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Monotone Operators as Convex Objects

AOC
Analyse – Optimisation – Contrôle

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"The formulas move in advance of thought, while the intuition often lags behind; in the oft-quoted words of d'Alembert, ***"L'algebre est genereuse, elle donne souvent plus qu'on lui demande.""***

Edward Kasner, "The Present Problems of Geometry," Bull. AMS (1905) vol XI, p.285.

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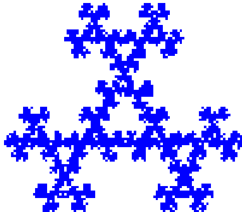
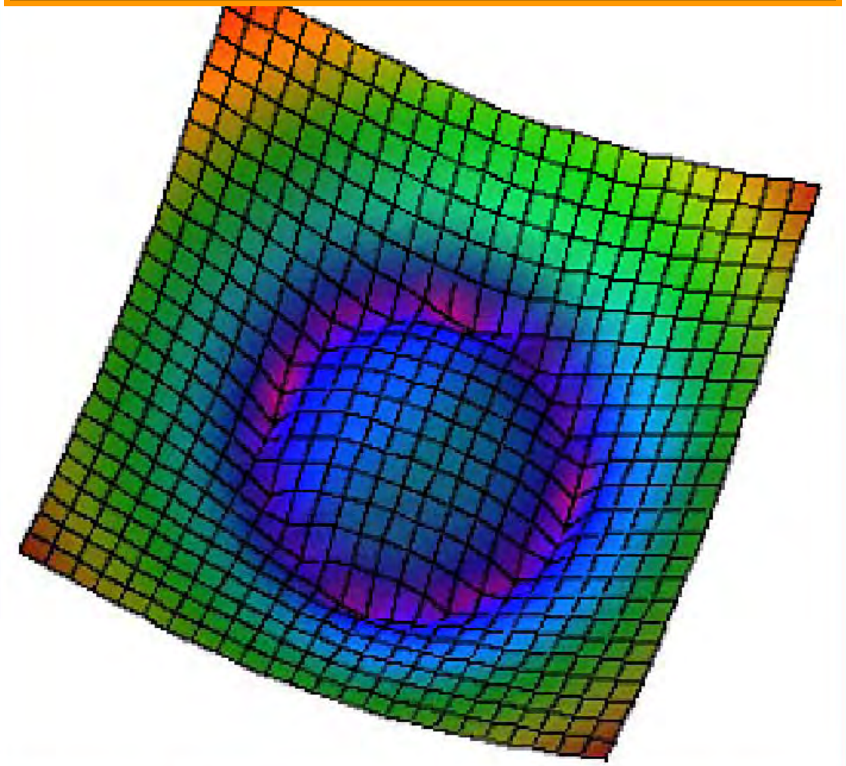


Convex Functions

Jonathan Borwein, FRSC
Dalhousie University

Encyclopedia of Mathematics and Applications



Jon Vanderwerff
La Sierra University

Coming in **2007** to a website near you. A Cambridge University Press Book provisionally entitled
Convex Functions: A Handbook.
This book is intended for:

- Researchers, practitioners and students
- In computation, optimization, analysis

Convex Functions

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Links

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Over the next few months this page will be populated with selected links to related material: nice books, reliable web sites and the like.

PLEASE ADD FUNCTIONS

- The Convex Function Database
- The Werner Fenchel Archive
- CVX: Steve Boyd's **Disciplined Convex Programming** site and his Convex programming course
- CECM's Computational Convex Analysis site
- CECM's Entropy Interface

Monotone Operators as Convex Objects

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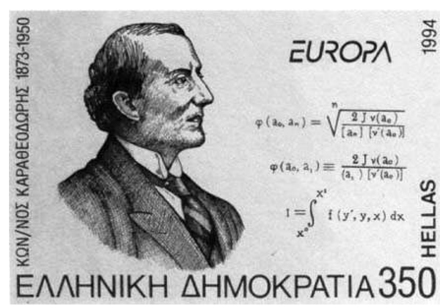
Fitzpatrick Memorial Workshop

Perth, September 25–26, 2005

> *I'll be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors, but this may occasionally be of use for the actual advancement of science.* (MAA 1936)



Constantin Carathéodory

Most details will appear in: J.M. Borwein
Maximal Monotonicity via Convex Analysis
Fitzpatrick Memorial, *JCA*, **13–14**, 2006.

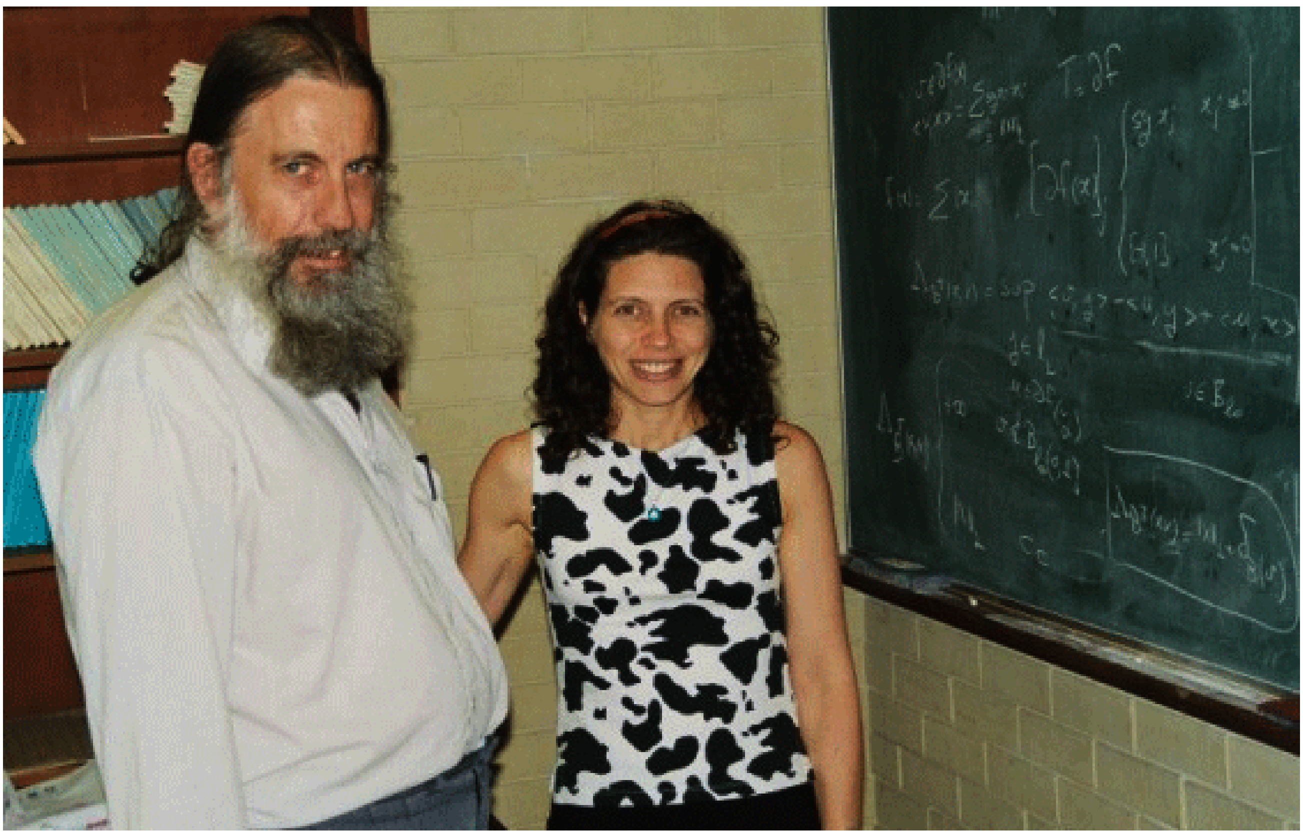
- http://users.cs.dal.ca/~jborwein/mon-jca2.pdf

Coxeter's favourite 4-D polytope
(with 120 dodecahedronal faces)

In Memoriam

In his ‘23’ “*Mathematische Probleme*” lecture to the Paris ICM in 1900[*], David Hilbert wrote

> “*Besides it is an error to believe that rigor in the proof is the enemy of simplicity.*”



Simon Fitzpatrick[†] (1953–2004).

[*]See Ben Yandell’s fine account of the *Hilbert Problems* and their solvers in *The Honors Class*, AK Peters, 2002. (He also died young in 2004.)

[†]At his blackboard with Regina Burachik

Journal of Convex Analysis

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MOTIVATION and GOALS

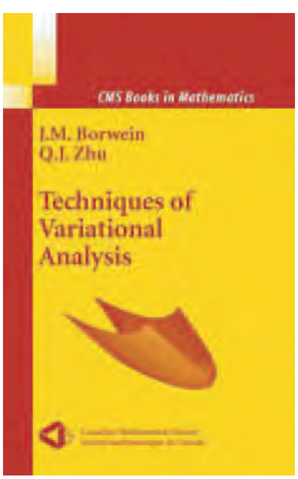
To reduce as much of monotone operator theory as possible to (elementary) convex analysis

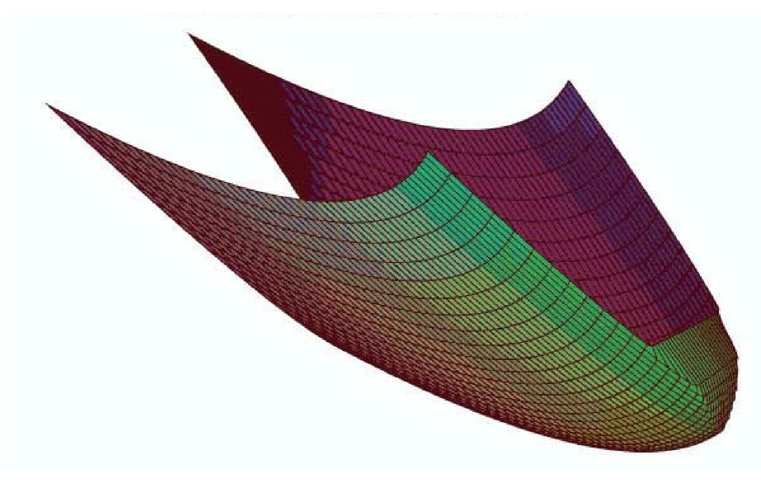
To thereby illustrate (some of) Simon Fitzpatrick's many fine contributions

To shed new light on the remaining open questions (in non-reflexive space)

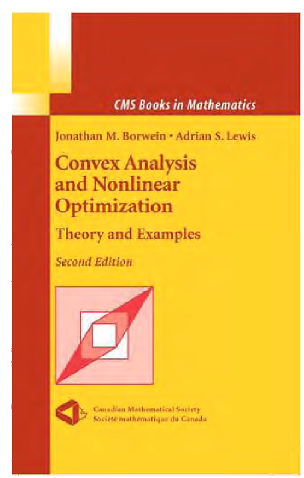
★ "Even convex objects are hard ..." ★

An essentially strictly convex function with non-convex subgradient domain and not strictly convex:





$\max\{(x-2)^2+y^2-1, -(xy)^{1/4}\}$



◀ JMB & J Zhu (Springer, 2005) JMB & A Lewis ▶

1. Preliminaries

Throughout X is a real Banach space. The *domain* of an extended valued convex function, dom (f), is the set of values less than $+\infty$. A point s is in the *core* of a set S ($s \in$ core S) when $X = \bigcup_{\lambda>0} \lambda(S-s)$.

Now $x^* \in X^*$ is a *subgradient* of $f : X \to (-\infty, +\infty]$ at $x \in$ dom f provided that

$$f(y) - f(x) \geq \langle x^*, y - x \rangle$$

for all y in Y. The set of all subgradients of f at x is the *subdifferential* of f at x, denoted $\partial f(x)$.

We need the *indicator function* $\iota_C(x)$ which is zero for x in C and $+\infty$ otherwise, the *Fenchel conjugate* $f^*(x^*) := \sup_x \{\langle x, x^* \rangle - f(x)\}$ and the *infimal convolution*

$$f^* \Box \frac{1}{2} \|\cdot\|_*^2(x^*) := \inf \left\{ f^*(y^*) + \frac{1}{2}\|z^*\|_*^2 : x^* = y^* + z^* \right\}$$

When f is convex and closed

$$x^* \in \partial f(x) \text{ exactly when } f(x) + f^*(x^*) = \langle x, x^* \rangle.$$

Finally, the *distance function* associated with a closed set C, given by $d_C(x) := \inf_{c \in C} \|x - c\|$, is convex if and only if C is. Moreover, $d_C = \iota_C \Box \|\cdot\|$.

We say $T : X \mapsto 2^{X^*}$ is *monotone* provided that for any $x, y \in X$, and $x^* \in T(x), y^* \in T(y)$,

$$\langle y - x, y^* - x^* \rangle \geq 0,$$

and that T is *maximal monotone* if its graph is not properly included in any other monotone graph.

- The *convex subdifferential* in Banach space* and a *skew linear matrix* are the canonical examples of maximal monotone multifunctions

We save the notation $J = J_X$ for the *duality map*

$$J_X(x) = \frac{1}{2}\partial\|x\|^2 = \left\{x^* \in X^* : \|x\|^2 = \|x^*\|^2 = \langle x, x^* \rangle\right\}$$

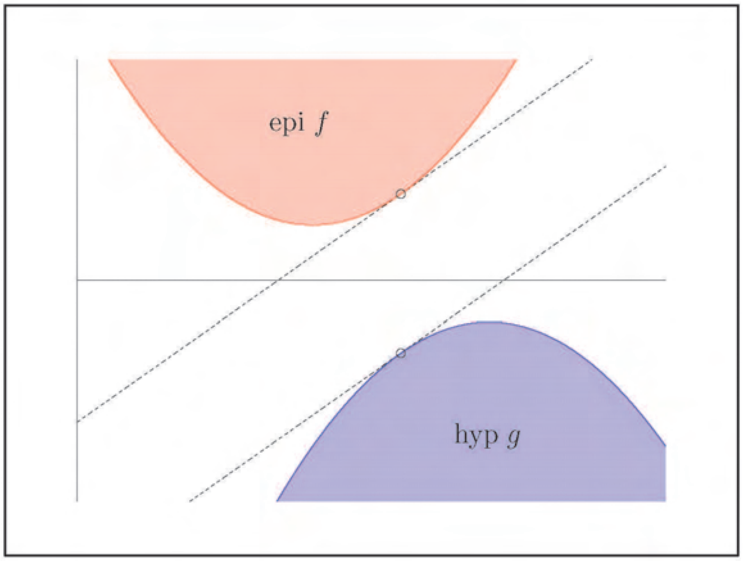
- It is not an exaggeration to say *the geometry of Banach space devolves to a deep study of J*

- The other foundational example is that of a second order nonlinear *elliptic PDE*

*There are several nice variational proofs. One based on the Mean value theorem follows.

Outline

Our goal is to derive *all* key results about maximal monotone operators *entirely from the existence of subgradients* and *Sandwich theorem* shown below



Section 2 considers general Banach spaces

~~**Section 3** looks at (a-)cyclic operators~~

Section 4 presents our central result on maximality of the sum in reflexive space

Section 5 looks at more applications of the technique of Section 4

~~**Section 6** provides limiting counter-examples.~~

2. Maximality in General Banach Space

For a monotone mapping T, we associate the *Fitzpatrick function* introduced in 1988 by Fitzpatrick. It is

$$\mathcal{F}_T(x,x^*) := \sup\{\langle x,y^*\rangle + \langle x^*,y\rangle - \langle y,y^*\rangle : y^* \in T(y)\}$$

which is clearly *lower semicontinuous and convex* as an affine supremum. Moreover,

Proposition 1 (Fitzpatrick) *For every maximal monotone operator* T *one has*

$$\mathcal{F}_T(x,x^*) \geq \langle x,x^*\rangle$$

with equality if and only if $x^* \in T(x)$.

- The equality $\mathcal{F}_T(x,x^*) = \langle x,x^*\rangle$ for $x^* \in T(x)$ requires only monotonicity not maximality.

- In generality, $\mathcal{F}_T$ is not useful for non-maximal operators. As an extreme example, on $\mathbb{R}$ if $T(0) = 0$ and $T(x) = \emptyset$ otherwise, then $\mathcal{F}_T \equiv 0$.

- The idea of associating a convex function to a monotone operator and exploiting the relationship was neglected for many years after its introduction until revisited by Penot, Simons, Simons and Zālinescu, Burachik and Svaiter etc.

Proposition 2 *A proper lsc convex function on a Banach space (i) is continuous throughout the core of its domain; and (ii) has a non-empty subgradient throughout the core of its domain.*

These two basic facts lead to:

Theorem 1 (**Hahn-Banach sandwich**) *Suppose $f, -g$ are lsc convex on a Banach space X and* $f(x) \geq g(x)$, *for all x in X. Assume (CQ) holds:*

$$0 \in \operatorname{core}\left(\operatorname{dom}(f) - \operatorname{dom}(-g)\right). \tag{1}$$

Then there is an affine continuous function a such that

$$f(x) \geq a(x) \geq g(x)$$

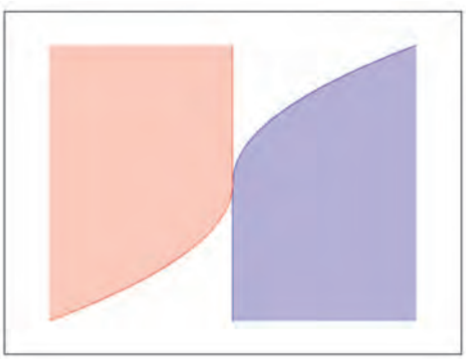
for all x in X.

Proof. The marginal, perturbation or *value function*

$$h(u) := \inf_{x \in X} f(x) - g(x-u)$$

is convex and (CQ) implies it is continuous at 0. Hence there is $-\lambda \in \partial h(0)$, which is the linear part of the affine separator. As needed, we have

$$f(x) - g(u-x) \geq h(u) - h(0) \geq \lambda(u). \qquad \blacksquare$$



$-\sqrt{-x} \geq \sqrt{x}$

- We refer to *constraint qualifications* like (1) as *transversality conditions*

◁ **CQ failure**

- It is easy to deduce complete *Fenchel duality theorem* from Thm 1

Proposition 3 *For a closed convex function f and $f_J := f + \frac{1}{2}\|\cdot\|^2$ we have that*

$$\left(f + \frac{1}{2}\|\cdot\|^2\right)^* = f^* \Box \frac{1}{2}\|\cdot\|_*^2$$

is everywhere continuous. Also

$$v^* \in \partial f(v) + J(v) \Leftrightarrow f_J^*(v^*) + f_J(v) - \langle v, v^* \rangle \leq 0.$$

2a. Representative Functions

A convex function $\mathcal{H}_T$ is a representative function for a monotone T on $X \times X^*$ if (i) $\mathcal{H}_T(x, x^*) \geq \langle x, x^* \rangle$ for all x, x^*; (ii) $\mathcal{H}_T(x, x^*) = \langle x, x^* \rangle$ **if** $x^* \in T(x)$.

For T maximal, Prop. 1 shows $\mathcal{F}_T$ is a representative function as is the (closed) convexification

$$\begin{aligned} \mathcal{P}_T(x, x^*) &= \inf \sum_{i=1}^{N} \lambda_i \langle x_i, x_i^* \rangle \\ \text{s.t.} \sum_i \lambda_i (x_i, x_i^*, 1) &= (x, x^*, 1), x_i^* \in T(x_i), \lambda_i \geq 0. \end{aligned}$$

Proposition 4 (Penot) *For any monotone mapping* T, $\overline{\mathcal{P}}_T$ *is a representative convex function.*

Proof. By monotonicity we have

$$\mathcal{P}_T(x, x^*) \geq \langle x^*, y \rangle + \langle y^*, x \rangle - \langle y^*, y \rangle,$$

for $y^* \in T(y)$. Thus, for *all* points

$$\mathcal{P}_T(x, x^*) + \mathcal{P}_T(y, y^*) \geq \langle x^*, y \rangle + \langle y^*, x \rangle.$$

By definition $\mathcal{P}_T(x, x^*) \leq \langle x^*, x \rangle$ for $x^* \in T(x)$. Setting $x = y$ and $x^* = y^*$ shows $\mathcal{P}_T(x, x^*) = \langle x^*, x \rangle$ for $x^* \in T(x)$ while $\mathcal{P}_T(z, z^*) \geq \langle z^*, z \rangle$ for (z^*, z) in $\operatorname{conv} \operatorname{graph} T$: (also for $\overline{\mathcal{P}}_T$). ■

2b. Monotone Extension Theorems

A direct calculation shows $(\mathcal{P}_T)^* = \mathcal{F}_T$ for any monotone T. This convexification originates with Simons but was much refined by Penot.

We illustrate its flexibility by proving a central case of the Debrunner-Flor theorem *without* Brouwer's theorem.

Theorem 2 *Suppose T is monotone on X with range contained in $\alpha\, B_{X^*}$, for some $\alpha > 0$. Then*

(a) *For every x_0 in X there is $x_0^* \in \overline{\text{conv}}^* R(T) \subset \alpha\, B_{X^*}$ such that (x_0, x_0^*) is monotonically related to* graph (T).

(b) *Hence, T has a bounded monotone extension $\overline{T}$ with* $\text{dom}\,(\overline{T}){=}X$ *and* $R(\overline{T}) \subset \overline{\text{conv}}^* R(T)$.

(c) *Thence, a maximal monotone T with bounded range has* $\text{dom}\,(T){=}X$.

Proof. (a) It is enough, after translation, to show $x_0 = 0 \in \mathrm{dom}\,(T)$. Fix $\alpha > 0$ with $R(T) \subset C := \overline{\mathrm{conv}}^* R(T) \subset \alpha B_{X^*}$.
Consider

$$\pi_T(x) := \inf \{\mathcal{P}_T(x, x^*) : x^* \in C\}.$$

Then π_T is convex since $\mathcal{P}_T$ is. Observe that

$$\mathcal{P}_T(x, x^*) \geq \langle x, x^* \rangle$$

and so $\pi_T(x) \geq \inf_{x^* \in C} \langle x, x^* \rangle \geq -\alpha \|x\|$ for all x in X. As $x \mapsto \inf_{x^* \in C} \langle x, x^* \rangle$ is concave and continuous the Sandwich Theorem 1 applies.

Thus, there exist w^* in X^* and γ in $\mathbf{R}$ with

$$\mathcal{P}_T(x, x^*) \geq \pi_T(x) \geq \langle x, w^* \rangle + \gamma \geq \inf_{x^* \in C} \langle x, x^* \rangle \geq -\alpha \|x\|$$

for all x in X and x^* in $C \subset \alpha B_{X^*}$.
Setting $x = 0$ shows $\gamma \geq 0$. Now, for any (y, y^*) in the graph of T we have $\mathcal{P}_T(y, y^*) = \langle y, y^* \rangle$. Thus,

$$\langle y - 0, y^* - w^* \rangle \geq \gamma \geq 0,$$

which shows that $(0, w^*)$ is monotonically related to the graph of T.

Finally, $\langle x, w^* \rangle + \gamma \geq \inf_{x^* \in C} \langle x, x^* \rangle \geq -\alpha \|x\|$ for all $x \in X$ involves three sublinear functions, and so implies that $w^* \in C \subset \alpha B_{X^*}$.

(b) Consider the set $\mathcal{E}$ of all monotone extensions of T with range in $C \subset \alpha B_{X^*}$, ordered by inclusion. By Zorn's lemma $\mathcal{E}$ admits a maximal member $\overline{T}$ and by (a) $\overline{T}$ has domain the whole space.
(c) follows immediately. ■

- ▶ $R(T) \subset MB_{X^*} \Rightarrow \pi_T := \inf_{X^*} \mathcal{P}_T(\cdot, x^*) \geq -M\|\cdot\|$

$$\boxed{x^* \in \partial\pi_T(x) \Leftrightarrow \pi_T(x) + \mathcal{F}_T(0, x^*) = \langle x, x^* \rangle}$$

- (a) holds on *any* w^*-closed convex set C in Hilbert space (Brezis). Our proof applies if

$$x_0 \in \text{core}\,(\text{dom}\,\pi_T + \text{dom}\,\sup_C).$$

The full Debrunner-Flor extension theorem is next:

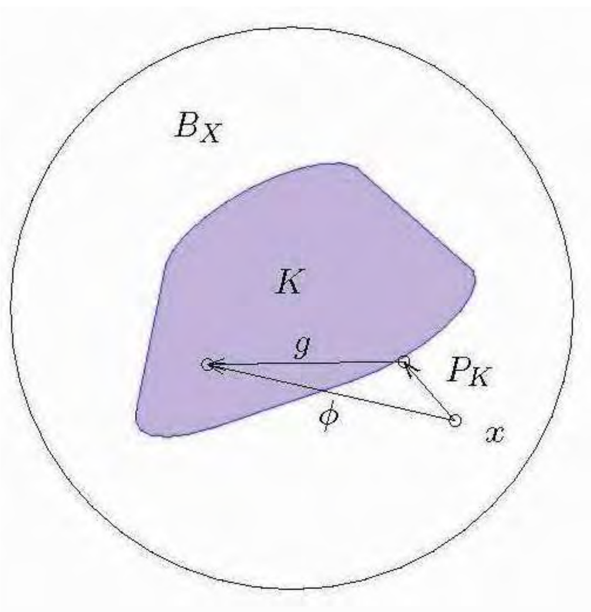
Theorem 3 (**Debrunner-Flor**) *Suppose T is a monotone operator on X with* range$T \subset C$ *for some weak-star compact and convex C. Suppose also $\varphi\colon C \mapsto X$ is weak-star to norm continuous.* Then *there is some $c^* \in C$ with*

$$\langle x - \varphi(c^*), x^* - c^* \rangle \geq 0$$

for all $x^* \in T(x)$.

Theorem 4 *The full Debrunner-Flor extension theorem is* equivalent *to Brouwer's theorem.*

Proof. Phelps derives Debrunner-Flor from Brouwer. Conversely, let g be a continuous self-map of a compact convex set $K \subset \operatorname{int} B_X$ in finite dimensions.



Apply Debrunner-Flor to the identity I *on* B_X and to $\varphi\colon B_X \mapsto X$ given by $\varphi(x) := g(P_K x)$, where P_K is the metric projection. We have $x_0^* \in B_X$, $x_0 := \varphi(x_0^*) = g(P_K x_0^*) \in K$,

$$\langle x - x_0, x - x_0^* \rangle \geq 0$$

for all $x \in B_X$.

Since $x_0 \in \operatorname{int} B_X$, for $h \in X$ and small $\epsilon > 0$ we have $x_0 + \epsilon h \in B_X$ and so $\langle h, x_0 - x_0^* \rangle \geq 0$ for all $h \in X$. Thus, $x_0 = x_0^*$ and so $P_K x_0^* = P_K x_0 = x_0 = g(P_K x_0^*)$, is a fixed point of the arbitrary self-map g. ■

2c. Local Boundedness Results

Recall that an operator T is *locally bounded* around a point x if $T(B_\varepsilon(x))$ is bounded for some $\varepsilon > 0$.

Theorem 5 (**Simons, Veronas**) *Let $S, T\colon X \to 2^{X^*}$ be monotone operators. Suppose*

$$0 \in \text{core}\,[\text{conv}\,\text{dom}\,(T) - \text{conv}\,\text{dom}\,(S)].$$

There exist $r, c > 0$ so that, for all x with $t^* \in T(x)$ and $s^* \in S(x)$,

$$\max(\|t^*\|, \|s^*\|) \le c\,(r + \|x\|)(r + \|t^* + s^*\|).$$

Proof. Consider the convex lsc function*

$$\sigma_T(x) := \sup_{z^* \in T(z)} \frac{\langle x - z, z^* \rangle}{1 + \lambda\|z\|}.$$

First, $\text{conv}\,\text{dom}\,(T) \subset \text{dom}\,\sigma_T$, and $0 \in$ core

$$\bigcup_{i=1}^{\infty} \left[\{x : \sigma_S(x) \le i, \|x\| \le i\} - \{x\colon \sigma_T(x) \le i, \|x\| \le i\}\right],$$

and apply conventional Baire category techniques–with some care. ■

*This is a refinement of the function SF-JMB used to prove local boundedness: $\mathcal{F}_T(x, 0) \approx \sigma_T(x)$

Corollary 1 *Let X be any Banach space. Suppose T is monotone and*

$$x_0 \in \operatorname{core}\operatorname{conv}\operatorname{dom}(T).$$

Then T is locally bounded around x_0.

Proof. Let $S = 0$ in Theorem 5 or directly apply Proposition 2 to σ_T. ■

We can also improve Theorem 2.

Corollary 2 *A monotone mapping T with bounded range admits an everywhere defined maximal monotone extension with bounded range contained in $\overline{\operatorname{conv}}^* R(T)$.*

Proof. Let $\widehat{T}$ denote the extension of Theorem 2 (b) Clearly it is everywhere locally bounded. The desired extension $\widetilde{T}(x)$ is the operator whose graph is the norm-weak-star closure of the graph of $x \mapsto \operatorname{conv}\widehat{T}(x)$, since this is both monotone and is a norm-w* cusco.
Explicitly,

$$\widetilde{T}(x) := \cap_{\varepsilon>0}\overline{\operatorname{conv}}^*\widehat{T}(B_\varepsilon(x))$$

(see ToVA). ■

A mapping is *locally maximal monotone*, or *type (FP)*, if (graph T^{-1})$\cap$($V\times X$) is maximal monotone in $V \times X$, for every convex open set V in X^* with $V \cap \text{range}\, T \neq \emptyset$.

- Simons showed subgradients are (FP). So are maximal monotones on reflexive space (SF-P).

We may usefully apply Corollary 2 to

$$T_n(x) := T(x) \cap n\, B_{X^*}.$$

Often the extension, $\widehat{T_n}$ is unique:

Proposition 5 (Fitzpatrick-Phelps) *Suppose T is maximal and n is such that $R(T) \cap n\,\text{int}\, B_{X^*} \neq \emptyset$.*
(a) There is a unique maximal monotone $\widehat{T}_n$ with

$$T_n(x) \subset \widehat{T_n}(x) \subset nB_{X^*}$$

whenever $M_n(x) :=$

$$\{x^* \in nB^* : \langle x^* - z^*, x - z\rangle \geq 0, \forall z^* \in T(z) \cap n\,\text{int}\, B_{X^*}\}$$

is monotone; in which case $M_n = \widehat{T}_n$.

(b) This holds if T is type (FP) and B_{X^*} is strictly convex; so for any maximal monotone on a rotund dual reflexive norm, e.g. Hilbert space.

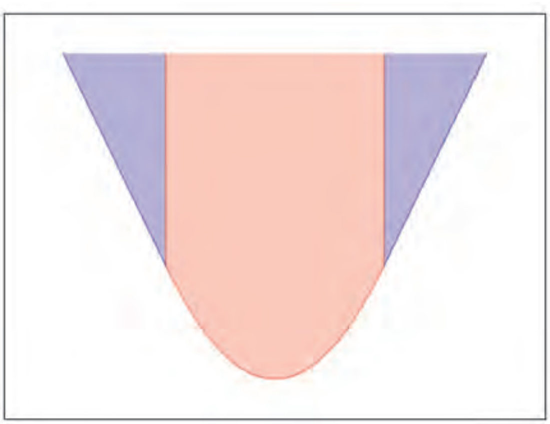
Proof. Since $\widehat{T_n}$ exists by Corollary 2 and since $\widehat{T_n}(x) \subset M_n(x)$, (a) follows. We refer to Fitzpatrick and Phelps for the fairly easy proof of (b). ■

★ $\{\widehat{T_n}\}_{n\in\mathbb{N}}$ is a non-reflexive generalization of the resolvent -based *Yosida approximate* or the *Hausdorff-Moreau* *Lipschitz regularization* of a convex function.

In the (FP) case one also easily shows (F-P) that:

(I) $\widehat{T_n}(x) = T(x) \cap n\,B_{X^*}$ if $T(x) \cap \operatorname{int} n\,B_{X^*} \neq \emptyset$

(II) $\widehat{T_n}(x) \setminus T(x) \subset n\,S_{X^*}$.



- cl $R(T)$ is convex if cl $R(\widehat{T_n})$ is for T type (II)

◀ function regularization

- For local properties (e.g. differentiability) one may replace T by $\widehat{T_n}$

2d. Maximality of Subgradients

Theorem 6 *Every closed convex function has a (locally) maximal monotone subgradient.**

Proof. (Sketch) Without loss we may suppose

$$\langle 0 - x^*, 0 - x \rangle \geq 0 \text{ for all } x^* \in \partial f(x)$$

but $0 \notin \partial f(0)$; so $f(\overline{x}) - f(0) < 0$ for some $\overline{x}$.

The *Approximate mean value theorem* (see [ToVA, Thm. 3.4.6]) lets us find $x_n \xrightarrow{f} c \in (0, \overline{x}]$ and $x_n^* \in \partial f(x_n)$ with

$$\limsup_n \langle x_n^*, x_n - c \rangle \leq 0, \limsup_n \langle x_n^*, \overline{x} \rangle \leq f(\overline{x}) - f(0) < 0.$$

Now $c = \theta\, \overline{x}$ for some $\theta > 0$. Hence,

$$\limsup_n \langle x_n^*, x_n \rangle < 0,$$

a contradiction. The locally maximal case follows 'similarly' on exploiting that $f(x_n) \to f(c)$, and that ∂f is dense type. ■

*This fails in *all* incomplete normed spaces and in *some* Fréchet spaces

2e. Convexity of Range and Domain

Corollary 3 *Let X be any Banach space. Suppose that T is maximal monotone with* core conv $D(T)$ *nonempty. Then*

$$\operatorname{core}\operatorname{conv} D(T) = \operatorname{int}\operatorname{conv} D(T) \subset D(T). \quad (2)$$

In consequence dom (T) *has both a convex closure and a convex interior.*

Proof. We first prove the inclusion in (2). Fix $x + \varepsilon B_X \subset \operatorname{int}\operatorname{conv}\operatorname{dom}(T)$ and, via Cor. 1, select $M := M(x, \varepsilon) > 0$ so that $T(x + \varepsilon B_X) \subset M B_{X^*}$. For $N > M$ define w^*-closed nested sets

$$T_N(x) := \{x^* : \langle x - y, x^* - y^* \rangle \geq 0, \forall y^* \in T(y) \cap N B_{X^*}\}.$$

By Theorem 2 (b), the sets are non-empty, and by the next lemma, bounded, hence w^*-compact. By maximality of T, $T(x) = \cap_N T_N(x) \neq \emptyset$, as a nested intersection, and x is in dom (T) as asserted.

Then $\operatorname{int}\operatorname{conv}\operatorname{dom}(T) = \operatorname{int}\operatorname{dom}(T)$ and so the final conclusion follows. ∎

Lemma 1 *For $x \in \operatorname{int}\operatorname{conv}\operatorname{dom}(T)$ and N sufficiently large, $T_N(x)$ is bounded.*

Proof. A Baire category argument shows for N large and $u \in 1/N\, B_X$ that $x + u \in \operatorname{cl}\operatorname{conv} D_N$ for

$$D_N := \{z \colon z \in D(T) \cap N\, B_X, T(z) \cap N\, B_{X^*} \neq \emptyset\}\,.$$

Now for each $x^* \in T_N(x)$, since $x + u$ lies in the closed convex hull of D_N, we have

$\langle u, x^* \rangle \leq \sup\{\langle z - x, z^* \rangle \colon z^* \in T(z) \cap N B_{X*}, z \in N B_X\}$
$\leq 2N^2$ and so $\|x^*\| \leq 2N^3$. ■

Another nice application is:

Corollary 4 (**Verona**) *Let X be Banach and let $S, T : X \to 2^{X^*}$ be maximal monotone. Suppose*

$$0 \in \operatorname{core}[\operatorname{conv}\operatorname{dom}(T) - \operatorname{conv}\operatorname{dom}(S)].$$

Then for any $x \in \operatorname{dom}(T) \cap \operatorname{dom}(S)$, $T(x) + S(x)$ is a w^*-closed subset of X^*.

Proof. Theorem 5 shows bounded w^*-convergent nets in $T(x) + S(x)$ have limits in $T(x) + S(x)$. We apply the *Krein-Smulian theorem*. ■

- Thus, we preserve some structure. It is still open if $T+S$ must actually be maximal.

We may neatly recover convexity of $\operatorname{int} D(T)$:

Theorem 7 (**Simons, 2005**) *Suppose T is maximal monotone and int* $\operatorname{dom}(T)$ *is nonempty.* *Then* $\operatorname{int}\operatorname{dom}(T) = \operatorname{int}\{x\colon (x,x^*) \in \operatorname{dom}\mathcal{F}_T\}$.

- Suppose T is *domain regularizable*: for $\varepsilon > 0$, there is a maximal T_ε with $H\left(D(T), D(T_\varepsilon)\right) \leq \varepsilon$ and $\operatorname{core} D(T_\varepsilon) \neq \emptyset$. In reflexive space we can use
$$T_\varepsilon := \left(T^{-1} + N_{\varepsilon B_X}^{-1}\right)^{-1}.$$
Then $\overline{\operatorname{dom}}(T)$ is convex.

3. Cyclic and Acyclic Monotone Operators

We recall that for $N = 2, 3, \ldots$, a multifunction T is *N-monotone* if

$$\sum_{k=1}^{N} \langle x_k^*, x_k - x_{k-1} \rangle \geq 0$$

whenever $x_k^* \in T(x_k)$ and $x_0 = x_N$.

T is *cyclically monotone* if T is N-monotone for all $N \in \mathbb{N}$, as hold for convex subgradients.

- Monotonicity and 2-monotonicity coincide
- $(N+1)$-monotone $\subsetneq$ N-monotone (Asplund)
- It is a classical result of Rockafellar that *every maximal cyclically monotone operator is the subgradient of a proper closed convex function* (and conversely).

We recast this result to make the parallel with the Debrunner-Flor Theorem 2 explicit.

Theorem 8 (**Rockafellar**) *Suppose C is cyclically monotone on a Banach space X.*

Then C has a maximal cyclically monotone extension $\overline{C}$, which is of the form $\boxed{\overline{C} = \partial f_C}$ for some proper closed convex function f_C.

Moreover $R(\overline{C}) \subset \overline{\mathrm{conv}}^* R(C)$.

Proof. We fix $x_0 \in \operatorname{dom} C, x_0^* \in C(x_0)$ and define

$$f_C(x) := \sup_{x_k^* \in C(x_k)} \{\langle x_n^*, x - x_n\rangle + \sum_{k=1}^{n-1} \langle x_{k-1}^*, x_k - x_{k-1}\rangle\}$$

where the 'sup' is over all $n \in \mathbb{N}$ and all such chains. The proof in Phelps' monograph shows that

$$C \subset \overline{C} := \partial f_C.$$

The range assertion follows because f_C is the supremum of affine functions whose linear parts all lie in range C. This is most easily seen by writing $f_C = g_C^*$ with

$$g_C(x^*) := \inf\{\sum_i t_i \alpha_i : \sum_i t_i x_i^* = x^*, \sum_i t_i = 1, t_i > 0\}$$

for appropriate $\alpha_i \in \mathbb{R}$. ■

The relationship of $\mathcal{F}_{\partial f}$ and ∂f is complicated:

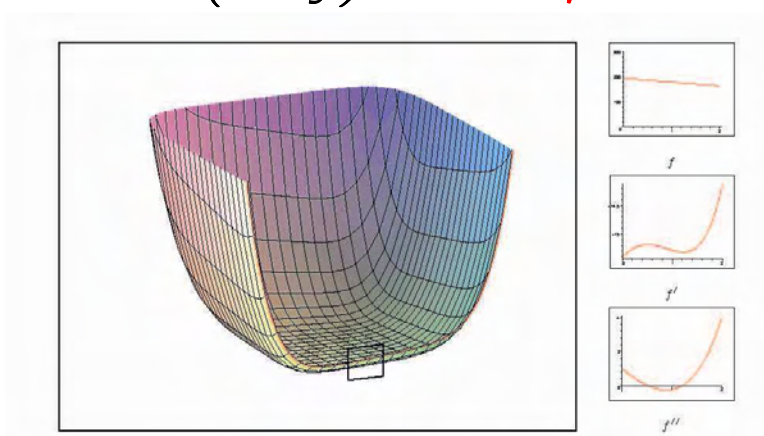
$$\begin{aligned}\langle x, x^* \rangle &\leq \mathcal{F}_{\partial f}(x, x^*) \leq f(x) + f^*(x^*) \leq \mathcal{F}^*_{\partial f}(x, x^*) \\ &\leq \langle x, x^* \rangle + \iota_{\partial f}(x, x^*),\end{aligned}$$

(see Bauschke et al.) Two central questions are:

Q1. **When is a maximal monotone operator T the sum of a subgradient ∂f and a skew linear S?** This is closely related to the behaviour of

$$\mathcal{FL}_T(x) := \int_0^1 \sup_{x^*(t) \in T(tx)} \langle x, x^*(t) \rangle \, dt$$

when $0 \in \operatorname{core} \operatorname{dom} T$, then $\mathcal{FL}_T = \mathcal{FL}_{\partial f} = f$ and we call T *(fully) decomposable*.



Fitzpatrick's Last Function *†

*The use of $\mathcal{FL}_T$ originates in discussions I had with Fitzpatrick shortly before his death.

†T *'inherits the differentiability'* of $\mathcal{FL}_T$.

A MONOTONE CONVERGENCE THEOREM FOR SEQUENCES OF NONLINEAR MAPPINGS

Edgar Asplund

In this paper we prove a theorem generalizing the elementary theorem on convergence of bounded, monotone sequences of real numbers, and also the theorem of Vigier and Nagy, cf. [**2**, Appendice II] on the convergence of certain sequences of symmetric linear operators on Hilbert space.

The paper consists of two sections. In the first we prove the main monotone convergence theorem (Theorem 1) and apply it to prove a decomposition for monotone operators which generalizes the decomposition of a linear operator into symmetric and antisymmetric parts. In the second section we apply Theorem 1

Q2. **How does one generalize the decomposition of a linear monotone operator L** into a symmetric (cyclic) and a skew (acyclic) part? Viz

$$L = \frac{1}{2}(L + L^*|_X) + \frac{1}{2}(L - L^*|_X).$$

3a. Asplund's approach to Q2

Every 3-monotone operator such that $0 \in T(0)$ has the local property that

$$\langle x, x^* \rangle + \langle y, y^* \rangle \geq \langle x, y^* \rangle \qquad (3)$$

whenever $x^* \in T(x)$ and $y^* \in T(y)$. We call a monotone operator satisfying (3), *3^--monotone*, and write $T \geq_N S$ if $T = S + R$ with R being N-monotone ($T \geq_{\omega_0} S$ if R is cyclically monotone.)

Proposition 6 (**Dini Property**) *Let N be $3^-, 3, 4,$ ..., or ω_0. Consider an increasing (infinite) net of monotone operators on a space X, satisfying*

$$\boxed{0 \leq_N T_\alpha \leq_N T_\beta \leq_2 T}$$

if $\alpha < \beta \in \mathcal{A}$. Suppose that $0 \in T_\alpha(0), 0 \in T(0)$ and that $0 \in \operatorname{core} \operatorname{dom} T$. Then

a) *There is a N-monotone $T_{\mathcal{A}}$ with $T_\alpha \leq_N T_{\mathcal{A}} \leq_2 T$, for all $\alpha \in \mathcal{A}$.*

b) *If $R(T) \subset MB_{X^*}$ for some $M > 0$ then one may suppose $R(T_{\mathcal{A}}) \subset MB_{X^*}$.*

Proof. a) **The single-valued case.** Since $0 \leq_2 T_\alpha \leq_2 T_\beta \leq_2 T$, while $T(0) = 0 = T_\alpha(0)$, we have

$$0 \leq \langle x, T_\alpha(x)\rangle \leq \langle x, T_\beta(x)\rangle \leq \langle x, T(x)\rangle,$$

for all x in $\operatorname{dom} T$. This shows $\langle x, T_\alpha(x)\rangle$ converges as α goes to ∞. Fix $\varepsilon > 0, M > 0$ with $T(\varepsilon B_X) \subset M B_{X^*}$. We write $T_{\beta\alpha} = T_\beta - T_\alpha$ for $\beta > \alpha$, so that $\langle T_{\beta\alpha} x, x\rangle \to 0$ for $x \in \operatorname{dom} T$ as $\alpha, \beta \to \infty$.
We appeal to (3) to obtain

$$\langle x, T_{\beta\alpha}(x)\rangle + \langle y, T_{\beta\alpha}(y)\rangle \geq \langle T_{\beta\alpha}(x), y\rangle, \qquad (4)$$

for $x, y \in \operatorname{dom} T$. Also, $0 \leq \langle x, T_{\beta\alpha}(x)\rangle \leq \varepsilon$ for $\beta > \alpha > \gamma(x)$ for all $x \in \operatorname{dom} T$.

Now, $0 \leq \langle y, T_{\beta\alpha}(y)\rangle \leq \langle y, T(y)\rangle \leq \varepsilon M$ for $\|y\| \leq \varepsilon^2$. Thus, for $\|y\| \leq \varepsilon$ and $\beta > \alpha > \gamma(x)$ we have

$$\begin{aligned} \varepsilon(M + \varepsilon) &\geq \langle x, T_{\beta\alpha}(x)\rangle + \langle y, T(y)\rangle \qquad (5) \\ &\geq \langle x, T_{\beta\alpha}(x)\rangle + \langle y, T_{\beta\alpha}(y)\rangle \\ &\geq \langle y, T_{\beta\alpha}(x)\rangle, \end{aligned}$$

from which we obtain $\|T_{\beta\alpha}(x)\| \leq M + \varepsilon$ for all $x \in \operatorname{dom} T$, while $\langle y, T_{\beta\alpha}(x)\rangle \to 0$ for all $y \in X$.

We conclude that $\{T_\alpha(x)\}_{\alpha \in \mathcal{A}}$ is a norm-bounded weak-star Cauchy net and so weak-star convergent to the desired N-monotone limit $T_{\mathcal{A}}(x)$.

<u>The set-valued case</u> uses (3) to deduce that $T_\beta = T_\alpha + T_{\beta\alpha}$ where (i) $T_{\beta\alpha} \subset (M+\varepsilon)B_{X^*}$ and (ii) for each $t^*_{\beta\alpha} \in T_{\beta\alpha}$ one has $t^*_{\beta\alpha} \to^* 0$ as $\alpha, \beta \to \infty$. The conclusion is as before but somewhat more technical.

b) Fix $x \in X$, and apply (3 to T_α to write

$$\langle Tx, x\rangle + \langle Ty, y\rangle \geq \langle T_\alpha x, x\rangle + \langle T_\alpha y, y\rangle \geq \langle T_\alpha x, y\rangle$$

for all $y \in D(T) = X$, by Theorem 2 (c). Hence

$$\langle Tx, x\rangle + M\|y\| \geq \|T_\alpha x\|\,\|y\|, \qquad \forall \|y\|$$

Let $\|y\| \to \infty$ to show $T_\alpha(x)$ lies in the M-ball, and since the ball is weak-star closed, so does $T_{\mathcal{A}}(x)$.

The set-valued case is analogous but *messier*. ■

- $0 \leq_2 (-ny, nx) \leq_2 (-y, x)$ for $n \in \mathbb{N}$, shows the need for (3) in the deduction that $T_{\beta\alpha}(x)$ are equi-norm bounded.

★ (**Daniel property**) If X is an *Asplund space*, the proof of Prop 6 can be adjusted to show

$$\boxed{T_{\mathcal{A}}(x) = \text{norm}-\lim_{\alpha\to\infty} T_\alpha(x)}$$

Definition 1 *We say a maximal monotone operator A is* acyclic *if whenever $A = \partial g + S$ with S maximal monotone and g closed and convex then g is necessarily linear.*

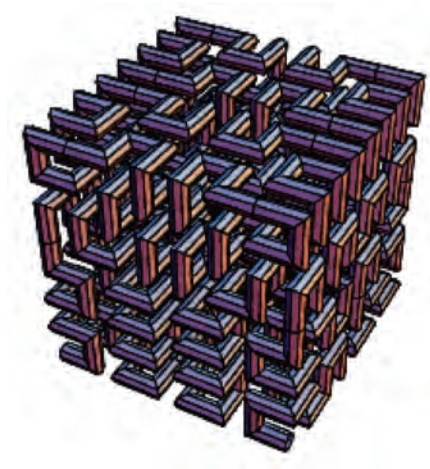
We provide a broad extension of Asplund's original idea:

Theorem 9 (**Asplund Decomposition**) *Suppose T is maximal monotone with $\operatorname{core}\operatorname{dom} T \neq \emptyset$.*

a) Then T may be decomposed as $T = \partial f + A$, where f is closed and convex while A is acyclic.

b) If the range of T lies in $M\,B_{X^*}$ then f may be assumed M-Lipschitz.

A Hilbert curve in 3D
is more constructive



Proof. a) We normalize so $0 \in T(0)$. Zorn's lemma applies to the cyclically monotone operators

$$\mathcal{C} := \{C\colon 0 \leq_{\omega_0} C \leq_2 T,\ 0 \in C(0)\}$$

in the cyclic order. By Prop. 6 every chain in $\mathcal{C}$ has a cyclically monotone upper-bound.

Fix a maximal $\overline{C}$ with $0 \leq_{\omega_0} \overline{C} \leq_2 T$. Hence $T = \overline{C} + A$ where by construction A is acyclic. Now, $T = \overline{C} + A \subset \partial f + A$, by Rockafellar's result. Since T is maximal the decomposition is as asserted.

b) We use the facts that (i) $0 \leq_{3^-} U \leq_2 T$ implies $\|U(x)\| \leq \|T(x)\|$ for all x and (ii) an M-bounded cyclically monotone operator extends to an M-Lipschitz subgradient—as Theorem 8 confirms. ∎

By way of application we offer:

Corollary 5 *Let T be an arbitrary maximal monotone operator T. For $\mu > 0$ one may decompose*

$$T \cap \mu B_{X^*} \subset \widehat{T_\mu} = \partial f_\mu + A_\mu,$$

where f_μ is μ-Lipschitz and A_μ is acyclic (with bounded range).

Proof. Combining Theorem 9 with Proposition 5 we deduce that the composition is as claimed. ■

- In Corollary 5, range A_μ is bounded. Thus, it is only skew and linear when T is cyclic—so a non-cyclic range bounded monotone operator is never fully decomposable in the sense of **Q1**.

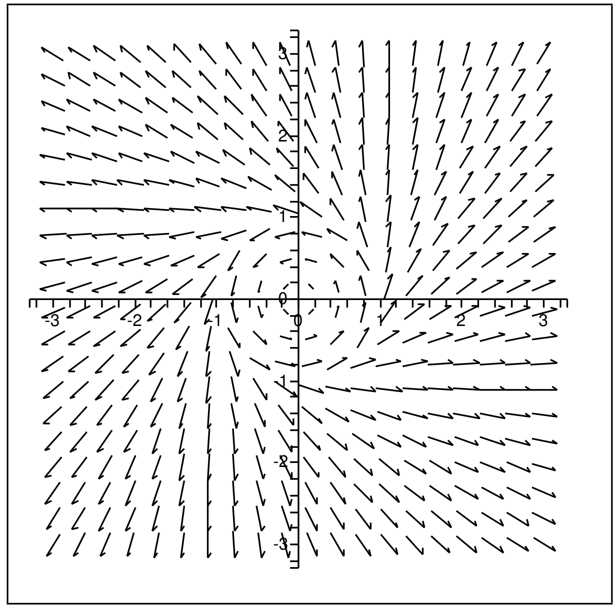
- Theorem 9 et al are entirely <u>existential</u>: can one prove Theorem 9 constructively in finite dimensions?

- How does one effectively diagnose acyclicity?

An Acyclic Monotone Operator

A concrete example in $\mathbb{R}^2$ is implicit in these observations (JMB-Wiersma).

- R_θ: rotation by $\theta < \pi/2$
- $\widehat{R_\theta}$: the range restriction to B_1 extended to be maximal with range in B_1.
- $\widehat{R_\theta}$ **is acyclic**: since any cyclic part P_θ has convex range while $R(P_\theta) \cap S_1 = \emptyset$.



For $\pi/2$, we obtain

$$\widehat{R}(x) = \alpha(x)\, R\left(\frac{x}{\|x\|}\right) + \beta(x)\,\frac{x}{\|x\|}$$

where

$$\alpha(x) := \sqrt{1 - 1 \wedge \frac{1}{\|x\|^2}}$$

and

$$\beta(x) := 1 \wedge \frac{1}{\|x\|}.$$

3b. Fitzpatrick Functions of Order N

- The *Fitzpatrick function of order N* is:

$$\mathcal{F}_T^N(x, x^*) := \sup_{x_N = x} \left\{ \langle x_1, x^* \rangle + \sum_{k=1}^{N-1} \langle x_{k+1} - x_k, x_k^* \rangle \right\}$$

where $x_k^* \in T(x_k)$ for $1 \leq k \leq N-1$.

- The *Rockafellar function of order N* is:

$$\mathcal{R}_T^N(x, x_1, x_1^*) : =$$

$$\sup \langle x - x_{N-1}, x_{N-1}^* \rangle + \sum_{i=1}^{N-2} \langle x_{i+1} - x_i, x_i^* \rangle,$$

for $x_1^* \in T(x_1)$, $x \in X$ and $N \geq 3$, over all $x_k^* \in T(x_k)$ (for $2 \leq k \leq N-1$).

Then $\mathcal{F}_T^\infty := \left(\mathcal{P}_T^\infty\right)^* = \sup \mathcal{F}_T^N$, $\mathcal{P}_T^\infty := \inf \mathcal{P}_T^N$, and $\mathcal{R}_T := \inf \mathcal{R}_T^N$. Moreover, for a maximal N-monotone T we have

$$\mathcal{F}_T^N(x, x^*) \geq \langle x, x^* \rangle$$

with equality if and only if $x^* \in T(x)$.

We recast Rockafellar's Theorem 8:

Theorem 10 *Suppose A is cyclically monotone. For $a_1^* \in A(a_1)$, $x \mapsto \mathcal{R}_A(x, a_1, a_1^*)$ is closed and convex and $\mathcal{R}_A(a_1, a_1, a_1^*) = 0$. Also for every $x \in X$, $A(x) \subset \partial \mathcal{R}_A(x, a_1, a_1^*)$. When A is maximal cyclically monotone one has $A = \partial \mathcal{R}_A$. Moreover, for every closed f satisfying $\partial f = A$, one has*

$$f(x) - f(a_1) = \mathcal{R}_A(x, a_1, a_1^*) \quad \text{for } x \in X.$$

We now connect the infinite Fitzpatrick function to the Rockafellar function.

Theorem 11 (**Bartz-Bauschke-Borwein-Reich - Wang**) *Let A be cyclically monotone. For each closed convex function f on X such that $A \subset \partial f$ one has*

$$\mathcal{F}_A^{\infty}(x, x^*) = f(x) + \sup_{a_1^* \in A(a_1)} \langle x^*, a_1 \rangle - f(a_1),$$

for $(x, x^*) \in X \times X^*$. If actually $\operatorname{dom} A = \operatorname{dom} \partial f$ *then*

$$\mathcal{F}_A^{\infty}(x, x^*) = (f \oplus f^*)(x, x^*) := f(x) + f^*(x^*),$$

for all $(x, x^*) \in X \times X^*$.

The Fitzpatrick Functions of a Rotation

Theorem 12 (BaBW) *Let $\theta \in [0, \pi/2]$ and*

$$A_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

1. $\theta = 0$. *then $A_\theta = I = \nabla\frac{1}{2}\|\cdot\|^2$ is cyclically monotone*, $F_I^\infty = \frac{1}{2}\|\cdot\|^2 \oplus \frac{1}{2}\|\cdot\|^2$, *and $n \geq 2$*

$$F_I^n\colon (x,u) \mapsto \frac{n-1}{2n}\left(\|x\|^2 + \|u\|^2\right) + \frac{1}{n}\langle x, u\rangle. \quad (6)$$

2. $\theta \in \,]0, \pi/2]$. *For $n \geq 2$, if $n \in [2, \pi/\theta[$, then A_θ is n-cyclically monotone and*

$$F_{A_\theta}^n\colon (x,u) \mapsto \frac{\sin(n-1)\theta}{2\sin n\theta}\left(\|x\|^2 + \|u\|^2\right) + \frac{\sin\theta}{\sin n\theta}\langle x, A_\theta^{n-1}u\rangle. \quad (7)$$

For $\pi/\theta \in \mathbb{N}$, *$A_\theta$ is (π/θ)-monotone and*

$$F_{A_\theta}^{\pi/\theta} = \iota_{\text{Graph}\, A_\theta} + \langle\cdot,\cdot\rangle. \quad (8)$$

If $n \in \,]\pi/\theta, +\inf[$, *then A_θ is not n-cyclically monotone since $F_{A_\theta}^n \equiv +\infty$.*

4. Maximality in Reflexive Banach Space

We begin with:

Proposition 7 *A monotone operator T on a reflexive Banach space is maximal iff the mapping $T(\cdot + x) + J$ is surjective for all x in X.*

Moreover, when J and J^{-1} are both single valued, a monotone mapping T is maximal if and only if $T + J$ is surjective.

Proof. We prove the 'if'. The 'only if' is completed in Corollary 8. Assume (w, w^*) is monotonically related to the graph of T. By hypothesis, we may solve $w^* \in T(x + w) + J(x)$. Thus $w^* = t^* + j^*$ where $t^* \in T(x + w), j^* \in J(x)$. Hence,

$$\begin{aligned} 0 &\leq \langle w - (w + x), w^* - t^* \rangle \\ &= -\langle x, w^* - t^* \rangle = -\langle x, j^* \rangle = -\|x\|^2 \leq 0. \end{aligned}$$

Thus, $j^* = 0, x = 0$. So $w^* \in T(w)$, and we are done. ∎

We now prove our central result whose proof—originally hard and due to Rockafellar—has been revisited over many years culminating in recent results of Simons, Penot, Zălinescu among others:

Theorem 13 (Sum) *Let X be reflexive, let T be maximal monotone and f closed and convex. Suppose* $0 \in \operatorname{core}\{\operatorname{conv}\operatorname{dom}(T) - \operatorname{conv}\operatorname{dom}(\partial f)\}$. *Then*

(a) $\partial f + T + J$ is surjective.

(b) $\partial f + T$ is maximal monotone.

(c) ∂f is maximal monotone.

Proof. (a) We consider the Fitzpatrick function $\mathcal{F}_T(x, x^*)$ and $f_J(x) := f(x) + 1/2\|x\|^2$.

Let $G(x, x^*) := -f_J(x) - f_J^*(-x^*)$. Observe that

$$\mathcal{F}_T(x, x^*) \geq \langle x, x^* \rangle \geq G(x, x^*)$$

pointwise thanks to the *Fenchel-Young inequality*

$$f_J(x) + f_J^*(-x^*) \geq \langle x, -x^* \rangle,$$

for all $x \in X, x^* \in X^*$, along with Proposition 1. The (CQ) assures the *Sandwich theorem* applies to $\mathcal{F}_T \geq G$ since f_J^* is everywhere finite by Prop. 3.

Then there are $w \in X$ and $w^* \in X^*$ such that

$$\mathcal{F}_T(x, x^*) - G(z, z^*) \geq w(x^* - z^*) + w^*(x - z) \quad (9)$$

for all x, x^* and all z, z^*. In particular, for $x^* \in T(x)$ and for all z^*, z we have

$$\begin{aligned}\langle x - w, x^* - w^* \rangle \quad &+ \quad [f_J(z) + f_J^*(-z^*) + \langle z, z^* \rangle] \\ &\geq \quad \langle w - z, w^* - z^* \rangle.\end{aligned}$$

Now use the fact that $-w^* \in \mathrm{dom}\,(\partial f_J^*)$, by Prop. 3, to deduce that $-w^* \in \partial f_J(v)$ for some v and so

$$\begin{aligned}\langle v - w, x^* - w^* \rangle \quad &+ \quad [f_J(v) + f_J^*(-w^*) + \langle v, w^* \rangle] \\ &\geq \quad \langle w - v, w^* - w^* \rangle = 0.\end{aligned}$$

The second term on the left is zero and so by maximality $w^* \in T(w)$. Substitution of $x = w$ and $x^* = w^*$ in (9), and rearranging yields

$$\begin{aligned}\langle w, w^* \rangle \quad &+ \quad \{\langle -z^*, w \rangle - f_J^*(-z^*)\} \\ &+ \quad \{\langle z, -w^* \rangle - f_J(z)\} \leq 0,\end{aligned}$$

for all z, z^*. Taking the supremum over z and z^* produces $\langle w, w^* \rangle + f_J(w) + f_J^*(-w^*) \leq 0$.

This shows $-w^* \in \partial f_J(w) = \partial f(w) + J(w)$ via the sum formula for subgradients, implicit in Prop. 3.

Thus, $0 \in (T + \partial f_J)(w)$. As all translations of $T + \partial f$ may be used, while (CQ) is undisturbed, we see that $(\partial f + T)(x + \cdot) + J$ is surjective which completes (a).

(b) $\partial f + T$ is maximal by Proposition 7.

(c) Setting $T \equiv 0$ we recover the reflexive case of the maximality for a lsc convex function. ■

Recall that the *normal cone* $N_C(x)$ to a closed convex set C at a point x in C is $N_C(x) = \partial \iota_C(x)$.

Corollary 6 *The sum of a maximal monotone operator T and a (necessarily maximal) normal cone N_C on a reflexive space is maximal monotone whenever the transversality condition*

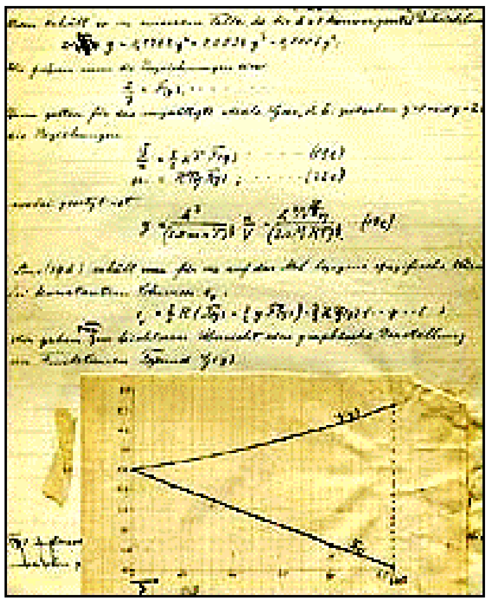
$$0 \in \mathrm{core}\,[C - \mathrm{conv}\,\mathrm{dom}\,(T)]$$

holds.

- In particular, if T is monotone and

$$C := \operatorname{cl}\operatorname{conv}\operatorname{dom}(T)$$

has nonempty interior, then for any maximal extension $\overline{T}$ the sum $\overline{T} + N_C$ is a *'domain preserving'* maximal monotone extension of T.



Einstein, 1924

- "Quantentheorie des einatomigen idealen Gases"

- On Bose-Einstein condensates, in Paul Ehrenfest' papers in Leiden. Confirmed in 1995.

Corollary 7 (Rockafellar) *The sum of maximal monotone operators* T_1 *and* T_2*, on a reflexive space, is maximal when the transversality condition* $0 \in \mathrm{core}\,[\mathrm{conv}\,\mathrm{dom}\,(T_1) - \mathrm{conv}\,\mathrm{dom}\,(T_2)]$ *holds.*

Proof. Theorem 13 applies to the product $T(x,y) := (T_1(x), T_2(y))$ and the indicator function $f(x,y) := \iota_{\{x=y\}}$ of the diagonal in $X \otimes X$.

We check that the given transversality condition implies the needed (CQ), as in Theorem 13. Hence, $T + J_{X\otimes X} + \partial\iota_{\{x=y\}}$ is surjective. Thus, so is

$$T_1 + T_2 + 2\,J$$

and we are done. ■

- One may easily replace the core condition by a relativized version—wrt the closed affine hull.

We re-record that $\mathcal{F}_{\partial f}(x, x^*) \le f(x) + f^*(x^*)$, and that we have exploited the beautiful inequality

$$\mathcal{F}_T(x, x^*) + f(x) + f^*(-x^*) \ge 0, \tag{10}$$

for all $x \in X, x^* \in X^*$, valid for *any* maximal monotone T and *any* convex function f.

4a. The Fitzpatrick Inequality

We have a *stronger* *Fitzpatrick inequality*

$$\mathcal{F}_{T_1}(x, x^*) + \mathcal{F}_{T_2}(x, -x^*) \geq 0 \qquad (11)$$

for all $x \in X, x^* \in X^*$, valid for *any* maximal monotone T_1, T_2. By Proposition 1

$$\begin{aligned} \mathcal{F}^*_T(x^*, x) &\geq \sup_{y^* \in T(y)} \langle x, y^* \rangle + \langle x^*, y \rangle - \mathcal{F}_T(y, y^*) \\ &= \mathcal{F}_T(x, x^*) \end{aligned} \qquad (12)$$

and we clearly have an extension of (11) in that

$$\mathcal{H}^1_T(x, x^*) + \mathcal{H}^2_S(x, -x^*) \geq 0,$$

for any representative functions $\mathcal{H}^1_T$ and $\mathcal{H}^2_S$. Letting $\widehat{\mathcal{F}_S}(x, x^*) := \mathcal{F}_S(x, -x^*)$, we may establish:

Theorem 14 (Sums) *Let S and T be maximal monotone on a reflexive space. Suppose that**

$0 \in \text{core}\,\{\text{dom}\,(\mathcal{F}_T) - \text{dom}\,(\widehat{\mathcal{F}_S})\}$ *as happens if*
$0 \in \text{core}\,\{\text{conv}\,\text{graph}\,(T) - \text{conv}\,\text{graph}\,(-S)\}$.

Then

$$0 \in \text{range}\,(T + S).$$

*This works for any representative functions.

Proof. We use Fenchel duality or follow the steps of Theorem 13. We have $\mu \in X, \lambda \in X^*, \beta \in \mathbb{R}$ with

$$\begin{aligned}\mathcal{F}_T(x,x^*) &- \langle x,\lambda\rangle - \langle \mu, x^*\rangle + \langle \mu,\lambda\rangle \geq \beta \\ &\geq -\mathcal{F}_S(y,-y^*) + \langle y,\lambda\rangle - \langle \mu, y^*\rangle - \langle \mu,\lambda\rangle,\end{aligned}$$

for all variables x, y, x^*, y^*. Hence for $x^* \in T(x)$ and $-y^* \in S(y)$ we obtain

$$\langle x-\mu, x^*-\lambda\rangle \geq \beta \geq \langle y-\mu, y^*+\lambda\rangle.$$

If $\beta \leq 0$, we derive that $-\lambda^* \in S(\mu)$ and so $\beta = 0$; consequently, $\lambda \in T(\mu)$ and since $0 \in (T+S)(\mu)$ we are done. If $\beta \geq 0$ we argue first with T. ■

- A graph (CQ) is formally tougher than a domain (CQ) as $\operatorname{conv}\operatorname{graph}(J_{\ell^2})$ is the diagonal in $\ell^2 \otimes \ell^2 = \operatorname{dom}(F_{J_{\ell^2}})$, while

  $$\mathcal{F}_{J_{\ell^2}}(x,x^*) = \frac{1}{4}\|x+x^*\|^2,$$

  yielding a simple proof in ℓ^2 of Cor. 8 below.

- Zalinescu has adapted this to extend results like those of Simons in the reflexive case: the sum has a *semi-convex graph*.

Corollary 8 (Rockafellar-Minty surjectivity theorem) *For a maximal monotone operator on a reflexive Banach space,* range $(T + J) = X^*$.

Proof. Let $f \equiv 0$ in Theorem 13. Alternatively, on noting that $\mathcal{F}_J(x, x^*) \le \frac{\|x\|^2+\|x^*\|^2}{2}$, we may apply Theorem 14. ■

4b. Extensions to Non-reflexive Space

Let $\overline{T}$ denote the *monotone closure* of T in $X^{**} \times X^*$. That is, $x^* \in \overline{T}(x^{**})$ when

$$\inf_{y^* \in T(y)} \langle x^* - y^*, x^{**} - y \rangle \ge 0.$$

Recall that T is *type (NI)* if

$$\inf_{y^* \in T(y)} \langle x^* - y^*, x^{**} - y \rangle \le 0$$

for all $x^{**} \in X^{**}, x^* \in X^*$:

Corollary 9 (Gossez for (D)). *For T type (NI)*

$$R(\overline{T} + \partial f^{**} + J^{**}) = X^*.$$

Proof. Mimic the steps of Theorem 13. ■

4c. A Non-reflexive Sum Rule

Theorem 15 *Suppose that A and B are maximal monotone in Banach space. If either*

a) $\operatorname{int}(D(A) \cap \operatorname{int} D(B)$ *is nonempty;*

b) $\operatorname{int}(D(A) \cap D(B) \neq \emptyset$ *while* $D(B)$ *is closed and convex; or*

c) (Voisei) Both $D(A), D(B)$ *are closed and convex and*

$$0 \in \operatorname{core}\operatorname{conv}\{D(A) - D(B)\}\,. \tag{13}$$

Then $A + B$ is maximal monotone.

Proof. Voisei (2005) shows, as in §6, that (13) implies

$$\begin{aligned}\Phi_{A,B}(x,x^*) &:= \mathcal{F}_A(x,\cdot)\Box\mathcal{F}_B(x,\cdot)(x^*)\\ &= \left(\mathcal{P}_A(x,\cdot)\Box\mathcal{P}_B(x,\cdot)\right)^*(x^*) \geq \langle x,x^*\rangle\end{aligned}$$

with equality if and only if $x^* \in (A+B)(x)$.

Moreover,

$$\mathcal{F}_{A+B} \leq \Phi_{A,B} \leq \mathcal{P}_{A+B}.$$

Hence $A+B$ is maximal **iff**

$$\mathcal{F}_{A+B}(x,x^*) \geq \langle x,x^*\rangle, \qquad (14)$$

for all x,x^*. Now all three conditions imply that

$$\overline{\mathrm{conv}}\, D(A) \cap \overline{\mathrm{conv}}\, D(B) \subset \overline{D(A+B)}^{alg},$$

since $\overline{D(A)}$ is convex when $D(A)$ has nonempty interior. This in turn implies (14). ∎

Corollary 10 *Suppose that T is maximal monotone, C is closed and convex while $C \cap \operatorname{int} D(T) \neq \emptyset$.*

Then $T+N_C$ is maximal monotone.

In particular, when $D(T)$ has nonempty interior, then T is of type (FPV).

5. Further Reflexive Applications

Another very useful result is:

Theorem 16 (Composition) *Suppose X and Y are Banach spaces with X reflexive, that T is maximal monotone operator on Y, and that $A\colon X \mapsto Y$, is a bounded linear mapping. Then*

$$T_A := A^* \circ T \circ A$$

is maximal monotone on X whenever

$$0 \in \operatorname{core}\left(\operatorname{range}(A) + \operatorname{conv}\operatorname{dom} T\right)$$

Proof. Monotonicity is clear. To obtain maximality, use the Fitzpatrick inequality (11) to write

$$f(x, x^*) + g(x, x^*) \geq 0,$$

where

$$f(x, x^*) := \inf\{\mathcal{F}_T(Ax, y^*)\colon A^* y^* = x^*\}$$

and

$$g(x, x^*) := \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2.$$

Apply Fenchel's duality theorem—or use the Sandwich theorem directly—to deduce the existence of $\overline{x} \in X, \overline{x}^* \in X^*$ with

$$f^*(\overline{x}^*, \overline{x}) + g^*(\overline{x}^*, \overline{x}) \leq 0. \tag{15}$$

Carefully using the standard formula for the conjugate of a convex composition —we have for some $\overline{y}^*$ with $A^*\overline{y}^* = \overline{x}^*$:

$$\begin{aligned} f^*(\overline{x}^*, \overline{x}) &= \inf\{\mathcal{F}_T^*(A\overline{x}, y^*) \colon A^*y^* = \overline{x}^*\} \\ &= \min\{\mathcal{F}_T^*(y^*, A\overline{x}) \colon A^*y^* = \overline{x}^*\} \\ &= \mathcal{F}_T^*(\overline{y}^*, A\overline{x}) \geq \mathcal{F}_T(A\overline{x}, \overline{y}^*), \end{aligned}$$

the last inequality following from (12). Moreover,

$$g^*(\overline{x}^*, \overline{x}) = \frac{1}{2}\|\overline{x}\|^2 + \frac{1}{2}\|A^*\overline{y}^*\|^2.$$

Thus, (15) implies that

$$\begin{aligned} & \left\{\mathcal{F}_T(A\overline{x}, \overline{y}^*) - \langle \overline{y}^*, A\overline{x}\rangle\right\} \\ + & \left\{\frac{1}{2}\|\overline{x}\|^2 + \frac{1}{2}\|A^*\overline{y}^*\|^2 + \langle \overline{y}^*, A\overline{x}\rangle\right\} \leq 0. \end{aligned}$$

We see that $\overline{y}^* \in T(A\overline{x})$, $-\overline{x}^* := -A^*\overline{y}^* \in J_X(\overline{x})$, since both bracketed terms are non-negative. Hence,

$$0 \in J_X(\overline{x}) + T_A(\overline{x}).$$

In the same way if we start with

$$f(x, x^*) := \inf\{\mathcal{F}_T(Ax, y^*) \colon A^*y^* = x^* + x_0^*\},$$

$$g(x, x^*) := \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 - \langle x, x_0^* \rangle,$$

we deduce, $x_0^* \in J_X(\overline{x}) + T_A(\overline{x})$. This applies to all *domain* translations of T. As in Theorem 13, this is sufficient to conclude T_A is maximal. ■

- This recovers the reflexive case of the formula that $A^*\partial f(Ax) = \partial(fA)(x)$ with the same (CQ).

- A recent paper [Bot et al] relaxes the (CQ) to

$$\{(A^*y^*, Ax, r) \colon \mathcal{F}_T^*(Ax, y^*) \leq r\} \tag{16}$$

  is relatively closed in $X^* \times R(A) \times \mathbb{R}$.

- Application of Theorem 16 to

$$T(x, y) := (T_1(x), T_2(y)),$$

  and $A(x) := (x, x)$ yields $T_A(x) = T_1(x) + T_2(x)$ and recovers Theorem 13. With more effort one may equally embed Theorem 16 in Theorem 13.

Note only X need be reflexive. A key case of Theorem 16 is a *reflexive injection*.

Corollary 11 *Let T be maximal monotone on a Banach space Y. Let ι denote the injection of a reflexive subspace $Z \subset Y$ into Y.*
Then $T_Z := \iota^* \circ T \circ \iota$ is maximal monotone on Z if

$$0 \in \operatorname{core}(Z + \operatorname{conv} \operatorname{dom} T).$$

Hence, if $0 \in \operatorname{core}(\operatorname{conv} \operatorname{dom} T)$, then T_Z is maximal for each reflexive Z.

- In this case, (16) implies the result holds when

  $$\{(y^*|_Z, z, r) \colon \mathcal{H}_T^*(z, y^*) \le r, z \in Z\}$$

  is relatively closed in $Z^* \times Z \times \mathbb{R}$
  *What happens generally?**

***Conjecture**: 'most' subspaces behave well $\Rightarrow T$ is (FPV) and so $\overline{D(T)}$ convex.

Conjectural Details

1. For a lsc representative $\mathcal{H}_T$ and dim $F < \infty$, if
$$\mathcal{H}_T^F(y, y^*) := \inf\{\mathcal{H}_T(y, x^*) \colon x^*|F = y^*\}$$
is lsc on $F \times F^*$ then T_F is maximal.

2. Equivalently, this holds if
$$\operatorname{epi}\mathcal{H} + \{0\} \times F^{\perp} \times \{0\} \tag{17}$$
is closed.

3. Hence, if (17) holds for 'most' F meeting dom T, we have a net of approximating 'nice' maximal monotone (e.g., FPV, FP) operators.

Example 1 *Consider* $T(x_1, x_2) := \partial f(x_1, x_2)$ *and* $\mathcal{H}_T(x_1, x_2, x_1^*, x_2^*) := f(x_1, x_2) + f^*(x_1^*, x_2^*)$ *where*

$$f(x_1, x_2) := \max\{|x_1|, 1 - \sqrt{x_2}\}, \qquad x_2 \geq 0$$

$$f^*(x_1^2, x_2^2) = \frac{\{(|x_1^*| - 1) \vee x_2^*\}^2}{4x_2} - (|x_1^*| - 1) \vee x_2^*,$$

and $|x_1^*| \leq 1, x_2^* < 0$. *Then (only)* $T_{R\times 0}$ *is not maximal and, necessarily,* $\mathcal{H}_T^{R\times 0}$ *is not lsc.*

A Dense Limiting Example

Example 2 *Let C be closed convex and bounded in an infinite dimensional Banach space X and fix $x_0 \neq 0$ in X. Define*

$$f_C(x) := \inf\{t \in \mathbb{R} : x + t\,x_0 \in C\}.$$

Set $c_x := x - f_C(x)x_0 \in C$. Then f_C is closed and convex and has no global minimum. Moreover, $\partial f_C(x) = \partial f_C(c_x)$. This implies that

$$\operatorname{dom} \partial f_C \subset \operatorname{supp} C.$$

Now arrange that $0 \in C$, that

$$Y \bigcap \operatorname{span}\,(C \cup \{x_0\}) = 0$$

for a dense subspace Y, while span C is also dense. It follows that $(\partial f_C)_F$ fails to be maximal for every non-trivial finite dimensional subspace $F \subset Y$.

Explicitly, take the (norm-compact) Hilbert cube $K := \{x \in \ell_2 : |x_n| \leq 1/2^n, \forall n \in \mathbb{N}\}$ and $x_0 := (1/2^n)$ so that

$$f_K(x) := \sup_{n \in \mathbb{N}} |2^n x_n - 1|,$$

and take $Y \setminus \{0\}$ to contain only more slowly decreasing sequences.

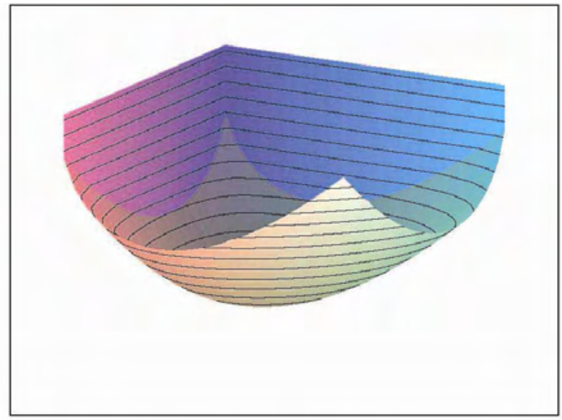
5a. Variational Inequalities

T is *coercive on C* if

$$\inf_{y^*\in T(y)+\partial\iota_C(y)} \langle y, y^*\rangle/\|y\| \to \infty$$

as $y \in C$ goes to infinity in norm.[a]

[a]This may be weakened significantly, especially if $0 \in C$.



A *variational inequality* **V(T,C)** requests a solution $y \in C$ and $y^* \in T(y)$ to

$$\langle y^*, x - y\rangle \geq 0 \qquad \forall x \in C.$$

Equivalently

$$0 \in T(y) + N_C(y)$$

or

$$0 \in T(y) + \partial\iota_C(y).$$

- This models the *necessary condition*

  $$\langle \nabla f(x), c - x\rangle \geq 0$$

  for all $c \in C$.

Corollary 12 *Suppose T is maximal monotone on a reflexive space and is coercive on the closed convex set C while $0 \in \mathrm{core}\,(C - \mathrm{conv}\,\mathrm{dom}\,(T))$. Then $V(T,C)$ has a solution.*

Proof. Let $f := \iota_C$, the indicator function. For $n = 1, 2, 3 \cdots$, let $T_n := T + J/n$. We solve

$$0 \in (T_n + \partial\iota_C)(y_n) = (T + \partial\iota_C) + \frac{1}{n}J(y_n) \quad (18)$$

and take limits as n goes to infinity.

More precisely, Theorem 13, yields y_n in C, and $y_n^* \in (T + \partial\iota_C)(y_n), j_n^* \in J(y_n)/n$ with $y_n^* = -j_n^*$. Then

$$\langle y_n^*, y_n\rangle = -\frac{1}{n}\langle j_n^*, y_n\rangle = -\frac{1}{n}\|y_n\|^2 \leq 0,$$

and so coercivity of $T + \partial\iota_C$ implies that $\|y_n\|$ remains bounded and so $j_n^* \to 0$. We may assume $y_n \rightharpoonup y$.

Since $T + \partial\iota_C$ is maximal monotone (again by Theorem 13), it is demi-closed. It follows that $0 \in (T + \partial\iota_C)(y)$, and y is as required. ■

Letting $C := X$ in Corollary 12 we deduce

Corollary 13 *Every coercive maximal monotone operator on a Banach space is surjective if (and only if) the space is reflexive.*

Proof. To complete the proof we recall that, by *James' theorem*, surjectivity of J is equivalent to reflexivity of the corresponding space. ■

We may improve Corollary 3 in the reflexive setting:

Theorem 17 *Suppose T is maximal monotone on a reflexive space. Then* dom (T) *and range* (T) *have convex closure (and interior).*

Proof. Without loss, we assume 0 is in the closure of conv dom (T). Fix $y \in \operatorname{dom}(T)$, $y^* \in T(y)$. Corollary 8 applied to T/n solves $w_n^*/n + j_n^* = 0$ with $w_n^* \in T(w_n), j_n^* \in J(w_n)$, for integer $n > 0$. By monotonicity

$$\frac{1}{n}\langle y^*, y - w_n\rangle \geq \frac{1}{n}\langle w_n^*, y - w_n\rangle = \|w_n\|^2 - \langle j_n^*, y\rangle$$

where $\|w_n\|^2 = \|j_n^*\|^2 = \langle j_n^*, w_n\rangle$ and $w_n \in \operatorname{dom}(T)$.

We deduce $\sup_n \|w_n\| < \infty$. Thus, (j_n^*) has a weak cluster point j^*. Thence, denoting $D := \mathrm{dom}\,(T)$

$$\begin{aligned} d_D^2(0) &\leq \liminf_{n\to\infty} \|w_n\|^2 \leq \inf_{y\in D} \langle j^*, y\rangle \\ &= \inf_{y\in \mathrm{conv}\, D} \langle j^*, y\rangle \leq \|j^*\|\, d_{\mathrm{conv}\, D}(0) = 0. \end{aligned}$$

We have shown that $\mathrm{cl}\,\mathrm{conv}\,\mathrm{dom}\,(T) \subset \mathrm{cl}\,\mathrm{dom}\,(T)$ and so $\mathrm{cl}\,\mathrm{dom}\,(T)$ is convex as required.
As $\mathrm{range}\,(T) = \mathrm{dom}\,(T^{-1})$ and X^* is reflexive we are done. ■

More generally:

Theorem 18 (**Fitzpatrick, Phelps**) *Every locally maximal monotone operator on a Banach space has* $\mathrm{cl}\,\mathrm{range}\, T$ *convex.*

Proof. We suppose not and then that there are $\pm x^*$ in $\mathrm{cl}\,\mathrm{range}\, T$ of unit-norm but with midpoint $0 \notin \mathrm{cl}\,\mathrm{range}\, T$.

Proof. We build the ball

$$B' := \text{conv}\,\{\pm 2x^*, \alpha\, B_X^*\}$$

where $0 < \alpha < 1/2$ is chosen with

$$(\text{range}\, T) \cap 2\alpha B_X^* = \emptyset.$$

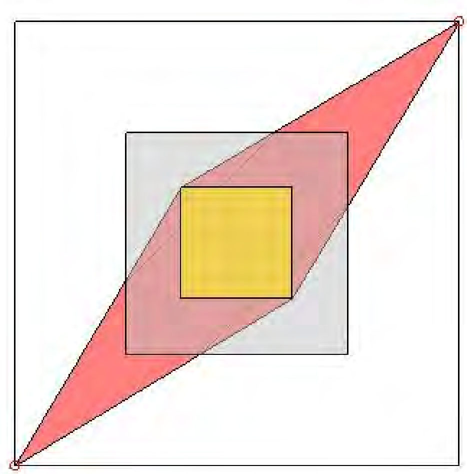
We extend $T \cap B'$ as in Prop. 5, so that

$R(\widehat{T}) \subset \text{cl conv}\,\{R(T) \cap B'\}$ and $R(\widehat{T}) \backslash R(T) \subset \text{bd}\, B'$.

It follows that

$\text{range}\,\widehat{T} \subset (R(T) \cap B') \bigcup (\text{cl conv}\,\{R(T) \cap B'\} \cap \text{bd}\, B')$.

Hence range $\widehat{T}$ is weak-star disconnected. As $\widehat{T}$ is a weak-star cusco it has a weak-star connected range which contradicts the construction. ∎



B' (red), $\alpha\, B_{X^*}$ (yellow) and $2\alpha\, B_{X^*}$ (grey)

Corollary 14 *Suppose T is maximal monotone on a reflexive Banach space X and is locally bounded at each point of* cl dom (T). *Then* dom $(T) = X$.

Proof. Observe dom (T) must be closed and so convex. By the Bishop-Phelps theorem, there is some boundary point $\overline{x} \in$ dom (T) with a non-zero support functional $\overline{x}^*$.
Then $T(\overline{x}) + [0, \infty)\, \overline{x}^*$ is monotonically related to the graph of T. By maximality

$$T(\overline{x}) + [0, \infty)\, \overline{x}^* = T(\overline{x})$$

which is non-empty and (linearly) unbounded. ■

6. Limiting Examples and Constructions

- It is unknown outside reflexive space whether cl dom (T) must always be convex for a maximal monotone operator

- Reflexivity in Theorem 17 may be relaxed to $R(T+J)$ is boundedly w^*-dense—as an examination of the proof will show

We do however have the following result:

Theorem 19 (JB-SF-Vanderwerff) *TFAE.*

(a) A Banach space X is reflexive

(b) int range (∂f) *is convex for each coercive lsc convex function f on X*

(c) int range (T) *is convex for each coercive maximal monotone mapping T.*

Proof. Suppose X is nonreflexive and $p \in X$ with $\|p\| = 5$ and $p^* \in Jp$ where J is the duality map. Define

$$f(x) := \max\left\{\frac{1}{2}\|x\|^2, \|x \mp p\| - 12 \pm \langle p^*, x\rangle\right\}$$

for $x \in X$. By the max-formula, for $x \in B_X$,

$$\partial f(\pm p) = B_{X^*} \pm p^*, \partial f(x) = Jx \tag{19}$$

using inequalities like $\|p - p\| - 12 + \langle p^*, p\rangle = 13 > \frac{25}{2} = \frac{1}{2}\|p\|^2$.

Moreover, $f(0) = 0$ and $f(x) > \frac{1}{2}\|x\|$ for $\|x\| > 1$, thus $\|x^*\| > \frac{1}{2}$ if $x^* \in \partial f(x)$ and $\|x\| > 1$. Combining this with (19) shows

$$\text{range}\,(\partial f) \cap \frac{1}{2}B_{X^*} = \text{range}\,(J) \cap \frac{1}{2}B_{X^*}.$$

Let $U := U_{X^*}$ denote the open unit ball in X^*. Now James' theorem gives $x^* \in \frac{1}{2}U_{X^*} \setminus \text{range}\,(J)$, thus $U_{X^*} \setminus \text{range}\,(\partial f) \neq \emptyset$. However, from (19)

$$U \subset \text{conv}\,\{(p^* + U) \cup (-p^* + U)\} \subset \text{conv}\,\text{int}\,R(\partial f)$$

so range (∂f) has non-convex interior. This shows that (b) implies (a) while (c) implies (b) is clear.

Finally (a) $\Rightarrow$ (c) follows from Theorem 17. ■

- Every locally maximal operator T has $\operatorname{cl}\operatorname{range} T$ convex (Fitzpatrick-Phelps)

Observe the two roles of convexity in the proof of (a) $\Leftrightarrow$ (c). One often uses the same logic to establish a result of the form

> *"Property P holds for all maximal monotone operators if and only if X is a Banach space with property Q."*

Two other examples are:

- *"Every monotone operator T on a space X is bounded on bounded subsets of $\operatorname{int}\operatorname{dom} T$ iff X is finite dimensional."*

- *"Every monotone operator T on a space X is single valued and norm-continuous on a generic subset of $\operatorname{int}\operatorname{dom} T$ iff X is an Asplund space."*

Example 3 *Most explicitly Fitzpatrick and Phelps used c_0, the space of null sequences, and*

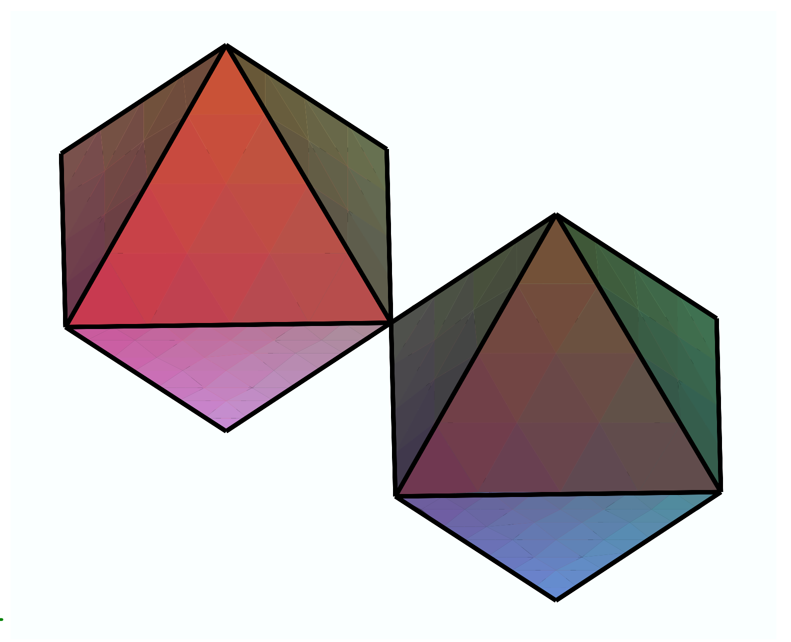
$$f(x) := \|x - e_1\|_\infty + \|x + e_1\|_\infty \tag{20}$$

where e_1 is first unit vector. Then int range ∂f is not convex (disconnected):

$$\text{int range}(\partial f) = \left\{U_{\ell_1} + e_1\right\} \cup \left\{U_{\ell_1} - e_1\right\}$$

$$\text{cl-int range}(\partial f) = \left\{B_{\ell_1} + e_1\right\} \cup \left\{B_{\ell_1} - e_1\right\}$$

both of which are far from convex. ■



The range of ∂f in ℓ^1

▼ It is instructive to compute cl-range (∂f)

Example 4 *Gossez gives a coercive maximal monotone operator T with full domain whose range has a non-convex closure.*

T is of the form $2^{-n}J_{\ell^1} + S$ for some $n > 0$ large with bounded linear $S : \ell_1 \to \ell_\infty$ given by

$$(Sx)_n := -\sum_{k<n} x_k + \sum_{k>n} x_k, \quad \forall x = (x_k) \in \ell_1, n \in N.$$

In fact, $\mp S : \ell_1 \mapsto \ell_\infty$ is skew bounded and S^* is not monotone but $-S^*$ is.

- Hence, $-S$ is both of dense type and locally maximal monotone (also called FP) while S is in neither class (Bauschke-JMB) ■

- Relatedly, let ι be the injection of ℓ^1 into ℓ^∞. For small $\epsilon > 0$

  $$S_\varepsilon := \varepsilon\iota + S$$

  is a coercive maximal monotone operator for which the closure $\overline{S_\varepsilon}$ fails to be coercive in X^{**}.

  One may use a smooth renorming of ℓ_1. This means $T + \lambda J$ is single-valued, demicontinuous.

Example 5 (Some further related results) *More abstractly, one can show that if the underlying space X is* rugged, *meaning* $\text{cl span range}\,(J - J) = X^*$, *then the following are equivalent whenever T is bounded linear and maximal monotone:*

i) T *is of dense type.*

ii) $\text{cl} - \text{range}\,(T + \lambda J) = X^*$, $\forall \lambda > 0$.

iii) $\text{cl} - \text{range}\,(T + \lambda J)$ *is convex,* $\forall \lambda > 0$.

iv) $T + \lambda J$ *is locally maximal monotone,* $\forall \lambda > 0$.

- Equivalences i)–iv) hold for the following rugged spaces: c_0, c, ℓ_1, ℓ_∞, $L_1[0,1]$, $L_\infty[0,1]$, $C[0,1]$.

  In cases like c_0 or $C[0,1]$, which contain no complemented copy of ℓ_1, a maximal monotone bounded linear T is always of dense type.*

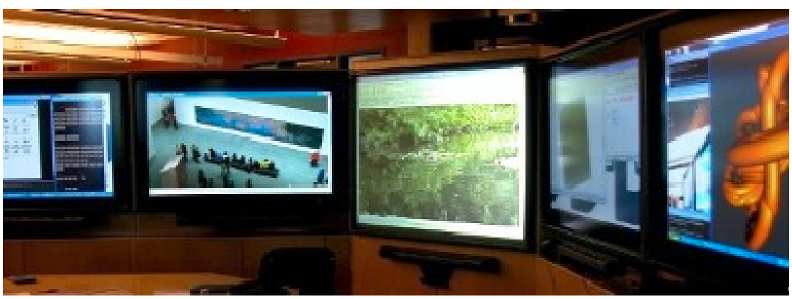
  In particular, S in Example 4 is necessarily not of dense type, etc.

*SF and JMB spent several weeks in 1994 looking for a counter-example in C[0,1].

7. Conclusion

Fitzpatrick's function was built to provide a transparent convex alternative to earlier saddle function constructions of Krauss. His interests were more in differentiation theory for Lipschitz functions.

Results relating when a maximal monotone T is single-valued to differentiability of $\mathcal{F}_T$ were not forthcoming, and he put the function aside.



D-Drive

- This is still the one area where to the best of my knowledge $\mathcal{F}_T$ has proved of little help—in part because generic properties of dom $\mathcal{F}_T$ and of dom (T) seem poorly related.

- By contrast, Fitzpatrick's function and its relatives now provide the easiest access to a gamut of solvability and boundedness results.

The clarity of the constructions also offers hope for resolving some of the most persistent open questions about maximal monotone operators such as:

Q3. Must $\mathrm{cl}\,\mathrm{dom}\,(T)$ always be convex? Simons shows this is so for operators of *dual type (FPV)*.

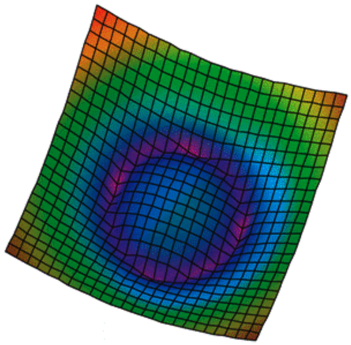
Q4. Can $T_1 + T_2$ fail be maximal when

$$0 \in \mathrm{core}\,\mathrm{conv}\,(\mathrm{dom}\,(T_1) - \mathrm{dom}\,(T_2))?$$

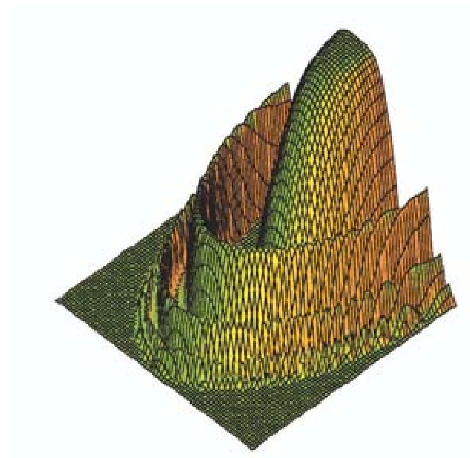
Q5. Given a maximal monotone T, can one associate a convex f_T with T in such fashion that $T(x)$ is singleton as soon as $\partial f_T(x)$ is?

Q6. Are there some nonreflexive spaces, such as c_0, for which such questions can be answered in the affirmative?*

*__Conjecture.__ On c_0 all maximal operators are type (NI).



Non-convex functions are hard too ...



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