

# Hilbert's Inequality and Witten's Zeta



**Jonathan M. Borwein, FRSC**



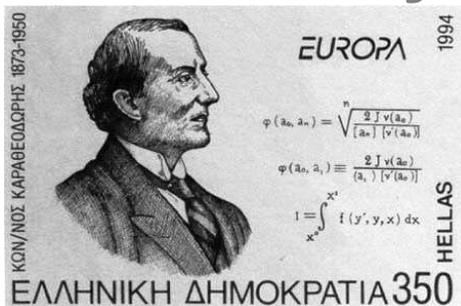
Research Chair in IT  
Dalhousie University

Halifax, Nova Scotia, Canada

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Constantin  
Carathéodory



MAA 1936

*I'll be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors, but this may occasionally be of use for the actual advancement of science.*

## **Abstract.**

We explore a variety of pleasing connections between analysis, number theory and operator theory, while exposing a number of beautiful inequalities originating with Hilbert. We shall first establish the afore-mentioned inequality [10,11,14] and then apply it to various multiple zeta values. In consequence we obtain the norm of Hilbert's matrix.

## **1. Hilbert's (easier) Inequality**

A useful preparatory lemma is

**Lemma 1** For  $0 < a < 1$  and  $n = 1, 2, \dots$

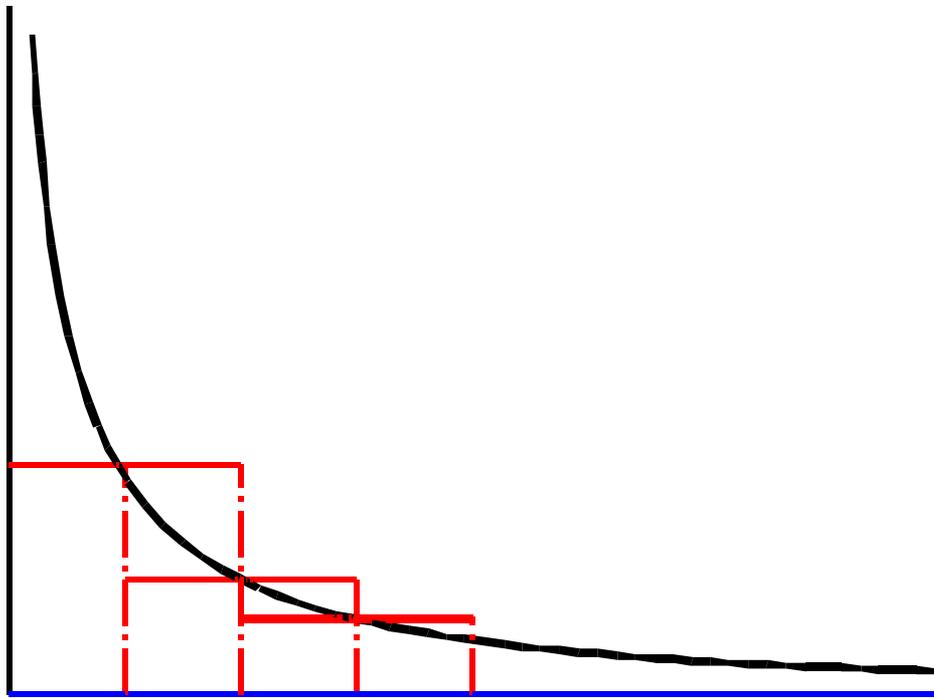
$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{(n+m)(m/n)^a} &< \int_0^{\infty} \frac{1}{(1+x)x^a} dx \\ &< \frac{(1/n)^{1-a}}{1-a} + \sum_{m=1}^{\infty} \frac{1}{(n+m)(m/n)^a}, \end{aligned}$$

and

$$\int_0^{\infty} \frac{1}{(1+x)x^a} dx = \pi \operatorname{csc}(a\pi).$$

**Proof.** The inequalities comes from: standard rectangular upper and lower approximations to a monotonic integrand; and overestimating the integral from 0 to  $1/n$ :

$$0 < \int_0^t \frac{1}{(1+x)x^a} dx \leq \frac{t^{1-a}}{1-a}$$



The evaluation is in tables and is known to *Maple* or *Mathematica*. We offer two other proofs: **(i)**

$$\begin{aligned}
 \int_0^\infty \frac{1}{(1+x)x^a} dx &= \int_0^1 \frac{x^{-a} + x^{a-1}}{1+x} dx \\
 &= \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{1}{n+1-a} + \frac{1}{n+a} \right\} \\
 &= \sum_{n=1}^{\infty} (-1)^n \left\{ \frac{1}{n+a} - \frac{1}{n-a} \right\} + \frac{1}{a} \\
 &= \frac{1}{a} + \sum_{n=1}^{\infty} \frac{(-1)^n 2a}{a^2 - n^2} = \pi \operatorname{csc}(a\pi)
 \end{aligned}$$

since the last equality is the classical *partial fraction* identity for  $\pi \operatorname{csc}(a\pi)$ .

**(ii)** Alternatively, we begin by letting  $1+x := 1/y$ ,

$$\begin{aligned}
 \int_0^\infty \frac{x^{-a}}{1+x} dx &= \int_0^1 y^{a-1} (1-y)^{-a} dy \\
 &= B(a, 1-a) = \Gamma(a) \Gamma(1-a) = \frac{\pi}{\sin(a\pi)}.
 \end{aligned}$$



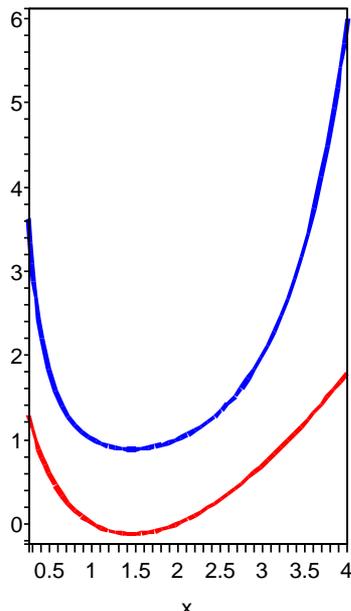
Combining the arguments in (i) and (ii) above actually derives the sine identity

$$\Gamma(a) \Gamma(1 - a) = \frac{\pi}{\sin(a\pi)},$$

from the partial fraction for cosec or vice versa

- Especially if we appeal to the result below to establish  $B(a, 1 - a) = \Gamma(a) \Gamma(1 - a)$

**Theorem 1 (Bohr-Mollerup, [1])** *The  $\Gamma$ -function is uniquely determined on  $(0, \infty)$  by*



1.  $G(1)=1$
2.  $G(x+1)=x G(x)$
3.  $\log G$  is convex

- So  $\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) dt$  (use Hölder for 3.)

## Theorem 2 (Hilbert Inequality)

For non-negative sequences  $(a_n)$  and  $(b_n)$ , not both zero, and for  $1 \leq p, q \leq \infty$  with  $1/p + 1/q = 1$  one has

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n+m} < \pi \operatorname{csc} \left( \frac{\pi}{p} \right) \|a_n\|_p \|b_n\|_q.$$

**Proof.** Fix  $\lambda > 0$ . We apply Hölder's inequality with '*compensating difficulties*' to obtain

$$\begin{aligned} & \sum_{n,m=1}^{\infty} \frac{a_n b_m}{n+m} \\ = & \sum_{n,m=1}^{\infty} \frac{a_n}{(n+m)^{1/p} (\mathbf{m/n})^{\lambda/p}} \frac{b_m}{(n+m)^{1/q} (\mathbf{n/m})^{\lambda/p}} \end{aligned} \quad (1)$$

$$\begin{aligned} & \leq \left( \sum_{n=1}^{\infty} |a_n|^p \sum_{m=1}^{\infty} \frac{(m/n)^{-\lambda}}{(n+m)} \right)^{1/p} \left( \sum_{m=1}^{\infty} |b_m|^q \sum_{n=1}^{\infty} \frac{(n/m)^{-\lambda q/p}}{(n+m)} \right)^{1/q} \\ & < \pi |\operatorname{csc}(\pi \lambda)|^{1/p} |\operatorname{csc}((q-1)\pi \lambda)|^{1/q} \|a_n\|_p \|b_m\|_q, \end{aligned} \quad (2)$$

so that the left hand side of (1) is no greater than  $\pi \operatorname{csc} \left( \frac{\pi}{p} \right) \|a_n\|_p \|b_n\|_q$  on setting  $\lambda = 1/q$  and appealing to symmetry in  $p, q$ . ■

- The integral analogue of (2) may likewise be established. There are numerous extensions.

One of interest for us later is

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{(n+m)^{\tau}} < \left\{ \pi \operatorname{csc} \left( \frac{\pi(q-1)}{\tau q} \right) \right\}^{\tau} \|a_n\|_p \|b_n\|_q \quad (3)$$

valid for  $p, q > 1, \tau > 0, 1/p + 1/q \geq 1$  and

$$\tau + 1/p + 1/q = 2.$$



- The best constant

$$C(p, q, \tau) \leq \left\{ \pi \operatorname{csc} \left( \frac{\pi(q-1)}{\tau q} \right) \right\}^{\tau}$$

is called a *Hilbert constant* [9]

For  $p = 2$ , (2) becomes Hilbert's original inequality:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n+m} \leq \pi \sqrt{\sum_{n=1}^{\infty} |a_n|^2} \sqrt{\sum_{n=1}^{\infty} |b_n|^2}, \quad (4)$$

though Hilbert only obtained the constant  $2\pi$ , [10].

# Fourier Series (Toeplitz)

A fine direct *Fourier analysis* proof starts from

$$\frac{1}{2\pi i} \int_0^{2\pi} (\pi - t) e^{int} dt = \frac{1}{n}$$

for  $n = 1, 2, \dots$ , and deduces

$$\sum_{n=1}^N \sum_{m=1}^N \frac{a_n a_m}{n+m} = \frac{1}{2\pi i} \int_0^{2\pi} (\pi - t) \sum_{k=1}^N a_k e^{ikt} \sum_{k=1}^N b_k e^{ikt} dt \quad (5)$$

We recover (4) by applying the integral form of the *Cauchy-Schwarz inequality* to the integral side of the representation (5).

**Example 1** Likewise

$$\sum_{n=1}^N \sum_{m=1}^N \frac{a_n a_m}{(n+m)^2} = \frac{1}{2\pi} \int_0^{2\pi} \left( \zeta(2) - \frac{\pi t}{2} + \frac{1}{4} \right) \sum_{k=1}^N a_k e^{ikt} \sum_{k=1}^N b_k e^{ikt}$$

and more generally

$$\sum_{n=1}^N \sum_{m=1}^N \frac{a_n a_m}{(n+m)^\sigma} = \frac{1}{2\pi i^\sigma} \int_0^{2\pi} \psi_\sigma \left( \frac{t}{2\pi} \right) \sum_{k=1}^N a_k e^{ikt} \sum_{k=1}^N b_k e^{ikt} dt$$

where

$$\psi_{2n}(x) = \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}}, \quad \psi_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n+1}}.$$



The  $\psi_\sigma(x)$  are related to *Bernoulli polynomials*:

$$\psi_\sigma(x) = (-1)^{\lfloor (1+\sigma)/2 \rfloor} B_\sigma(x) \frac{(2\pi)^\sigma}{2\sigma!},$$

for  $0 < x < 1$ .

- It follows that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n a_m}{(n+m)^\sigma} \leq \|\psi_\sigma\|_{[0,1]} \|a\|_2 \|b\|_2,$$

where for  $n > 0$  we compute

- $\|\psi_{2n}\|_{[0,1]} = \psi_{2n}(0) = \zeta(2n)$
- $\|\psi_{2n+1}\|_{[0,1]} = \psi_{2n+1}(1/4) = \beta(2n+1)$ .
- Hence, there is no known closed-form for  $\zeta(3)$ .

# Godfrey Harold Hardy



This and much more of the early 20th century history, and philosophy, of the “*bright and amusing*” subject of inequalities charmingly discussed in Hardy’s retirement lecture as London Mathematical Society Secretary, [10]. GHH comments (p. 474) that:

*Harald Bohr is reported to have remarked “Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove.”*

This remains true, though more recent inequalities often involve less linear objects such as *entropies* and log-barrier functions [1,5] like the *divergence estimate* [4, p. 63]:

$$\sum_{n=1}^N p_i \log \left( \frac{p_i}{q_i} \right) \geq \frac{1}{2} \left( \sum_{n=1}^N |p_i - q_i| \right)^2 ,$$

for positive sequences with  $\sum_{i=1}^N p_i = \sum_{i=1}^N q_i = 1$ .

► Two other high-spots in Hardy's essay are:

### *Carleman's inequality*

$$\sum_{n=1}^{\infty} (a_1 \cdot a_2 \cdot \dots \cdot a_n)^{1/n} \leq e \sum_{n=1}^{\infty} a_n,$$

see also [2, p. 284],

and

### *Hardy's inequality*

$$\sum_{n=1}^{\infty} \left( \frac{a_1 + a_2 + \dots + a_n}{n} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p, \quad (6)$$

for  $p > 1$ .

Hardy remarks [10, p. 485] that

[his] *“own theorem was discovered as a by-product of my own attempt to find a really simple and elementary proof of Hilbert's.”*

► For  $p = 2$ , Hardy offers Elliott's proof of (7), writing "*it can hardly be possible to find a proof more concise or elegant*".

**Theorem 3 (Hardy)** For all positive sequences

$$\sum_{n=1}^{\infty} \left( \frac{a_1 + a_2 + \cdots + a_n}{n} \right)^2 \leq 4 \sum_{n=1}^{\infty} a_n^2 \quad (7)$$

**Proof.** Set  $A_n := a_1 + a_2 + \cdots + a_n$  and write

$$\begin{aligned} \frac{2a_n A_n}{n} - \left( \frac{A_n}{n} \right)^2 &= \frac{A_n^2}{n} - \frac{A_{n-1}^2}{n-1} + (n-1) \left( \frac{A_n}{n} - \frac{A_{n-1}}{n-1} \right)^2 \\ &\geq \frac{A_n^2}{n} - \frac{A_{n-1}^2}{n-1} \end{aligned}$$

- something easy to check symbolically, and sum to obtain

$$\sum_n \left( \frac{A_n}{n} \right)^2 \leq 2 \sum_n \frac{a_n A_n}{n} \leq 2 \sqrt{\sum_n a_n^2} \sqrt{\sum_n \left( \frac{A_n}{n} \right)^2}$$

which proves (7) for  $p = 2$ . ■

► This easily adapts to the general case

Finally we record the (harder) *Hilbert inequality*:

$$\left| \sum_{n \neq m \in \mathbf{Z}} \frac{a_n b_m}{n - m} \right| < \pi \sqrt{\sum_{n=1}^{\infty} |a_n|^2} \sqrt{\sum_{n=1}^{\infty} |b_n|^2} \quad (8)$$

- The best constant  $\pi$  is due to Schur in (1911), [13]. There are many extensions—with applications to prime number theory, [13].

## 2. Witten $\zeta$ -functions

Let us recall that initially for  $r, s > 1/2$ :

$$\mathcal{W}(r, s, t) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^r m^s (n+m)^t}$$

is a *Witten  $\zeta$ -function*, [15,8,7]. We refer to [15] for a description of the uses of more general Witten  $\zeta$ -functions. Ours are also called *Tornheim double sums*, [8]. Correspondingly

$$\zeta(t, s) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^s (n+m)^t} = \sum_{n > m > 0} \frac{1}{n^t m^s}$$

is an *Euler double sum* or a *multi zeta-value* (MZV), [1,2, 7,15].

# Algebraic Relations

There is a simple algebraic relation

$$\mathcal{W}(r, s, t) = \mathcal{W}(r - 1, s, t + 1) + \mathcal{W}(r, s - 1, t + 1) \quad (9)$$

This is based on writing

$$\frac{m + n}{(m + n)^{t+1}} = \frac{m}{(m + n)^{t+1}} + \frac{n}{(m + n)^{t+1}}$$

$$\text{Also } \mathcal{W}(r, s, t) = \mathcal{W}(s, r, t), \quad (10)$$

and

$$\mathcal{W}(r, s, 0) = \zeta(r) \zeta(s) \text{ while } \mathcal{W}(r, 0, t) = \zeta(r, t) \quad (11)$$

- Hence,  $\mathcal{W}(s, s, t) = 2 \mathcal{W}(s, s - 1, t + 1)$  and so

$$\mathcal{W}(1, 1, 1) = 2 \mathcal{W}(1, 0, 2) = 2 \zeta(2, 1) = 2 \zeta(3).$$

- Reference [3] has *many* instructive proofs of basic identity

$$\zeta(2, 1) := \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{n-1} \frac{1}{m} = \sum_{n=1}^{\infty} \frac{1}{n^3} =: \zeta(3)$$

The *reflection formula* analogue to (9),

$$\zeta(s, t) + \zeta(t, s) = \zeta(s) \zeta(t) - \zeta(s + t),$$

shows  $\mathcal{W}(s, 0, s) = 2 \zeta(s, s) = \zeta^2(s) - \zeta(2s)$ .

- So  $\mathcal{W}(2, 0, 2) = 2 \zeta(2, 2) = \frac{\pi^4}{36} - \frac{\pi^4}{90} = \frac{\pi^4}{72}$ .

**“Lisez Euler, lisez Euler, c’est notre maitre a tous.”** (Bill Dunham) A letter follows in which Goldbach precisely formulates the series which sparked Euler’s investigations into what would become known as the Zeta-function. He confirmed Goldbach’s evaluation  $\zeta(3, 1) + \zeta(4) = \frac{\pi^4}{72}$ . This was all due to serendipitous mistake:

When I recently considered further the indicated sums of the last two series in my previous letter, I realized immediately that the same series arose due to a mere writing error, from which indeed the saying goes, “Had one not erred, one would have achieved less.” (*Si non errasset, fecerat ille minus*).

**Example 2** Let  $a_n := 1/n^r$ ,  $b_n := 1/n^s$ . Then inequality (4) becomes

$$\mathcal{W}(r, s, 1) \leq \pi \sqrt{\zeta(2r)} \sqrt{\zeta(2s)}. \quad (12)$$

Similarly, the inequality (1) becomes

$$\mathcal{W}(r, s, 1) \leq \pi \operatorname{csc} \left( \frac{\pi}{p} \right) \sqrt[p]{\zeta(pr)} \sqrt[q]{\zeta(qs)}. \quad (13)$$

Also (3) allows estimates of  $\mathcal{W}(r, s, \tau)$  for some  $\tau \neq 1$ .

- Note (12) implies  $\zeta(3) \leq \pi^3/3$ , on using (16).

More generally, recursive use of (9) and (10), along with initial conditions (11) shows that *all integer  $\mathcal{W}(s, r, t)$  values are expressible in terms of double (and single) Euler sums* (are *reducible*).

As we shall see in (18) representations are necessarily homogeneous polynomials of *weight*  $r + s + t$ .

- All double sums of weight less than 8 and all those of *odd* weight reduce to sums of products of single variable zeta values. The first irreducibles are  $\zeta(6, 2), \zeta(5, 3)$ , [2].

# Analytic Relations

In terms of the *polylogarithm*  $Li_s(t) := \sum_{n>0} t^n/n^s$  for real  $s$ , representation (5) yields

$$\mathcal{W}(r, s, 1) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \sigma Li_r(-e^{i\sigma}) Li_s(-e^{i\sigma}) d\sigma. \quad (14)$$

This representation is not numerically effective. It is better to start with

$$\Gamma(s)/(m+n)^t = \int_0^1 (-\log \sigma)^{t-1} \sigma^{m+n-1} d\sigma$$

and so to obtain

$$\mathcal{W}(r, s, t) = \frac{1}{\Gamma(t)} \int_0^1 Li_r(\sigma) Li_s(\sigma) \frac{(-\log \sigma)^{t-1}}{\sigma} d\sigma \quad (15)$$

- This real variable analogue of (14) is somewhat more satisfactory computationally: we recover an analytic proof of

$$\begin{aligned} 2 \zeta(2, 1) = \mathcal{W}(1, 1, 1) &= \int_0^1 \frac{\ln^2(1-\sigma)}{\sigma} d\sigma \\ &= 2 \zeta(3) \end{aligned} \quad (16)$$

- Integration by parts yields

$$\begin{aligned}\mathcal{W}(r, s + 1, 1) &= \mathcal{W}(r + 1, s, 1) \quad (17) \\ &= \text{Li}_{r+1}(1) \text{Li}_{s+1}(1) = \zeta(r + 1) \zeta(s + 1)\end{aligned}$$

So, in particular,  $\mathcal{W}(s + 1, s, 1) = \zeta^2(s + 1)/2$ .

- Symbolically, *Maple* immediately evaluates  $\mathcal{W}(2, 1, 1) = \pi^4/72$ .
- It fails directly with  $\mathcal{W}(1, 1, 2)$ , but we know it must be a multiple of  $\pi^4$  or equivalently  $\zeta(4)$ ; and numerically obtain

$$\mathcal{W}(1, 1, 2)/\zeta(4) = .499999999999999999999999998 \dots$$

Continuing, for  $r + s + t = 5$  the only base terms to consider are  $\zeta(5)$ ,  $\zeta(2)\zeta(3)$ , and *PSLQ* yields the weight five relations (as predicted):

$$\mathcal{W}(2, 2, 1) = \int_0^1 \frac{\text{Li}_2(x)^2}{x} dx = 2 \zeta(3) \zeta(2) - 3 \zeta(5)$$

$$\mathcal{W}(2, 1, 2) = \int_0^1 \frac{\text{Li}_2(x) \log(1-x) \log(x)}{x} dx = \zeta(3) \zeta(2) - \frac{3}{2} \zeta(5)$$

$$\mathcal{W}(1, 1, 3) = \int_0^1 \frac{\log^2(x) \log^2(1-x)}{2x} dx = -2 \zeta(3) \zeta(2) + 4 \zeta(5)$$

$$\mathcal{W}(3, 1, 1) = \int_0^1 \frac{\text{Li}_3(x) \log(1-x)}{x} dx = -\zeta(3) \zeta(2) + 3 \zeta(5).$$

Likewise, for  $r + s + t = 6$  the only terms we need to consider are  $\zeta(6)$ ,  $\zeta^2(3)$  since  $\zeta(6)$ ,  $\zeta(4) \zeta(2)$  and  $\zeta^3(2)$  are all rational multiples of  $\pi^6$ .



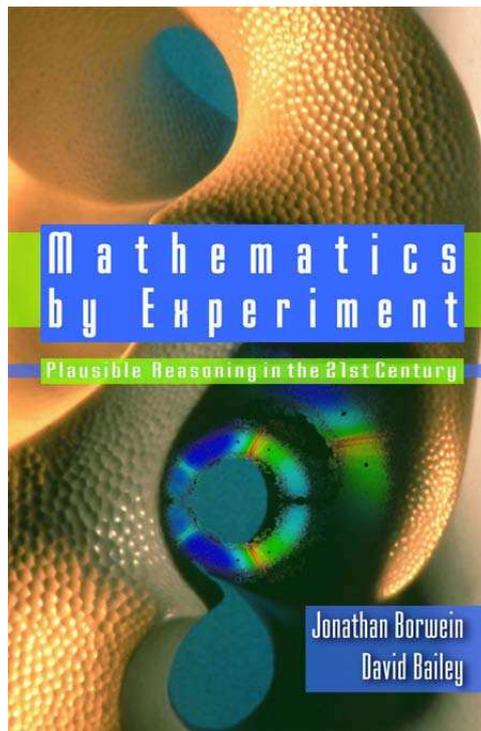
- We recover identities like

$$\mathcal{W}(3, 2, 1) = \int_0^1 \frac{\text{Li}_3(x) \text{Li}_2(x)}{x} dx = \frac{1}{2} \zeta^2(3),$$

consistent with the equation below (17).

The general form of the reduction, for integer  $r, s$  and  $t$ , is due to Tornheim and expresses  $\mathcal{W}(r, s, t)$  in terms of  $\zeta(a, b)$  with weight  $a + b = N := r + s + t$ :

$$\mathcal{W}(r, s, t) = \sum_{i=1}^{r \vee s} \left\{ \binom{r+s-i-1}{s-1} \binom{r+s-i-1}{r-1} \right\} \zeta(i, N-i)$$



- This and various other general formulas for classes of sums such as  $\mathcal{W}(2n+1, 2n+1, 2n+1)$  and  $\mathcal{W}(2n, 2n, 2n)$  are given in [8]

# The Best Constant

The constant  $\pi$  in Theorem 2 is best possible, [10].

**Example 3** Let us numerically explore the ratio, as  $s \rightarrow 1/2^+$ , of

$$\mathcal{R}(s) := \frac{\mathcal{W}(s, s, 1)}{\pi \zeta(2s)}$$

Note that  $\mathcal{R}(1) = 12 \zeta(3)/\pi^3 \sim 0.4652181552\dots$

Initially, we may directly sum as follows:

$$\begin{aligned} \mathcal{W}(s, s, 1) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^{-s} n^{-s}}{m+n} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \sum_{m=1}^{n-1} \frac{1/n}{(m/n)^s (m/n + 1)} + \frac{\zeta(2s+1)}{2} \\ &\leq 2 \zeta(2s) \int_0^1 \frac{x^{-s}}{1+x} dx + \frac{\zeta(2s+1)}{2} \\ &\leq 2 \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \sum_{m=1}^n \frac{1/n}{(m/n)^s (m/n + 1)} + \frac{\zeta(2s+1)}{2} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \sum_{m=1}^{n-1} \frac{1/n}{(m/n)^s (m/n + 1)} + \frac{3\zeta(2s+1)}{2} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^{-s} n^{-s}}{m+n} + \zeta(2s+1). \end{aligned}$$

We deduce that

$$\mathcal{R}(s) \sim \mathcal{I}(s) := \frac{2}{\pi} \int_0^1 x^{-s}/(1+x) dx$$

as  $s \rightarrow 1/2$ .

Further numerical explorations seem in order.

- For  $1/2 < s < 1$ , (15) is hard to use numerically and led us to look for a more sophisticated attack like the Hurwitz-zeta and Bernoulli polynomial integrals used in [8]. More fruitful were the expansions in [7, (2.6) and (2.9)]:

$$\mathcal{W}(r, s, t) = \int_0^1 E(r, x) E(s, x) \overline{E(t, x)} dx \quad (18)$$

where  $E(s, x) := \sum_{n=1}^{\infty} e^{2\pi i n x} n^{-s} = \text{Li}_s(e^{2\pi i x})$ , via

$$E(s, x) = \sum_{m=0}^{\infty} \zeta(s-m) \frac{(2\pi i x)^m}{m!} + \Gamma(1-s) (-2\pi i x)^{s-1}$$

for  $|x| < 1$  and with  $\eta(s) := (1 - 2^{1-s}) \zeta(s)$  for  $0 < x < 1$

$$E(s, x) = - \sum_{m=0}^{\infty} \eta(s-m) \frac{(2x-1)^m (\pi i)^m}{m!}$$

Indeed, carefully expanding (18) with a free parameter  $\theta \in (0, 1)$ , leads to the following efficient formula *when neither  $r$  nor  $s$  is an integer*:

$$\begin{aligned}
\Gamma(t)\mathcal{W}(r, s, t) &= \sum_{m, n \geq 1} \frac{\Gamma(t, (m+n)\theta)}{m^r n^s (m+n)^t} \\
&+ \sum_{u, v \geq 0} (-1)^{u+v} \frac{\zeta(r-u)\zeta(s-v)\theta^{u+v+t}}{u!v!(u+v+t)} \\
&+ \Gamma(1-r) \sum_{v \geq 0} (-1)^v \frac{\zeta(s-v)\theta^{r+v+t-1}}{v!(r+v+t-1)} \\
&+ \Gamma(1-s) \sum_{u \geq 0} (-1)^u \frac{\zeta(r-u)\theta^{s+u+t-1}}{u!(s+u+t-1)} \\
&+ \Gamma(1-r)\Gamma(1-s) \frac{\theta^{r+s+t-2}}{r+s+t-2} \quad (19)
\end{aligned}$$

When one or both of  $r, s$  is an integer, a limit formula with a few more terms results.

We can now accurately plot  $\mathcal{R}$  and  $\mathcal{I}$  on  $[1/3, 2/3]$ , as shown in Figure 1, and so are lead to:

**Conjecture.**  $\lim_{s \rightarrow 1/2} \mathcal{R}(s) = 1.$

**Proof.** To establish this, we denote

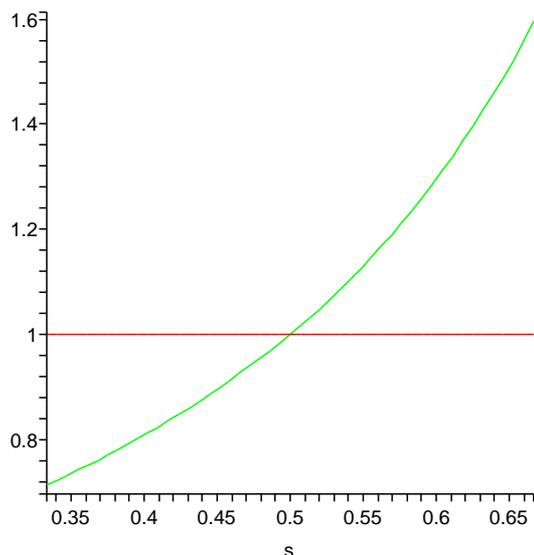
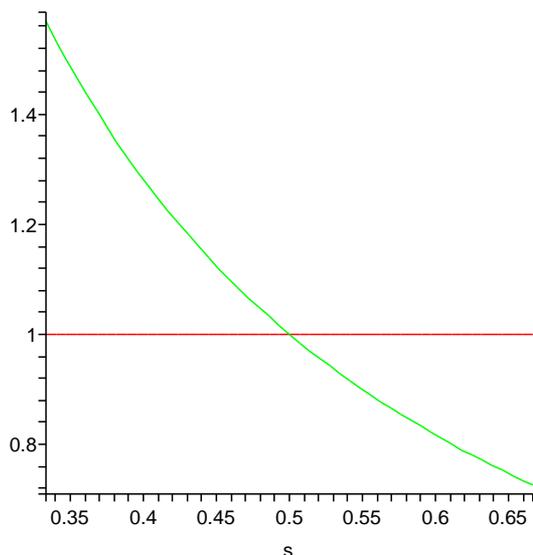
$$\sigma_n(s) := \sum_{m=1}^{\infty} n^s m^{-s} / (n + m)$$

and appeal to Lemma 1 to write

$$\begin{aligned} \mathcal{L} &:= \lim_{s \rightarrow 1/2} (2s - 1) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^{-s} m^{-s}}{n + m} \\ &= \lim_{s \rightarrow 1/2} (2s - 1) \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \sigma_n(s) \\ &= \lim_{s \rightarrow 1/2} (2s - 1) \sum_{n=1}^{\infty} \frac{\{\sigma_n(s) - \pi \operatorname{CSC}(\pi s)\}}{n^{2s}} \\ &\quad + \lim_{s \rightarrow 1/2} \pi (2s - 1) \zeta(2s) \operatorname{CSC}(\pi s) = 0 + \pi, \end{aligned}$$

since, by another appeal to Lemma 1, the parenthetic term is  $O(n^{s-1})$  while in the second  $\zeta(2s) \sim 1/(2s - 1)$  as  $s \rightarrow 1/2^+$ .

In consequence, we see that  $\mathcal{L} = \lim_{s \rightarrow 1/2} \mathcal{R}(s) = 1$ , and—at least to first-order—inequality (4) is best possible, see also [12]. ■



**$\mathcal{R}$ (left) and  $\mathcal{I}$  (right) on  $[1/3, 2/3]$**

Likewise, the constant in (2) is best possible. Motivated by above argument we consider

$$\mathcal{R}_p(s) := \frac{\mathcal{W}((p-1)s, s, 1)}{\pi \zeta(ps)},$$

and observe that with

$$\sigma_n^p(s) := \sum_{m=1}^{\infty} (n/m)^{-(p-1)s} / (n+m) \rightarrow \pi \operatorname{csc} \left( \frac{\pi}{q} \right),$$

we have:

$$\begin{aligned}
\mathcal{L}_p &:= \lim_{s \rightarrow 1/p} (ps - 1) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^{-s} m^{-(p-1)s}}{n+m} \\
&= \lim_{s \rightarrow 1/p} (ps - 1) \sum_{n=1}^{\infty} \frac{1}{n^{ps}} \sigma_n^p(s) \\
&= \lim_{s \rightarrow 1/p} (ps - 1) \sum_{n=1}^{\infty} \frac{\left\{ \sigma_n^p(s) - \pi \operatorname{CSC}(\pi/q) \right\}}{n^{ps}} \\
&\quad + \lim_{s \rightarrow 1/p} (2s - 1) \zeta(ps) \pi \operatorname{CSC}\left(\frac{\pi}{q}\right) \\
&= 0 + \pi \operatorname{CSC}\left(\frac{\pi}{q}\right).
\end{aligned}$$

Setting

$$r := (p-1)s, s \rightarrow 1/p^+$$

we check that  $\zeta(ps)^{1/p} \zeta(qr)^{1/q} = \zeta(ps)$  and hence the best constant in (13) is the one given.  $\blacksquare$

To recapitulate in terms of the celebrated infinite *Hilbert matrices* (see [2, pp. 250–252])

$$\mathcal{H}_0 := \left\{ \frac{1}{m+n} \right\}_{m,n=1}^{\infty}, \quad \mathcal{H}_1 := \left\{ \frac{1}{m+n-1} \right\}_{m,n=1}^{\infty}$$

we have actually proven:

**Theorem 4** *For  $1 < p, q < \infty$  with  $1/p + 1/q = 1$ , the Hilbert matrices  $\mathcal{H}_0$  and  $\mathcal{H}_1$  determine bounded linear mappings from  $\ell^p$  to itself such that*

$$\begin{aligned} \|\mathcal{H}_1\|_{p,p} = \|\mathcal{H}_0\|_{p,p} &= \lim_{s \rightarrow 1/p} \frac{\mathcal{W}(s, (p-1)s, 1)}{\zeta(ps)} \\ &= \pi \operatorname{CSC} \left( \frac{\pi}{p} \right). \end{aligned}$$

**Proof.** Appealing to the isometry between  $(\ell^p)^*$  and  $\ell^q$ , we directly compute the operator norm of  $\mathcal{H}_0$  as

$$\begin{aligned} \|\mathcal{H}_0\|_{p,p} &= \sup_{\|x\|_p=1} \|\mathcal{H}_0 x\|_p \\ &= \sup_{\|y\|_q=1} \sup_{\|x\|_p=1} \langle \mathcal{H}_0 x, y \rangle \\ &= \pi \operatorname{CSC} \left( \frac{\pi}{p} \right). \end{aligned}$$

Now clearly  $\|\mathcal{H}_0\|_{p,p} \leq \|\mathcal{H}_1\|_{p,p}$ . For  $n \geq 2$  we have

$$\sum_{m=1}^{\infty} \frac{1}{(n+m-1)(m/n)^a} \leq \sum_{m=1}^{\infty} \frac{1}{(n-1+m)(m/(n-1))^a} \\ \leq \pi \operatorname{csc}(\pi a)$$

an appeal to Lemma 1 and Theorem 2 shows

$$\|\mathcal{H}_0\|_{p,p} \geq \|\mathcal{H}_1\|_{p,p}. \quad \blacksquare$$

- A delightful operator-theoretic introduction to the Hilbert matrix  $\mathcal{H}_0$  is given by Choi in his Chauvenet prize winning article [6] while there is a nice recent set of notes by G.J. O. Jameson, see [12].
- In the case of (3), Finch ([9, §4.3]) comments that the issue of best constants is unclear in the literature. He remarks that even the case  $p = q = 4/3, \tau = 1/2$  appears to be open.

*It seems improbable* that the techniques of this note can be used to resolve the question.

Indeed, consider  $\mathcal{R}_{1/2}(s, \alpha) := \mathcal{W}(s, s, 1/2) / \zeta(4s/3)^\alpha$ , with the critical point in this case being  $s = 3/4$ . Numerically, using (19) we discover that

$$\log(\mathcal{W}(s, s, 1/2)) / \log(\zeta(4s/3)) \rightarrow 0.$$

Hence, for any  $\alpha > 0$ ,  $\lim_{s \rightarrow 3/4} \mathcal{R}_{1/2}(s, \alpha) = 0$ , which is certainly not the norm.

- We are exhibiting that the subset of sequences  $(a_n) = (n^{-s})$  for  $s > 0$  is *norming* in  $\ell^p$  for  $\tau = 1$  but not apparently for general  $\tau > 0$ .

**The corresponding behaviour of Hardy's inequality.** Setting  $a_n := 1/n$  in (7) and notating  $H_n := \sum_{k=1}^n 1/k$  yields

$$\sum_{n=1}^{\infty} \left( \frac{H_n}{n} \right)^p \leq \left( \frac{p}{p-1} \right)^p \zeta(p).$$

Application of the integral test and the evaluation

$$\int_1^{\infty} \left( \frac{\log x}{x} \right)^p dx = \frac{\Gamma(1+p)}{(p-1)^{p+1}},$$

for  $p > 1$  shows the constant is best possible. ■



**Coxeter's favourite 4-D polytope**  
(with 120 dodecahedral faces)

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