Uniform Bounds for the Incomplete Complementary Gamma Function

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Abstract. We prove upper and lower bounds for the complementary incomplete gamma function $\Gamma(a,z)$ with complex parameters a and z. Our bounds are refined within the circular hyperboloid of one sheet $\{(a,z) : |z| > c|a-1|\}$ with a real and z complex. Our results show that within the hyperboloid, $|\Gamma(a,z)|$ is of order $|z|^{a-1}e^{-\operatorname{Re}(z)}$, and extends an upper estimate of Natalini and Palumbo to complex values of z.

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1. Introduction

Euler's gamma function

$$\Gamma(a) := \int_0^\infty e^{-x}x^{a-1}dx, \tag{1.1}$$

defined for complex a, with simple poles at $-1, -2, \ldots$, is an important object in many areas of mathematics and has been widely studied. The incomplete gamma function $\gamma(a,z)$ and its complement $\Gamma(a,z)$, defined for $z > 0$ by

$$\gamma(a,z) := \int_0^z e^{-x}x^{a-1}dx \tag{1.2}$$

and

$$\Gamma(a,z) := \int_z^\infty e^{-x}x^{a-1}dx, \tag{1.3}$$

also appear in many different contexts and applications. For example, $\gamma(a,z)$ appears in the asymptotic expansions of the Bessel functions [10, pp. 204–205], and $\Gamma(a,z)$ is closely related to the generalized complementary error function

$$\operatorname{erfc}_p(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^p}dt = \frac{2}{p\sqrt{\pi}}\Gamma(\frac{1}{p}, x^p). \tag{1.4}$$

One can find an extended and highly readable overview on $\gamma(a,z)$ and $\Gamma(a,z)$ in [3].

Our present investigation was motivated by the need for explicit upper bounds on $|\Gamma(a,z)|$ in large regions of the complex z-plane in order to give effective asymptotic formulas for Laguerre Polynomials [2]. While there are many asymptotic formulas [5], [7], [8], [9], and inequalities [1], [4], [6] for $\Gamma(a,z)$ in the literature, these results were

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not applicable in the context of [2] either because the errors to the asymptotics were non-effective, or because the inequalities were only proved for restricted domains.

For example, Alzer [1, eq. 2.6] derived inequalities for the generalized complementary error function, and thus for $\Gamma(1/p, x^p)$ by (1.4), but only for real positive x and p. Of particular note are the inequalities of Natalini and Palumbo [4], which include the following: for $a > 1$, $B > 1$ and $x > \dfrac{B}{B-1}(a-1)$, we have

$$x^{a-1}e^{-x} < |\Gamma(a,x)| < Bx^{a-1}e^{-x}. \tag{1.5}$$

The inequalities of both Alzer and Natalini-Palumbo were proved for *real* a and z. In this note, we prove upper and lower bounds for $|\Gamma(a,z)|$ that are uniform in the region $|z| > c|a-1|$, where c is a constant and both a and z are complex. We also prove that our upper bound is asymptotically tight.

2. Bounds on $\Gamma(a,z)$

We begin by making a change of variable in (1.3) to obtain the following alternative definition for $\Gamma(a,z)$.

$$\Gamma(a,z) = z^a e^{-z} \int_0^\infty e^{-zs}(1+s)^{a-1}ds. \tag{2.1}$$

It is clear that (2.1) can be analytically continued for complex values of both a and z, and converges for all a when $\mathrm{Re}(z) > 0$, and at least for $\mathrm{Re}(a) < 0$ if $\mathrm{Re}(z) = 0$. With $\Gamma(a,z)$ expressed as in (2.1), estimating $|\Gamma(a,z)|$ reduces to estimating the integral

$$I(a,z) := \int_0^\infty e^{-zs}(1+s)^{a-1}ds. \tag{2.2}$$

For convenience, we write $z = x + iy$, $a = u + iv$, where u, v, x, $y \in \mathbb{R}$. We begin with two trivial estimates.

Theorem 2.1. *When $x > 0$, we have*

$$|I(a,z)| \le \begin{cases} \dfrac{1}{x-(u-1)}, & \text{if } u \ge 1, \\ \dfrac{1}{x}, & \text{if } u \le 1. \end{cases} \tag{2.3}$$

Proof. When $u \ge 1$, we have $0 < (1+s)^{u-1} \le (e^s)^{u-1}$ for $s \in [0,\infty)$. Therefore,

$$\begin{aligned} |I(a,z)| &\le \int_0^\infty \left|e^{-zs}(1+s)^{a-1}\right| ds \\ &\le \int_0^\infty e^{-xs}e^{(u-1)s}ds = \frac{1}{x-(u-1)}. \end{aligned}$$

When $u \le 1$, we use the estimate $0 \le (1+s)^{u-1} \le 1$. □

Theorem 2.2. *For $x \ge 0$ and $u < 0$, we have*

$$|I(a,z)| \le -\frac{1}{u}. \tag{2.4}$$

Proof. We apply the same argument as that of Theorem 2.1 except we bound $|e^{-zs}| < 1$ instead of $(1+s)^{u-1}$. This yields

$$\begin{aligned} |I(a,z)| &\leq \int_0^\infty \left| e^{-zs}(1+s)^{a-1} \right| ds \\ &\leq \int_0^\infty (1+s)^{u-1} ds = -\frac{1}{u}. \end{aligned}$$

□

We may now prove our key upper bound.

Theorem 2.3. *Let a and z be complex numbers with $\mathrm{Re}(z) > 0$, and set $\theta_n := \arg(a-n)$, where n is any positive integer. Then, for $u = \mathrm{Re}(a) \notin \mathbb{Z}$, we have,*

$$|I(a,z)| \leq \begin{cases} \dfrac{1}{|z|}\left(1 + |\sec\theta_1|\right), & \text{if } u < 1, \\ \dfrac{1}{|z|}\left(\displaystyle\sum_{k=0}^{N-2} \left| \frac{a-1}{z} \right|^k + (1 + |\sec\theta_N|) \left| \frac{a-1}{z} \right|^{N-1} \right), & \text{if } u > 1, \end{cases} \tag{2.5}$$

where $N = \lceil u \rceil$ is the smallest integer greater than or equal to u.

Proof. Since we are integrating with respect to the real variable s, we integrate by parts to find that

$$\begin{aligned} I(a,z) &= \int_0^\infty e^{-zs}(1+s)^{a-1} ds \\ &= -\frac{e^{-zs}}{z}(1+s)^{a-1}\Big|_{s=0}^{\infty} + \int_0^\infty \frac{a-1}{z} e^{-zs}(1+s)^{a-2} ds \\ &= \frac{1}{z} + \frac{a-1}{z} I(a-1,z). \end{aligned} \tag{2.6}$$

For $u < 1$, we have, by (2.6) and Theorem 2.2,

$$\begin{aligned} |I(a,z)| &\leq \frac{1}{|z|} + \left| \frac{1-a}{z} \right| \cdot \frac{1}{1-u} \\ &= \frac{1}{|z|}(1 + \sqrt{1 + \tan^2\theta_1}) = \frac{1}{|z|}(1 + |\sec\theta_1|). \end{aligned} \tag{2.7}$$

Now, suppose $u > 1$. By iterating (2.6), we find that for any fixed integer $N > 0$, we have

$$I(a,z) = \sum_{k=0}^{N-1} \frac{(a-1)(a-2)\cdots(a-k)}{z^{k+1}} + \frac{(a-1)\cdots(a-N)}{z^N} I(a-N,z). \tag{2.8}$$

Set $N = \lceil u \rceil$. Then for $1 \leq k < N$ we have,

$$|(a-1)\cdots(a-k)| \leq |a-1|^k.$$

since $u > 1$. Clearly, we also have

$$|(a-1)\cdots(a-N)| \leq |a-1|^{N-1} |a-N|. \tag{2.9}$$

Thus, using (2.8), (2.9), and Theorem 2.2, we find that

$$\begin{aligned}|I(a,z)| &\le \sum_{k=0}^{N-1}\left|\frac{(a-1)(a-2)\cdots(a-k)}{z^{k+1}}\right| + \left|\frac{(a-1)\cdots(a-N)}{z^N}I(a-N,z)\right| \\ &\le \frac{1}{|z|}\sum_{k=0}^{N-1}\frac{|a-1|^k}{|z|^k} + \frac{|a-1|^{N-1}|a-N|}{|z|^N}|I(a-N,z)| \\ &\le \frac{1}{|z|}\left(\sum_{k=0}^{N-1}\left|\frac{a-1}{z}\right|^k + \left|\frac{a-1}{z}\right|^{N-1}|\sec\theta_N|\right). \qquad (2.10)\end{aligned}$$

□

Note that θ_N tends to zero as $\mathrm{Im}(a)$ approaches zero, provided that $\mathrm{Re}(a) > 1$ and is not an integer. Hence, for real a, we may refine the estimate in (2.5) by improving the bound in (2.9).

Theorem 2.4. *Let z be complex with $\mathrm{Re}(z) > 0$ and a be real. Let $N = \lceil a \rceil$ as in Theorem 2.3. Then we have*

$$|I(a,z)| \le \begin{cases} \dfrac{2}{|z|}, & \text{if } a < 1, \\ \dfrac{1}{|z|}\displaystyle\sum_{k=0}^{N-1}\left|\frac{a-1}{z}\right|^k, & \text{if } a \ge 1, a \in \mathbb{Z}, \\ \dfrac{1}{|z|}\left(\displaystyle\sum_{k=0}^{N-1}\left|\frac{a-1}{z}\right|^k + \left|\frac{a-1}{z}\right|^{N-1}\frac{(N-1)!}{(N-1)^{N-1}}\right), & \text{if } a > 1, a \notin \mathbb{Z}. \end{cases} \qquad (2.11)$$

Proof. When $a < 1$, the inequality follows from Theorem 2.3 since $\theta_1 = \pi$. When $a > 1$, we follow the proof of Theorem 2.3 except that instead of the estimate (2.9), we use

$$|(a-1)\cdots(a-N+1)| \le |a-1|^{N-1}\prod_{k=0}^{N-1}\left|\frac{a-1-k}{a-1}\right| \le |a-1|^{N-1}\frac{(N-1)!}{(N-1)^{N-1}}, \qquad (2.12)$$

since for any $0 < m \le n$ and $\varepsilon > 0$ we have

$$\frac{m}{n} \le \frac{m+\varepsilon}{n+\varepsilon}.$$

Applying (2.12) and Theorem 2.2 to the Nth iterate of (2.6) gives the desired result, since when a is an integer, the product $(a-1)\cdots(a-N) = 0$. □

Given the form of the bound in Theorem 2.4, it is natural to consider what happens within the circular hyperboloid of one sheet given by $\{(a,z) : |z| > c|a-1|\}$. In fact, in this region, we obtain a very clean upper bound. The original requirement in [2] was met with $c = 2$.

Corollary 2.5. *Let $c > 1$. For complex z and real a with $\mathrm{Re}(z) > 0$, $a \ge 1$, and $|z| \ge c(a-1)$, we have*

$$|I(a,z)| \le \frac{1}{|z|}\cdot\frac{c}{c-1}. \qquad (2.13)$$

This is valid for $a \geq 1$ if $1 < c \leq 440$ and valid for $a > 2$ for all $c > 1$.

Proof. Since $|z| \geq c(a-1)$ we may bound (2.11) by

$$|I(a,z)| \leq \frac{1}{|z|}\left(\sum_{k=0}^{N-1} c^{-1} + \frac{(N-1)!}{(N-1)^{N-1}c^{N-1}}\right)$$
$$= \frac{1}{|z|}\left(\frac{c-c^{-N}}{c-1} + \frac{(N-1)!}{(N-1)^{N-1}c^{N-1}}\right).$$

By Stirling's Formula, we know that

$$n! < n^n e^{-n+1/12n}\sqrt{2\pi n}.$$

For convenience, let $n = N-1$. Thus

$$|I(a,z)| \leq \frac{1}{|z|}\left(\frac{c-c^{-n+1}}{c-1} + c^{-n}e^{-n+1/12n}\sqrt{2\pi n}\right)$$
$$= \frac{1}{|z|}\left(\frac{c}{c-1} + \frac{c^{-n}(c-1)e^{-n+1/12n}\sqrt{2\pi n} - c^{-n+1}}{c-1}\right)$$
$$\leq \frac{1}{|z|}\cdot\frac{c}{c-1}$$

whenever

$$(c-1)e^{-n+1/12n}\sqrt{2\pi n} - c \leq 0.$$

Or, equivalently,

$$e^{-n+1/12n}\sqrt{2\pi n} \leq 1 + \frac{1}{c-1}.$$

It is easy to see that the left-hand side is a decreasing function of n for $n \geq 1$, and is less than one if $n \geq 2$. The inequality is valid for $n = 1$ if

$$1 < c \leq 1 + \frac{1}{\sqrt{2\pi}e^{-11/12} - 1} \approx 440.66\ldots.$$

□

The upper bound in Corollary 2.5 is best possible in the following limiting sense.

Theorem 2.6.

$$\lim_{a\to\infty}(a-1)I(a,c(a-1)) = \frac{1}{c-1}. \tag{2.14}$$

Proof. We derive the result from the following well-known [3, Eq. 2.12], [9] asymptotic formula for $\Gamma(a,z)$.

$$\Gamma(a+1,x) = \frac{e^{-x}x^a}{x-a}\left(1 - \frac{a}{(x-a)^2} + \frac{2a}{(x-a)^3} + O\left(\frac{a^2}{(x-a)^4}\right)\right), \tag{2.15}$$

as $\sqrt{a}/(x-a)$ tends to zero. Thus, by the definition of $I(a,z)$, we find that,

$$\begin{aligned}\lim_{a\to\infty} aI(a+1,ca) &= \lim_{a\to\infty} a\Gamma(a+1,ca)e^{ca}\,(ca)^{-a} \\ &= \lim_{a\to\infty} \frac{a}{ca-a}\left(1+O\left(\frac{a}{(ca-a)^2}\right)\right) \\ &= \frac{1}{c-1}, \end{aligned} \tag{2.16}$$

since $\sqrt{a}/\,(ca-a) \to 0$ as $a\to\infty$. Replacing a with $a-1$ gives the stated claim. □

3. A lower bound

To determine a lower bound for $|I(a,z)|$, we apply the functional equation (2.6).

Theorem 3.1. *Let $c \geq 2$. Let z be complex and a be real with* $\mathrm{Re}(z) > 0$*, $a > 3$, and $|z| \geq c(a-1)$. Then we have*

$$|I(a,z)| \geq \frac{1}{|z|}\cdot\frac{c-2}{c-1}. \tag{3.1}$$

Proof. By (2.6), we have

$$I(a,z) - \frac{a-1}{z}I(a-1,z) = \frac{1}{z}.$$

Therefore, by the triangle inequality we find that

$$|I(a,z)| \geq \frac{1}{|z|} - \frac{|a-1|}{|z|}|I(a-1,z)|.$$

Since $|z| \geq c(a-1) > c(a-1-1)$ and $a-1 > 2$ we may apply the upper bound in Corollary 2.5 to $|I(a-1,z)|$. The result follows after some simplification. □

We close with a two-sided corollary.

Corollary 3.2. *Let c, z, and a be as in Theorem 3.1, and recall that $x = \mathrm{Re}(z)$. Then*

$$|z|^{a-1}e^{-x}\cdot\frac{c-2}{c-1} \leq |\Gamma(a,z)| \leq |z|^{a-1}e^{-x}\cdot\frac{c}{c-1}. \tag{3.2}$$

Let us compare (3.2) with the Natalini-Palumbo bound (1.5). If we let $c = B/(B-1)$, we readily find that $B = c/(c-1)$. Thus, we see that the upper bound in (3.2) extends (1.5) to complex z. However, even though the lower bound in (3.2) is of the same order as that of (1.5), it is much weaker when c is near 2.

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