{\frac{n+1}{2}-1} \,{}_2F_1\left(\left.{\frac12 - s, \frac12 + s \atop 1}\right| x\right) \mathrm{d}x \\
&= \frac{\pi}{2(n+1)} \,{}_3F_2\left(\left.{\frac12 - s, \frac12 + s, 1 \atop 1, \frac{n+3}{2}}\right| 1\right) \\
&= \frac{\pi}{2(n+1)} \,{}_2F_1\left(\left.{\frac12 - s, \frac12 + s \atop \frac{n+3}{2}}\right| 1\right),
\end{aligned}$$

which is summable, by Gauss' formula (3), to the desired result.

The proof of (39) is similar, and the recursions follow. □

Example 5 (Complementary closed forms) Thence, with $s = 0$ and $n = 0, 1$ we recover

$$K_0' = \frac{\pi^2}{4}, \quad E_0' = \frac{\pi^2}{8}, \quad K_1' = 1, \quad E_1' = \frac{2}{3},$$

as discussed in [7, p. 198]. Correspondingly

$$K'_{0,1/6} = \frac{3\pi}{4}, \quad K'_{1,1/6} = \frac{9\sqrt{3}}{16}, \quad E'_{0,1/6} = \frac{9\pi}{28}, \quad K'_{1,1/6} = \frac{27\sqrt{3}}{80},$$

$$K'_{0,1/3} = \frac{3\sqrt{3}\pi}{8}, \quad K'_{1,1/3} = \frac{9}{10}, \quad E'_{0,1/3} = \frac{9\sqrt{3}\pi}{64}, \quad E'_{1,1/3} = \frac{27}{55}.$$

We note that π, not π^2 appears in these evaluations, since in (40, c), $\sin(\pi s)/s \to \pi$ as $s \to 0$. ◇

2.2.1 Connecting moments and complementary moments

We first remark that a comparison of Theorems 3 and 4 shows that for all s we have

$$K'_{1,s} = K_{1,s} \qquad \text{and} \qquad E'_{1,s} = E_{1,s}.$$

The formula

$$\int_0^1 K(k)\,\frac{\mathrm{d}k}{1+k} = \int_0^1 K\left(\frac{1-h}{1+h}\right)\frac{\mathrm{d}h}{1+h} = \frac12 \int_0^1 K'(k)\,\mathrm{d}k \tag{42}$$

is recorded in [7, p. 199]. It is proven by using the quadratic transform [7, Thm 1.2 (b), p. 12] for the second equality and a substitution for the first. This implies

$$2\sum_{n=0}^{\infty}(-1)^n K_n = \frac{\pi^2}{4} = K_0', \tag{43}$$

on appealing to Theorem 4.

The corresponding identity for $s = 1/6$ is best written

$$\int_0^1 {}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle|1-t^3\right)\mathrm{d}t \;=\; 3\int_0^1 {}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\middle|t^3\right)\frac{\mathrm{d}t}{1+2\,t}, \tag{44}$$

which follows analogously from the cubic transformation [9, Eqn 2.1] and a change of variables. This is a beautiful counterpart to (42) especially when the latter is written in hypergeometric form:

$$\int_0^1 {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle|1-k^2\right)\mathrm{d}k \;=\; 2\int_0^1 {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix}\middle|k^2\right)\frac{\mathrm{d}k}{1+k}. \tag{45}$$

We further evaluate equation (44) in (99) of section 5.4.

Additionally, [7, p. 188] outlines how to derive

$$\int_0^1 \frac{K(k)\,\mathrm{d}k}{\sqrt{1-k^2}} = K\left(\frac{1}{\sqrt{2}}\right)^2.$$

Using the same technique, we generalize this to

$$\int_0^1 \frac{K^s(k)\,\mathrm{d}k}{\sqrt{1-k^2}} = K^s\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\cos^2(\pi s)}{16\pi}\Gamma^2\left(\frac{1+2s}{4}\right)\Gamma^2\left(\frac{1-2s}{4}\right). \tag{46}$$

Here we have used Gauss' formula (3) for the evaluation

$$K^s\left(\frac{1}{\sqrt{2}}\right) = \frac{\cos \pi s}{4}\beta\left(\frac{1+2s}{4},\frac{1-2s}{4}\right).$$

By the generalized Legendre's identity (13), which simplifies as the complementary integrals coincide with the original ones at $1/\sqrt{2}$, we obtain

$$E^s\left(\frac{1}{\sqrt{2}}\right) = \frac{K^s\left(\frac{1}{\sqrt{2}}\right)}{2} + \frac{\pi\cos\pi s}{4(2s+1)K^s(\frac{1}{\sqrt{2}})}.$$

2.3 Analytic continuation of results

We finish this section by recalling a useful theorem:

Theorem 5 (Carlson (1914)) *Let f be analytic in the right half-plane $\Re z \geq 0$ and of exponential type (meaning that $|f(z)| \leq Me^{c|z|}$ for some M and c), with the additional requirement that*

$$|f(z)| \leq Me^{d|z|}$$

for some $d < \pi$ on the imaginary axis $\Re z = 0$. If $f(k) = 0$ for $k = 0, 1, 2, \ldots$ then $f(z) = 0$ identically.

Carlson's dissertation result [15, 5.81] allows us to prove that many of the results in this paper hold generally as soon as they hold for integer n. For example, the equations (75) or (76) hold generally as soon as the integral cases hold: once we check growth on the imaginary axis which is easy for hypergeometric functions. This matter is discussed at some length in [3, Thm 2.8.1 and sequel] — including an elegant 1941 proof by Selberg for the case where f is bounded in the right half-plane.

3 Closed form initial-values for various s

Many results work for all s (as we have seen) but a few others are more satisfactory when $s \in \Omega$ — since these four K^s are the only modular functions ([7, Prop 5.7], [9]) amongst the generalized elliptic integrals K^s.

Empirically, we discovered an algebraic relation

$$2(1+s)\,E_{0,s} - (1+2s)\,K_{0,s} = \frac{\cos \pi s}{1+2s}. \tag{47}$$

Equivalently, we exhibit a parametric series for $1/\pi$:

$$\frac{1}{\pi} = \frac{(1+2s)(2+2s)\,{}_3F_2\left(\left.{\frac{1}{2},\frac{1}{2}+s,-\frac{1}{2}-s \atop 1,\frac{3}{2}}\right|1\right) - (1+2s)^2\,{}_3F_2\left(\left.{\frac{1}{2},\frac{1}{2}+s,\frac{1}{2}-s \atop 1,\frac{3}{2}}\right|1\right)}{2\cos(\pi s)}.$$

On using (11) to eliminate $E_{0,s}$ in (47), it becomes

$$K_{2,s} = \frac{K_{0,s} + \cos(\pi\, s)}{4-4s^2} \tag{48}$$

which in turn is a special case of (76) with $r = \frac{1}{2}$ (as is justified by Carlson's Theorem 5), thus proving our empirical observation.

Hence, to resolve all integral values for a given s, we are left with looking for satisfactory representations only for $K_{0,s}$. We will write

$$G_s := \frac{1}{2}K_{0,s} = \frac{\pi}{4}\,{}_3F_2\left(\left.{\frac{1}{2},\frac{1}{2}-s,\frac{1}{2}+s \atop 1,\frac{3}{2}}\right|1\right).$$

and call this the associated or *generalized Catalan* constant. For various reasons, the results for $s = 1/6$ are especially interesting. This is the case corresponding to the cubic AGM [9].

3.1 Evaluation of G_s

From (20) we obtain

$$\begin{aligned} K_{0,s} = \frac{\pi}{2}\,{}_3F_2\left(\left.{\frac{1}{2},\frac{1}{2}-s,\frac{1}{2}+s \atop 1,\frac{3}{2}}\right|1\right) &= \frac{\cos \pi s}{2}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+n+s\right)\Gamma\left(\frac{1}{2}+n-s\right)}{(2\,n+1)\,(n!)^2} \\ &= \frac{\cos \pi\, s}{2}\sum_{n=0}^{\infty}\beta\left(n+\frac{1}{2}-s, n+\frac{1}{2}+s\right)\frac{\binom{2\,n}{n}}{2\,n+1} \\ &= \frac{\cos \pi s}{4}\int_0^1 \frac{\arcsin\left(2\sqrt{t-t^2}\right)}{t^{1+s}\,(1-t)^{1-s}}\mathrm{d}t \\ &= \frac{\cos \pi s}{2}\int_0^{\pi/2}\left\{\tan^{2s}\left(\frac{\theta}{2}\right) + \cot^{2s}\left(\frac{\theta}{2}\right)\right\}\frac{\theta}{\sin\theta}\,\mathrm{d}\theta. \end{aligned} \tag{49}$$

This uses the definition directly, see also [7, Prop 5.6], to attain the first identity after writing the rising factorials in terms of the β function, whose integral representation we use here:

$$\beta(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t.$$

We exchange integral and sum to arrive at the penultimate integral. Moving the integral to $[-1/2, 1/2]$ and then making various trig substitutions, we arrive at the final result in (49). For example, we have

$$K_{0,0} = \int_0^{\pi/2} \frac{\theta}{\sin\theta}\,\mathrm{d}\theta = 2G.$$

The final equality has various derivations [7, 1]. These include contour integration as explored in section 4.

If we now make the trigonometric substitution $t = \tan(\theta/2)$ in (49), and integrate the two resulting terms separately, we arrive at a central result.

Theorem 6 (Generalized Catalan constants for $0 \le s \le \frac{1}{2}$)

$$\begin{aligned} K_{0,s} &= \cos\pi s \int_0^1 \left(t^{2s-1} + t^{-2s-1}\right) \arctan t\,\mathrm{d}t \\ &= \frac{\cos\pi s}{8s}\left\{\Psi\left(\frac{3-2s}{4}\right) + \Psi\left(\frac{1+2s}{4}\right) - \Psi\left(\frac{1-2s}{4}\right) - \Psi\left(\frac{3+2s}{4}\right)\right\} \quad (50) \\ &= \frac{\cos\pi s}{4\,s}\left\{\Psi\left(\frac{s}{2}+\frac{1}{4}\right) - \Psi\left(\frac{s}{2}+\frac{3}{4}\right)\right\} + \frac{\pi}{4\,s} = 2\,G_s. \quad (51) \end{aligned}$$

Note that for $s = 0$, applying L'Hôpital's rule to (50) yields

$$K_{0,0} = \frac{1}{8}\Psi'\left(\frac{1}{4}\right) - \frac{1}{8}\Psi'\left(\frac{3}{4}\right)$$

which is precisely $2G$.

The digamma expression in (51) simplifies entirely when $s \in \Omega$ to the forms originally discovered in the next section. We now obtain complete evaluations for $s \in \Omega$, as was our goal.

Corollary 1 (Generalized Catalan values for s in Ω)

$$G_0 = G, \quad G_{1/6} = \frac{3}{4}\sqrt{3}\,\log 2, \quad G_{1/4} = \log\left(1+\sqrt{2}\right), \quad G_{1/3} = \frac{3}{8}\sqrt{3}\,\log\left(2+\sqrt{3}\right). \quad (52)$$

Mathematica, which currently knows more about the Ψ function than *Maple*, can evaluate the integral in Theorem 6 symbolically for some s. For example, if $s = 1/12$, after simplification we have the very nice expression:

$$G_{1/12} = 3\left(\sqrt{3}+1\right)\left\{\log\left(\sqrt{2}-1\right) + \frac{\sqrt{3}}{2}\log\left(\sqrt{3}+\sqrt{2}\right)\right\}.$$

More generally, the evaluation requires only knowledge of $\sin(\pi s/2)$, and hence we can determine which s give a reduction to radicals. As a last example,

$$G_{1/5} = \frac{5}{8}\sqrt{5+2\sqrt{5}}\left\{\frac{\sqrt{5}-1}{2}\operatorname{arcsinh}\left(\sqrt{5+2\sqrt{5}}\right) - \operatorname{arcsinh}\left(\sqrt{5-2\sqrt{5}}\right)\right\}.$$

3.2 Other generalizations of G

Two other famous representations of G are:

$$G = -\int_0^{\pi/2} \log\left(2\sin\frac{t}{2}\right)\,\mathrm{d}t \tag{53}$$

$$= \int_0^{\pi/2} \log\left(2\cos\frac{t}{2}\right)\,\mathrm{d}t \tag{54}$$

and

$$G = -\int_0^{\pi/2} \log(\tan t)\,\mathrm{d}t, \tag{55}$$

which easily follows from (53) and (54) . To prove (53) we integrate by parts and obtain

$$\begin{aligned} -\int_0^{\pi/2} \log\left(2\sin\frac{t}{2}\right)\,\mathrm{d}t &= 2\int_0^{\pi/4} t\cot t\,\mathrm{d}t - \frac{\pi}{4}\log 2 \\ &= 2\int_0^{\pi/4} {}_2F_1\left(\left.{\frac{1}{2},\frac{1}{2} \atop \frac{3}{2}}\right|\sin^2 t\right)\cos t\,\mathrm{d}t - \frac{\pi}{4}\log 2 \\ &= 2\int_0^{1/\sqrt{2}} \frac{\arcsin x}{x}\,\mathrm{d}x - \frac{\pi}{4}\log 2 \\ &= \left(G + \frac{\pi}{4}\log 2\right) - \frac{\pi}{4}\log 2 = G. \end{aligned}$$

The second and third equalities hold since $x\,{}_2F_1\left(\left.{\frac{1}{2},\frac{1}{2} \atop \frac{3}{2}}\right|x^2\right) = \arcsin x$. The final equality follows on integrating $\arcsin(x)/x$ term by term. *Inter alia*, we have shown that

$$G = \int_0^{\pi/2} \frac{t}{\sin t}\,\mathrm{d}t = \int_0^{\pi/2} {}_2F_1\left(\left.{\frac{1}{2},\frac{1}{2} \atop \frac{3}{2}}\right|\sin^2 t\right)\,\mathrm{d}t. \tag{56}$$

We may generalize (53) or equivalently (56) to:

Proposition 1

$$G_s = \frac{\cos\pi s}{2}\int_0^{\pi/2} \tan^{2s} t\,{}_2F_1\left(\left.{\frac{1}{2},\frac{1}{2}-s \atop \frac{3}{2}}\right|\sin^2 t\right)\,\mathrm{d}t. \tag{57}$$

Proof. We write

$$
\begin{aligned}
G_s &= \frac{1}{2}\int_0^1 K^s(k)\,\mathrm{d}k = \frac{\pi}{4}\int_0^1 {}_2F_1\left(\left.{\frac{1}{2}-s, \frac{1}{2}+s \atop 1}\right| k^2\right)\mathrm{d}k \\
&= \frac{\cos \pi s}{4}\int_0^1 t^{s-1/2}(1-t)^{-s-1/2}\,\mathrm{d}t\int_0^1 (1-k^2t)^{s-1/2}\,\mathrm{d}k \\
&= \frac{\cos \pi s}{4}\int_0^1 t^{s-1/2}(1-t)^{-s-1/2}\,{}_2F_1\left(\left.{\frac{1}{2}, \frac{1}{2}-s \atop \frac{3}{2}}\right| t\right)\mathrm{d}t \\
&= \frac{\cos \pi s}{2}\int_0^{\pi/2} \tan^{2s} u\,{}_2F_1\left(\left.{\frac{1}{2}, \frac{1}{2}-s \atop \frac{3}{2}}\right| \sin^2 u\right)\mathrm{d}u.
\end{aligned}
$$

□

Note that Theorem 2 gives a series for G_s for $0 \le s \le 1/2$:

$$
\begin{aligned}
\frac{4}{\pi} G_s &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}-s\right)_n \left(\frac{1}{2}+s\right)_n}{(n!)^2\,(2\,n+1)} \\
&= {}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}+s, \frac{1}{2}-s \atop 1, \frac{3}{2}}\right| 1\right). \qquad (58)
\end{aligned}
$$

Recalling (29) we recover Theorem 6 in the equivalent form

$$
G_s = \frac{\pi}{4}\,{}_3F_2\left(\left.{\frac{1}{2}-s, \frac{1}{2}+s, \frac{1}{2} \atop 1, \frac{3}{2}}\right| 1\right) = \frac{\cos \pi s}{4s}\,\gamma\left(\frac{1}{2}+s\right) - \frac{\pi}{8s}. \qquad (59)
$$

From (58) it is clear that G_s is monotonically decreasing from G to $\pi/4$ as s runs from 0 to $1/2$. In fact, G_s is concave on $[0, 1/2]$, as illustrated in Figure 1.

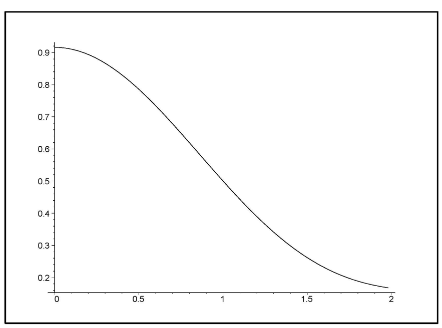


Figure 1: (58) plotted on $[0, 2]$.

4 Contour integrals for $K_{0,s}$

By contour integration on the infinite rectangle above $[0, \pi/2]$ we obtain

$$\begin{aligned} G_0 &= \frac{1}{2}\int_0^\infty \frac{t}{\cosh t}\,\mathrm{d}t \\ &= \int_0^\infty \frac{te^{-t}}{1+e^{-2t}}\,\mathrm{d}t = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2} = G. \end{aligned} \tag{60}$$

Here we have used the geometric series and integrated term by term the Γ function terms that we obtain. The final evaluation is definitional.

Done carefully, contour integration over the same rectangle, converting to exponentials, and then integrating term by term, provides a fine general integral evaluation:

Theorem 7 (Contour integral for G_s) *For $0 \leq s < 1/2$ we have*

$$\begin{aligned} 2\,G_s = K_{0,s} &= 2^{2s}\sin(2\pi s)\int_0^\infty \frac{(\cosh t)^{4s}-(\sinh t)^{4s}}{(\sinh 2t)^{2s+1}}\,t\,\mathrm{d}t + \\ &\quad \cos(\pi s)\int_0^\infty \frac{\cos(2\,s\arctan(\sinh t))}{\cosh t}\,t\,\mathrm{d}t. \end{aligned} \tag{61}$$

Example 6 (Experimentally obtained evaluations) For $s = 1/4$, equation (61) becomes

$$K_{0,1/4} = \sqrt{2}\int_0^\infty \frac{\cosh t - \sinh t}{(\sinh 2\,t)^{3/2}}\,t\,\mathrm{d}t + 2\,\sqrt{2}\int_0^\infty \frac{\cosh t}{(\cosh 2\,t)^{3/2}}\,t\,\mathrm{d}t, \tag{62}$$

with numerical value ≈ 1.7627471740392. Here for the first time the specific form of the root of unity has played a role. Quite remarkably, if we — much as before — convert the integrand to exponential form and apply the binomial theorem, we obtain Γ function values which become:

$$\begin{aligned} G_{1/4} &= \sum_{n=0}^{\infty}\binom{-\frac{3}{2}}{n}\frac{12\,n+8\,n^2+5+(-1)^n\,(2\,n+1)^2}{8\,(n+1)^2\,(2\,n+1)^2} \\ &= \log\left(1+\sqrt{2}\right). \end{aligned} \tag{63}$$

Having first proven this, we then discovered using the integer relation algorithm `PSLQ` and the *Maple* `identify` function that:

$$K_{0,1/6} = \frac{3}{2}\sqrt{3}\,\log 2, \tag{64}$$

with numerical value ≈ 1.8008492007794, and a similar evaluation:

$$K_{0,1/3} = \frac{3}{2}\sqrt{3}\,\log\left(1+\sqrt{3}\right) - \frac{3}{4}\sqrt{3}\,\log(2), \tag{65}$$

with numerical value ≈ 1.7107784916770. ◇

Example 7 (Further integrals) We have discovered additionally, using inverse symbolic computational methods (http://carma.newcastle.edu.au/isc2), that

$$\int_0^\infty \frac{(\cosh t)^{4/3} - (\sinh t)^{4/3}}{(\sinh t \cosh t)^{5/3}}\, t\, \mathrm{d}t = \frac{9}{4}\log(3),$$

and

$$\int_0^\infty \frac{(\cosh t)^{2/3} - (\sinh t)^{2/3}}{(\sinh t \cosh t)^{4/3}}\, t\, \mathrm{d}t = \frac{3}{2}\log\left(\frac{27}{16}\right).$$

In light of Corollary 1 these are now proven. ◇

4.1 Contour integral based series for $K_{0,s}$

Let us write

$$K_{0,s} = \sin(2\pi s)\, S(s) + \cos(\pi s)\, C(s) \tag{66}$$

where

$$S(s) := 2^{2s} \int_0^\infty \frac{(\cosh t)^{4s} - (\sinh t)^{4s}}{(\sinh 2t)^{2s+1}}\, t\, \mathrm{d}t \tag{67}$$

$$C(s) := \int_0^\infty \frac{\cos(2\, s \arctan(\sinh t))}{\cosh t}\, t\, \mathrm{d}t. \tag{68}$$

To evaluate $S(s)$ we make a substitution $u = \tanh(t)$. We obtain

$$\begin{aligned} S(s) &= \frac{1}{2}\int_0^1 (u^{-2s-1} - u^{2s-1}) \operatorname{arctanh}(u)\, \mathrm{d}u \\ &= \frac{-1}{8s}\left(2\gamma + 4\log(2) + \Psi\left(\frac{1}{2} - s\right) + \Psi\left(\frac{1}{2} + s\right)\right). \end{aligned} \tag{69}$$

Here γ denotes the *Euler-Mascheroni* constant.

To evaluate $C(s)$ we note that

$$\cos(2\, s \arctan(\sinh t)) = \cos(2\, s \arcsin(\tanh t)) = {}_2F_1\left(\begin{matrix} s, -s \\ \frac{1}{2} \end{matrix}\middle| \tanh^2 t\right) \tag{70}$$

and so we obtain a converging (finite if $s = 0$) series

$$C(s) = \int_0^\infty \frac{\cos(2\, s \arctan(\sinh t))}{\cosh t}\, t\, \mathrm{d}t = \sum_{n=0}^\infty \frac{(s)_n\, (-s)_n}{\left(\frac{1}{2}\right)_n} \frac{\tau_n}{n!}$$

where

$$\tau_n := \int_0^\infty \frac{x^{2n}}{(1+x^2)^{n+1}} \operatorname{arcsinh}(x)\, \mathrm{d}x, \tag{71}$$

and where we have expanded termwise. Moreover,

$$\tau_{m+2} = \frac{(13 + 8\,m^2 + 20\,m)\,\tau_{m+1} - 2\,(m+1)\,(2\,m+1)\,\tau_m}{2\,(m+2)\,(2\,m+3)} \tag{72}$$

where $\tau_0 = K_0 = 2G$ and $\tau_1 = E_0 = G + \frac{1}{2}$. In particular $C(0) = 2G$.

A closed form for τ_n is easily obtained. It is

$$\tau_n = \beta\left(n + \frac{1}{2}, \frac{1}{2}\right)\left\{\frac{2\,G}{\pi} + \frac{1}{4}\sum_{k=1}^{n}\frac{\Gamma\left(k\right)^2}{\Gamma\left(k+\frac{1}{2}\right)^2}\right\}. \tag{73}$$

Collecting up evaluations, we deduce that

$$\begin{aligned} K_{0,s} \;&=\; \sin\left(2\,\pi\,s\right)\left\{\frac{-1}{8s}\left(2\gamma + 4\log(2) + \Psi\left(\frac{1}{2} - s\right) + \Psi\left(\frac{1}{2} + s\right)\right)\right\} + \\ &\quad \frac{\sin(2\pi\,s)}{\pi s}\left\{G - \pi\sum_{k=0}^{\infty}\frac{\Gamma\left(k+s+1\right)\Gamma\left(k-s+1\right) - k!^2}{8\,\Gamma\left(k+\frac{3}{2}\right)^2}\right\}, \end{aligned}$$

since on interchanging order of summation

$$\frac{\pi}{4}\,\cos\left(\pi s\right)\sum_{n=1}^{\infty}\frac{(s)_n\,(-s)_n}{n!^2}\,\sum_{k=1}^{n}\frac{\Gamma\left(k\right)^2}{\Gamma\left(k+\frac{1}{2}\right)^2} = -\frac{\sin 2\pi\,s}{8s}\,\sum_{k=1}^{\infty}\frac{\Gamma\left(k+s\right)\Gamma\left(k-s\right) - \Gamma(k)^2}{\Gamma\left(k+\frac{1}{2}\right)^2}.$$

This ultimately yields:

Theorem 8 (Contour series for G_s)

$$G_s \;=\; \frac{\sin 2\pi s}{16s}\left(\sum_{k=1}^{\infty}\frac{\Gamma(k)^2 - \Gamma(k+s)\Gamma(k-s)}{\Gamma\left(k+\frac{1}{2}\right)^2} + 2\Psi\left(\frac{1}{2}\right) - 2\Psi\left(s + \frac{1}{2}\right) + \pi\tan(\pi s) + \frac{8G}{\pi}\right). \tag{74}$$

Example 8 (A related series) Note for $s = 0$ we obtain precisely $G_0 = G$ as all other terms in (74) are zero. Comparing, (74) to (50) leads to a closed form for the infinite series $Q(s)$ given by

$$\begin{aligned} Q(s) \;&:=\; \sum_{k=1}^{\infty}\frac{\Gamma\left(k+s\right)\Gamma\left(k-s\right) - \Gamma(k)^2}{\Gamma\left(k+\frac{1}{2}\right)^2} \\ &=\; \frac{8}{\pi}\int_0^{\pi/4}\frac{\left(\tan t\right)^{2\,s} + \left(\cot t\right)^{2\,s} - 2}{\cos 2t}\,t\,\mathrm{d}t \\ &=\; \frac{8}{\pi}\int_0^{1}\frac{\left(x^s - x^{-s}\right)^2}{1 - x^2}\,\arctan x\,\mathrm{d}x. \end{aligned}$$

The integrals above are obtained much as in the derivation of (74). For example,

$$Q\left(\frac{1}{4}\right) = \frac{8\,G}{\pi} - 4\,\log\left(1 + \frac{1}{\sqrt{2}}\right),$$

and there other nice evaluations. ◇

5 Closed forms at negative integers

We observe that (20) and (21) give analytic continuations which allow us to study negative moments. In [1] Adamchik studies such moments of K.

5.1 Negative moments

Adamchik's starting point is the study of $K_n = K_{n,0}$ for which Ramanujan appears to have known that

$$(2r+1)^2 K_{2r+1} - (2r)^2 K_{2r-1} = 1, \tag{75}$$

for $\Re r > -1/2$. For integer r this is a direct consequence of (24).

Experimentally, we found the following extension for general s by using integer relation methods with $s := 1/n$ to determine the coefficients:

$$\left((2r+1)^2 - 4s^2\right) K_{2r+1,s} - (2r)^2 K_{2r-1,s} = \cos \pi s. \tag{76}$$

For integer r this is established as follows — the general case then follows by Carlson's Theorem 5. Using (24) and the functional relation for the Γ function, we have:

$$\begin{aligned}
&\left((2r+1)^2 - 4s^2\right) K_{2r+1,s} - 4r^2 K_{2r-1,s} \\
=\; & \frac{\pi\,(r!)^2}{\Gamma(\frac{1}{2}+r-s)\Gamma(\frac{1}{2}+r+s)} \left\{ \sum_{k=0}^{r} \frac{(\frac{1}{2}-s)_k(\frac{1}{2}+s)_k}{(k!)^2} - \sum_{k=0}^{r-1} \frac{(\frac{1}{2}-s)_k(\frac{1}{2}+s)_k}{(k!)^2} \right\} \\
=\; & \frac{\pi\,(r!)^2}{\Gamma(\frac{1}{2}+r-s)\Gamma(\frac{1}{2}+r+s)} \frac{(\frac{1}{2}-s)_r(\frac{1}{2}+s)_r}{(r!)^2} \\
=\; & \frac{\pi}{\Gamma(\frac{1}{2}-s)\Gamma(\frac{1}{2}+s)} = \cos(\pi s).
\end{aligned}$$

From (76) by creative telescoping one again deduces

$$K_{2n+1,s} = \frac{\cos \pi s}{4} \frac{n!^2}{\Gamma\left(n+\frac{3}{2}+s\right)\Gamma\left(n+\frac{3}{2}-s\right)} \sum_{k=0}^{n} \frac{\Gamma\left(k+\frac{1}{2}+s\right)\Gamma\left(k+\frac{1}{2}-s\right)}{k!^2}. \tag{77}$$

This provides another proof of Theorem 3.

Equation (12), when combined with (76), implies

$$E_{n,s} = \frac{(2s+1)^2 K_{n,s} + \cos \pi s}{(2s+1)(2s+n+2)}, \tag{78}$$

which extends (16) and completes the proof in Example 2.

Adamchik also develops a reflection formula which in our terms is

$$K^*_{-1-2r} + K_{2r} = -\frac{\pi}{4^{2r}} \binom{2r}{r}^2 \{\log 2 + H_r - H_{2r}\} \tag{79}$$

for $r = 0, 1, 2, \ldots$. Here

$$K^*_{-1-2r} \quad := \quad \lim_{t \to r} \left\{ K_{-1-2t} - \frac{\binom{2n}{n}^2}{4^{2n+1}} \frac{\pi}{t-r} \right\}. \tag{80}$$

Note that, as examined in Theorem 9 of the next subsection, K^*_{-2r-1} removes the singularity at $-2r-1$. Hence, it can be written as an infinite sum [1].

Example 9 (Terminating sums) While studying [1] we found the following results.

1. For $0 < a \le 1$
$$ {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, a \\ 1, 1+a \end{matrix} \right| 1 \right) = \frac{4a}{\pi} \, {}_3F_2\left(\left. \begin{matrix} 1, 1, 1-a \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \right| 1 \right). \tag{81}$$

   In particular when $a = 1/2$ then

$$ {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \right| 1 \right) \quad = \quad \frac{2}{\pi} \, {}_3F_2\left(\left. \begin{matrix} 1, 1, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \right| 1 \right) = \frac{4}{\pi} G, \tag{82}$$

$$ {}_3F_2\left(\left. \begin{matrix} \frac{3}{4}, 1, 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \right| 1 \right) \quad = \quad \frac{\Gamma^4(1/4)}{16\pi}. \tag{83}$$

2. Moreover, for $n = 1, 2, 3, \ldots$

$$ {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, n \\ 1, 1+n \end{matrix} \right| 1 \right) \tag{84}$$

   always terminates. For example,

$$ {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, 1 \\ 1, 2 \end{matrix} \right| 1 \right) = \frac{4}{\pi}, \qquad {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, 2 \\ 1, 3 \end{matrix} \right| 1 \right) = \frac{40}{9\pi}. \tag{85}$$

3. Also for $n = 1, 2, \ldots$

$$ (2n+1)^2 \, {}_3F_2\left(\left. \begin{matrix} 1, 1, -n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \right| 1 \right) \quad - \quad 4\,n^2 \, {}_3F_2\left(\left. \begin{matrix} 1, 1, 1-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \right| 1 \right) = 1, \tag{86}$$

$$ {}_3F_2\left(\left. \begin{matrix} 1, 1, 1-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \right| 1 \right) \quad = \quad \frac{4^{2n-1}}{n^2 \binom{2n}{n}^2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{4^{2k}}, \tag{87}$$

   and

$$ {}_3F_2\left(\left. \begin{matrix} 1, 1, \frac{1}{2}-n \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \right| 1 \right) = \frac{\binom{2n}{n}^2}{4^{2n}} \left\{ 2G + \sum_{k=0}^{n-1} \frac{4^{2k}}{\binom{2k}{k}^2 (2k+1)^2} \right\}. \tag{88}$$

4. For $0 < a \le 1$ and $n = 1, 2, \ldots$

$$ {}_3F_2\left(\left.\begin{matrix}1,1,1-n-a\\ \frac{3}{2},\frac{3}{2}\end{matrix}\right|1\right) = \frac{(a)_n^2}{(a+\frac{1}{2})_n^2}\left\{ {}_3F_2\left(\left.\begin{matrix}1,1,1-a\\ \frac{3}{2},\frac{3}{2}\end{matrix}\right|1\right) + \frac{1}{4\,a^2}\sum_{k=0}^{n-1}\frac{(a+\frac{1}{2})_k^2}{(a+1)_k^2}\right\}, \tag{89}$$

and

$$ {}_3F_2\left(\left.\begin{matrix}1,1,-a\\ \frac{3}{2},\frac{3}{2}\end{matrix}\right|1\right) = \left(\frac{2a}{2a+1}\right)^2 {}_3F_2\left(\left.\begin{matrix}1,1,1-a\\ \frac{3}{2},\frac{3}{2}\end{matrix}\right|1\right) + \frac{1}{(2a+1)^2}. \tag{90}$$

5. Finally

$$\sum_{k=0}^{n}(-1)^k\frac{k!}{\Gamma^2(k+\frac{3}{2})(n-k)!} = \frac{n!}{\pi\Gamma^2(n+\frac{3}{2})}\sum_{k=0}^{n}\frac{\Gamma^2(k+\frac{1}{2})}{(k!)^2}. \tag{91}$$

$\diamond$

5.2 Analyticity of $K_{\cdot,s}$ for $0 \le s < 1/2$

The analytic structure of $r \mapsto K_{r,s}$ is similar qualitatively for all values of s. This is illustrated in Figure 2 for $s = 1/3$ and $s = 1/\pi$ both superimposed on $s = 0$ (red). In all cases there are simple poles at odd negative integers with computable residues.

Theorem 9 (Poles of $K_{\cdot,s}$) *Let $R_{n,s}$ denote the residue of $K_{\cdot,s}$ at $r = -2n+1$. Then*

$$(a)\ R_{n+1,s} = \frac{\left(n-\frac{1}{2}\right)^2 - s^2}{n^2}\,R_{n,s}, \qquad (b)\ R_{1,s} = \frac{\pi}{2}. \tag{92}$$

Explicitly

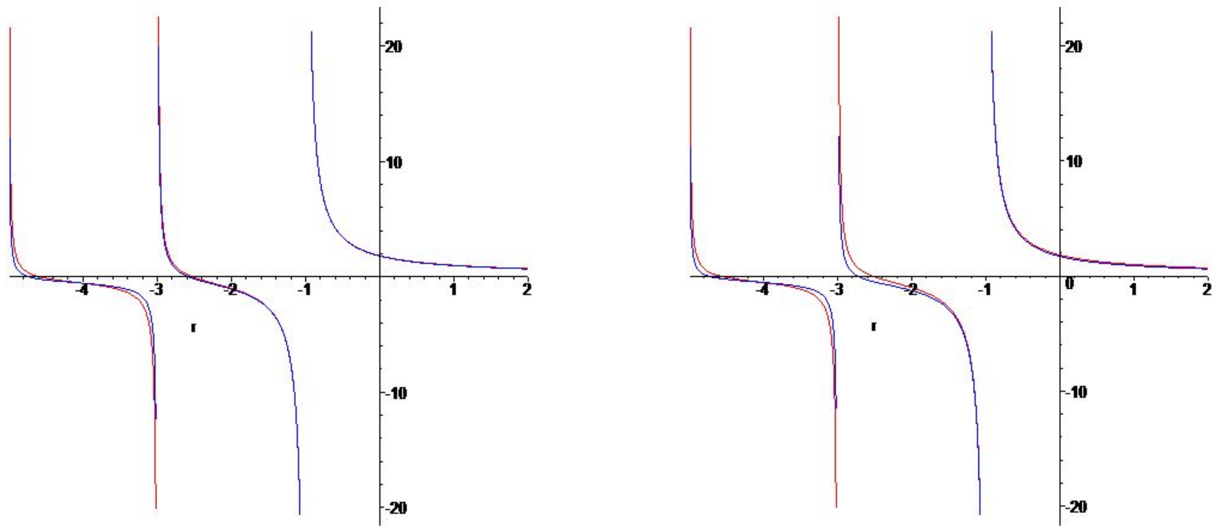
$$(c)\ R_{n,s} = \frac{\cos \pi s\,\Gamma\left(n-\frac{1}{2}+s\right)\Gamma\left(n-\frac{1}{2}-s\right)}{2\,\Gamma^2(n)}. \tag{93}$$

Proof. Recursion (92, a) follows from multiplying (76) by $2(r+n) = (2r+1)-(1-2n) = (2r-1)-(-2n-1)$ and computing the limits as $r \to -n$.

Directly from Theorem 2, we have the

$$R_{1,s} = \frac{\pi}{2}\lim_{r\to -1}\frac{r+1}{r+1}\,{}_3F_2\left(\left.\begin{matrix}\frac{1}{2}-s,\frac{1}{2}+s,\frac{r+1}{2}\\ 1,\frac{r+3}{2}\end{matrix}\right|1\right) = \frac{\pi}{2},$$

which is (b); part (c) follows easily as a telescoping product. $\square$



(a) $s = 0, 1/\pi$ (b) $s = 0, 1/3$

Figure 2: $r \mapsto K_{r,s}$ analytically continued to the real line.

5.3 Other rational values of s

Generally, directly integrating (1) or appealing to Theorem 2 yields the Saalschützian evaluation:

$$K_{(-1/2),s} = \pi\, {}_3F_2\left(\begin{matrix}\frac{1}{2}+s, \frac{1}{2}-s, \frac{1}{4}\\ 1, \frac{5}{4}\end{matrix}\middle|1\right). \tag{94}$$

For $s = 0$ only, $K_{-1/2,s}$ reduces to a case of Dixon's theorem [14, Eqn. (2.3.3.5)] and yields

$$K_{(-1/2),0} = \frac{\Gamma\left(\frac{1}{4}\right)^4}{16\,\pi}, \tag{95}$$

a result known to Ramanujan. Indeed, the two relevant specializations of Dixon's theorem are

$${}_3F_2\left(\begin{matrix}\frac{1}{2}+s, \frac{1}{2}-s, \frac{1}{4}\\ 1-2s, \frac{5}{4}\,s-1\end{matrix}\middle|1\right) = \frac{\Gamma\left(\frac{5}{4}-\frac{1}{2}\,s\right)\Gamma\left(\frac{1}{2}-\frac{3}{2}\,s\right)\Gamma\left(1-2\,s\right)\Gamma\left(\frac{5}{4}-s\right)}{\Gamma\left(\frac{3}{2}-s\right)\Gamma\left(\frac{3}{4}-2\,s\right)\Gamma\left(\frac{3}{4}-\frac{3}{2}\,s\right)\Gamma\left(1-\frac{1}{2}\,s\right)}$$

and more pleasingly,

$${}_3F_2\left(\begin{matrix}\frac{1}{4}, \frac{1}{2}-s, \frac{1}{2}+s\\ \frac{3}{4}+s, \frac{3}{4}-s\end{matrix}\middle|1\right) = \frac{\sqrt{2}\,\pi}{\Gamma^2\left(\frac{5}{8}\right)}\,\frac{\Gamma\left(\frac{3}{4}+s\right)\Gamma\left(\frac{3}{4}-s\right)}{\Gamma\left(\frac{5}{8}+s\right)\Gamma\left(\frac{5}{8}-s\right)}.$$

In the same way, we should like to be able to evaluate $K_{-1/3,1/6}$ and $K'_{-1/3,1/6}$ or equivalently

$$H_0 = \frac{\pi}{2}\int_0^1 {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix}\middle|t^3\right)\mathrm{d}t \quad \text{and} \quad H_0^* = \frac{\pi}{2}\int_0^1 {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix}\middle|1-t^3\right)\mathrm{d}t, \tag{96}$$

respectively. So far we have met with partial success, see (97) and (99) below.

5.4 Moments with respect to t^3 instead

To evaluate H_0^* we first write

$$H_0^* = \frac{\pi}{6}\int_0^1 x^{-\frac{2}{3}}\,{}_2F_1\left(\begin{matrix}\frac{\pi}{6},\frac{2}{3}\\ 1\end{matrix}\middle|1-x\right)\mathrm{d}x = \frac{\pi}{6}\int_0^1 (1-x)^{-\frac{2}{3}}\,{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\middle|x\right)\mathrm{d}x.$$

Now the integral (22) shows this is $\frac{\pi}{2}\,{}_3F_2\left(\begin{matrix}\frac{1}{3},\frac{2}{3},1\\ \frac{2}{3},\frac{4}{3}\end{matrix}\middle|1\right) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{3},1\\ \frac{4}{3}\end{matrix}\middle|1\right)$. By Gauss' formula (3) we arrive at

$$H_0^* = \frac{\pi}{2}\,\frac{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} = \frac{\sqrt{3}}{12}\,\Gamma^3\left(\frac{1}{3}\right). \tag{97}$$

This also follows directly from the analytic continuation of the formula in (38) of Theorem 4. Similarly,

$$H_0 = \frac{\pi}{6}\int_0^1 x^{-\frac{2}{3}}\;{}_2F_1\left(\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\middle|x\right)\mathrm{d}x = \frac{\pi}{3}\,{}_3F_2\left(\begin{matrix}\frac{1}{3},\frac{1}{3},\frac{2}{3}\\ 1,\frac{4}{3}\end{matrix}\middle|1\right).$$

If we use Bailey's identity:

$${}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\middle|1\right) = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)}\frac{\Gamma(s)}{\Gamma(b+s)\Gamma(c+s)}\,{}_3F_2\left(\begin{matrix}d-a,e-a,s\\ s+b,s+c\end{matrix}\middle|1\right)$$

for $s = d+e-a-b-c$, when $\mathrm{Re}(s>0)$, $\mathrm{Re}(a)>0$ [14, Eqn. (2.3.3.7)], this can be transformed to

$$H_0 = \frac{\pi}{6}\,\frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} - \frac{3\sqrt{3}}{16}\,{}_3F_2\left(\begin{matrix}1,1,1\\ \frac{5}{3},\frac{5}{3}\end{matrix}\middle|1\right)$$

which seems more promising. Next, applying (16.4.11) in the *Digital Library of Math Functions*

$${}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\middle|1\right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)}\,{}_3F_2\left(\begin{matrix}a,d-b,d-c\\ d,d+e-b-c\end{matrix}\middle|1\right),$$

we arrive at

$$H_0 \;=\; \frac{\pi}{6}\,\frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{2}{3})} - \frac{3\sqrt{3}}{4}\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{n}\frac{3k-1}{3k+1}}{3n+2}, \tag{98}$$

while

$$G \;=\; \sum_{n=0}^{\infty}\frac{\prod_{k=1}^{n}\frac{1-2k}{1+2k}}{2n+1}.$$

Finally, we also arrive at a reworking of equation (44):

$$3\sum_{k=0}^{\infty}(-2)^n H_n = 3\,H_0^* = \frac{\sqrt{3}}{4}\,\Gamma^3\left(\frac{1}{3}\right), \tag{99}$$

as a companion to (43).

6 Conclusion and open questions

Another impetus for this study was a query from Roberto Tauraso regarding whether, for integer $m = 0, 1, 2, \ldots$, one can find closed forms for

$$T(m,s) \quad := \quad \sum_{k=1}^{\infty} \frac{(\frac{1}{2}+s)_k\,(\frac{1}{2}-s)_k}{(1)_k^2}\frac{1}{k^m}. \tag{100}$$

We are able to write, more generally, that

$$\begin{aligned} T(m,s,\alpha) \quad &:= \quad \sum_{k=1}^{\infty} \frac{(\frac{1}{2}+s)_k\,(\frac{1}{2}-s)_k}{(1)_k^2}\frac{1}{(k+\alpha)^m} & (101)\\ &= \quad \frac{\frac{1}{4}-s^2}{(\alpha+1)^m}\,{}_{m+2}F_{m+1}\left(\left.\begin{matrix} \frac{3}{2}+s, \frac{3}{2}-s, \alpha+1, \cdots, \alpha+1 \\ 2, \alpha+2, \cdots, \alpha+2 \end{matrix}\right| 1\right). & (102) \end{aligned}$$

- Sad to say, we have nothing better to provide than the hypergeometric form of (102).
- We should also very much like to know if one can evaluate the cubic moment $H_0 = \frac{2}{3}K_{-1/3,1/6}$ other than in (96), (98) as we were able to do for $K_{-1/2,0}$. Both reduce to evaluation of cases of $\frac{\pi}{1+2s}\,{}_3F_2\left(\left.\begin{matrix}\frac{1}{2}-s, \frac{1}{2}+s\ \frac{s}{2}+\frac{1}{4}\\ 1, \frac{s}{2}+\frac{5}{4}\end{matrix}\right|1\right)$ $(s = 0, 1/6)$.
- Are there other non-trivial explicit fractional evaluations?
- What is the correct s-generalization of the reflection formula (80)?
- Finally, how do the connection results of (43), (99) generalize?

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