Effective Computation of Bessel Functions

David Borwein, Jonathan M. Borwein, and O-Yeat Chan

AMS-SIAM Session on *Asymptotic Methods in Analysis with Apps*, Jan 6th 2008

Talk at www.cs.dal.ca/~jborwein

"Harald Bohr is reported to have remarked 'Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove.' "

J.H. Garling)

R … e *Cauchy Schwarz Master Class in* the M … , 575-579.

Joint Mathematics Meetings
San Diego
January 6–9, 2008
Sunday - Wednesday
San Diego Convention Center

Harald Bohr
1887-1951

Revised 04/01/2007

http://ddrive.cs.dal.ca/~isc/portal.html

Jon Borwein's Math Resource Portal

The following is a list to useful math tools.

Utilities

1. ISC2.0: The Inverse Symbolic Calculator
2. EZ Face : An interface for evaluation of Euler sums and Multiple Zeta Values
3. 3D Function Grapher
4. GraPHedron: Automated and computer assisted conjectures in graph theory
5. Julia and Mandelbrot Set Explorer

Reference

6. The On-Line Encyclopedia of Integer Sequences
7. Finch's Mathematical Constants
8. The Digital Library of Mathematical Functions

NIST Digital Library of Mathematical Functions

1 Algebraic & Analytic Methods
2 Asymptotic Approximations
3 Numerical Methods
4 Elementary Functions
5 Gamma Function
6 Exponential, Logarithmic, Sine & Cosine Integrals
7 Error Functions, Dawson's & Fresnel Integrals
8 Incomplete Gamma & Related Functions
9 Airy & Related Functions
10 Bessel Functions
11 Struve & Related Functions
12 Parabolic Cylinder Functions
13 Confluent Hypergeometric Functions
14 Legendre & Associated Legendre Functions
15 Hypergeometric Function
16 Generalized Hypergeometric Functions & Meijer G-Function
17 q-Hypergeometric Functions
18 Orthogonal Polynomials
19 Elliptic Integrals
20 Theta Functions
21 Multidimensional Theta Functions
22 Jacobian Elliptic Functions
23 Weierstrass Elliptic & Modular Functions
24 Bernoulli & Euler Polynomials
25 Zeta & Related Functions
26 Combinatorial Analysis
27 Functions of Number Theory
28 Mathieu Functions & Hill's Equation
29 Lamé Functions
30 Spheroidal Wave Functions
31 Heun Functions
32 Painlevé Transcendents
33 Coulomb Functions
34 3j, 6j, 9j Symbols
35 Functions of Matrix Argument
36 Integrals with Coalescing Saddles
37 Computer Algebra
Bibliography

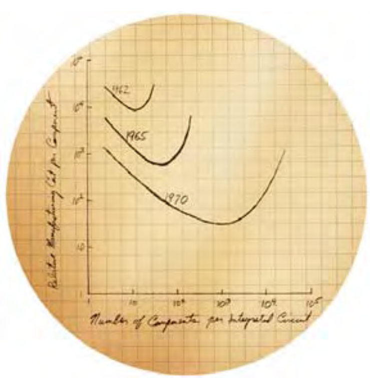
Karl Dilcher

Current Content Status

Abstract and Outline

Bessel functions are among the most important functions in mathematical physics and the theory of special functions.The ability to compute their values is equally important.

The standard method of evaluating the Bessel functions has been to use an ascending series for small argument, and the **asymptotic** (but divergent) **series** for large argument. In this talk, we describe a new series (based on arc-trig series) that is *geometrically convergent* in the number of summands, with explicitly computable error estimates for the tails.



Moore's Law

- **Motivation and Context (JMB)**
  - **Earlier Talk on Laguerre Asymptotics**
- **Our New Algorithms (O-YC)**
  - **Preprint related to Current Talk**

Motivation and Context

A primary research motive for providing effective asymptotics lies in a beautiful Laguerre series for the incomplete gamma function (see [1]), namely [14]

$$\begin{aligned}\Gamma(a,z) &= z^a e^{-z}\cfrac{1}{z+\cfrac{1-a}{1+\cfrac{1}{z+\cfrac{2-a}{1+\cdots}}}}\\ &= \sum_{n=0}^{\infty}\frac{(1-a)_n}{(n+1)!}\,\frac{1}{L_n^{(-a)}(-z)\,L_{n+1}^{(-a)}(-z)},\end{aligned} \tag{5}$$

where $(c)_n := c(c+1)\cdots(c+n-1)$ is the *Pochhammer symbol*. This series is valid whenever none of the Laguerre denominators has a zero. Thus an interesting sidelight is the research problem of establishing zero-free regions for Laguerre polynomials (see our Open Problems section).

This was needed by Crandall to perform **high precision** computation of *Riemann-Zeta*, say near $(1+10^{10}i, 1+2\cdot 10^{10}i)$ to treat primes around 10^{20}

Motivation and Context

For example we obtain the large n asymptotic

$$L_n^{(-a)}(-z) \sim S_n(a,z)\left(1 + O(m^{-1/2})\right), \tag{3}$$

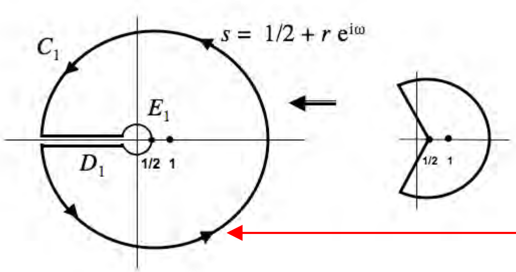
where the sub-exponential term S is

$$S_n(a,z) := \frac{e^{-z/2}}{2\sqrt{\pi}} \frac{e^{2\sqrt{mz}}}{z^{1/4-a/2}\ m^{1/4+a/2}}. \tag{4}$$

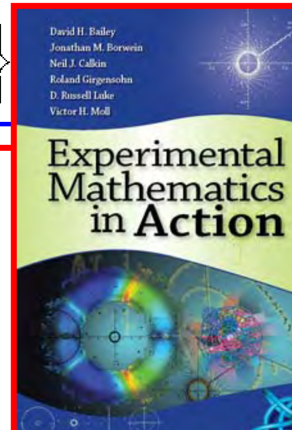
In such expressions, $\Re\left(\sqrt{mz}\right)$ is taken to be $\sqrt{m|z|}\cos(\theta/2)$ where $\theta := \arg(z) \in (-\pi, \pi]$ (we hereby adopt the convention $\arg(-1) := \pi$), and so for $(a,z) \in \mathcal{D}$ the expression (4) involves genuinely diverging growth in n, due to the sub-exponential $\exp(2\sqrt{mz})$ factor.

What we seek are effective bounds, for example to replace a logical error-bounding statement for an expression E in the following way:

$$\left\{E = O\left(\frac{1}{\sqrt{m}}\right)\right\} \quad \text{is replaced by} \quad \left\{E < \frac{C}{\sqrt{m}} \text{ for } m > m'\right\}$$



We manage this in part by finding the most effective contour for Laguerre polynomials experimentally (C_1 dominates)



Motivation and Context

Ch.10. Bessel Functions
Properties: Bessel and Hankel Functions
§10.1. Notation §10.3. Graphs and Visualizations

§10.2. Definitions

Contents

§10.2(i). Bessel's Equation

10.2.1

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0.$$

This differential equation has a regular singularity at $z = 0$ with indices $\pm\nu$, and an irregular singularity at $z = \infty$ of rank 1; compare §§2.8(i) and 2.8(ii).

§10.2(ii). Standard Solutions

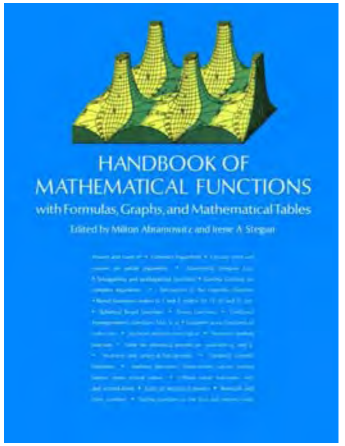
Bessel Function of the First Kind

10.2.2

$$J_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k!\,\Gamma(\nu + k + 1)}.$$

This is what the new DLMF (A&S) provides

(with metadata suppressed)



This solution of (10.2.1) is an analytic function of $z \in \mathbb{C}$, except for a branch point at $z = 0$ when ν is not an integer. The *principal branch* of $J_\nu(z)$ corresponds to the principal value of $\left(\frac{1}{2}z\right)^\nu$ (§4.2(iv)) and is analytic in the z-plane cut along the interval $(-\infty, 0]$.

When $\nu = n$ $(\in \mathbb{Z})$, $J_\nu(z)$ is entire in z.

For fixed z $(\neq 0)$ each branch of $J_\nu(z)$ is entire in ν.

http://dlmf.nist.gov/X/

Motivation and Context

Inter alia we obtained expressions for J_ν (and I_ν) at **integer** order:

$$J_n(z) = \frac{2}{\pi}\sum_{k=0}^{\infty} g_k(-2in)\left(b_k \cos\chi - c_k \sin\chi\right), \qquad (66)$$

with angle

$$\chi := z - \pi n/2 - \pi/4,$$

$$g_k(\tau) := \prod_{i=1}^{k} \tau^2 + (2j-1)^2$$

and the coefficients b_k, c_k determined by

$$b_k := B_k(iz)e^{i\pi/4} + B_k(-iz)e^{-i\pi/4},$$
$$ic_k := B_k(iz)e^{i\pi/4} - B_k(-iz)e^{-i\pi/4}.$$

Note that if z is real then each b_k, c_k is real, whence our series here has all real terms. Note that our recursion (45) likewise ignites a recursion amongst the b_k, c_k.

B_k is an error-function-class integral

$$B_k(p) := \beta_{2k-1}(p, \pi/4) = \int_0^{1/\sqrt{2}} x^{2k} e^{-2px^2}\, dx. \qquad (43)$$

It is both computationally and theoretically important that B_k can be given a closed form (in terms of Γ- and incomplete Γ-functions) as well as a recursion relation. Namely, we have

$$B_k(p) = \frac{1}{2}\frac{1}{(2p)^{k+1/2}}\left\{\Gamma(k+1/2) - \Gamma(k+1/2, p)\right\}, \qquad (44)$$

Motivation and Context

This development depended critically on the following **exp-arc** expansions:

For any complex τ and $x \in [-1,1]$, one has a remarkable expansion :

$$e^{\tau \arcsin x} = 1 + \sum_{k=1}^{\infty} r_k(\tau) \frac{x^k}{k!}, \qquad (1)$$

where the coefficients depend on the parity of the index, as

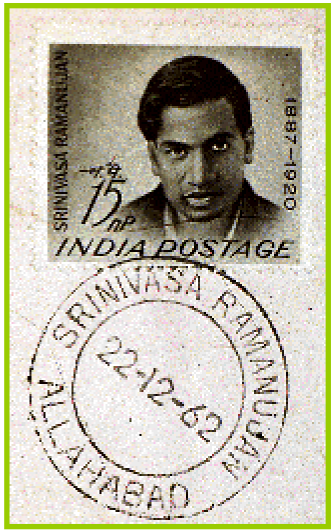
$$r_{2m+1}(\tau) := \tau \prod_{j=1}^{m} \left(\tau^2 + (2j-1)^2\right), \qquad r_{2m}(\tau) := \prod_{j=1}^{m} \left(\tau^2 + (2j-2)^2\right).$$

By differentiating with respect to x we obtain

$$\frac{e^{\tau \arcsin x}}{\sqrt{1-x^2}} = \frac{1}{\tau} \sum_{k=0}^{\infty} r_{k+1}(\tau) \frac{x^k}{k!},$$

valid for $x \in (-1,1)$.

I learned (1) from Ramanujan and Berndt while doing number theory



Motivation and Context

We noted more generally, for z and ν having positive real part, that

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu t - z\sin t)\, dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu t - z\sinh t}\, dt, \tag{68}$$

with a corresponding representation

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos t}\cos(\nu t)\, dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu t - z\cosh t}\, dt, \tag{69}$$

itself valid for the same cases of z, ν. One wonders whether an exp-arc approach can be used to resolve the integrals here—which contribute when ν is not an integer—as exp-arc series.

The paper concluded with several open questions: notably

- It would be useful to establish the very most efficient way to calculate $J_n(z)$ with our converging series (66) and to know, for given arguments n, z how many terms of the exp-arc sum yield b good bits in the answer for $J_n(z)$. It should also be possible to extract the classical ascending series for J_n directly from our converging series.
- Can the integral pieces of (68, 69) be resolved as exp-arc series, to provide even more general, universally convergent I, J series (i.e. for noninteger ν)?

Drive

And this is what we now consider …

Effective Computation of Bessel Functions, Part II

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For any complex pair (p, q) and real numbers $\alpha, \beta \in (-\pi, \pi)$, let

$$\mathcal{I}(p, q, \alpha, \beta) := \int_{\alpha}^{\beta} e^{-iq\omega} e^{p\cos\omega} d\omega.$$

Then we have the absolutely convergent representation

$$\mathcal{I}(p, q, \alpha, \beta) = \frac{ie^p}{q} \sum_{k=0}^{\infty} \frac{r_{k+1}(-2iq)}{k!} \int_{\sin\frac{\alpha}{2}}^{\sin\frac{\beta}{2}} x^k e^{-2px^2} dx,$$

where

$$r_{2m+1}(\nu) := \nu \prod_{j=1}^{m} \left(\nu^2 + (2j-1)^2\right), \quad r_{2m}(\nu) := \prod_{j=1}^{m} \left(\nu^2 + (2j-2)^2\right).$$

These are, you may recall, the coefficients in the series expansion of $\exp(\arcsin x)$.

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In particular, for the case where $(\alpha,\beta) = (-\pi/2,\pi/2)$, we have

$$\mathcal{I}(p,q) := \mathcal{I}(p,q,-\pi/2,\pi/2) = \frac{2ie^p}{q}\sum_{k=0}^{\infty}\frac{r_{2k+1}(-2iq)}{(2k)!}B_k(p),$$

with

$$B_k(p) := \int_0^{1/\sqrt{2}} x^{2k}e^{-2px^2}dx = \frac{1}{2^{k+1}\sqrt{2}}\int_0^1 e^{-pu}u^{k-\frac{1}{2}}du$$

$$= -\frac{e^{-p}}{p2^{k+1}\sqrt{2}} + \left(k-\frac{1}{2}\right)\frac{B_{k-1}(p)}{2}.$$

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$$= -\frac{e^{-p}}{p 2^{k+1}\sqrt{2}} + \left(k - \frac{1}{2}\right) \frac{B_{k-1}(p)}{2}.$$

For integral order, we have from the Laguerre paper

$$J_n(z) = \frac{1}{2\pi}\left(e^{-in\pi/2}\mathcal{I}(iz,n) + e^{in\pi/2}\mathcal{I}(-iz,n)\right),$$

and

$$I_n(z) = \frac{1}{2\pi}\left(\mathcal{I}(z,n) + \cos(\pi n)\mathcal{I}(-z,n)\right).$$

As Jon mentioned, we want to use the integral representations to get expressions for general ν.

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The integral representations are:

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu t - z\sin t)dt - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\nu t - z\sinh t}dt,$$

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin t - \nu t)dt$$

$$- \frac{1}{\pi}\int_0^\infty (e^{\nu t} + e^{-\nu t}\cos\nu\pi)e^{-z\sinh t}dt,$$

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos t}\cos\nu t\, dt - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-z\cosh t - \nu t}dt,$$

and

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh\nu t\, dt = \frac{1}{2}\int_{-\infty}^\infty e^{-z\cosh t - \nu t}dt.$$

The integrals on $[0, \pi]$ can be expressed in terms of the $\mathcal{I}$ function. Specifically,

$$J_\nu(z) = \frac{1}{2\pi}\left(e^{-i\nu\pi/2}\mathcal{I}(iz,\nu) + e^{i\nu\pi/2}\mathcal{I}(-iz,\nu)\right) - \frac{\sin\nu\pi}{\pi}\int_0^\infty \cdots$$

$$Y_\nu(z) = \frac{1}{2\pi i}\left(e^{-i\nu\pi/2}\mathcal{I}(iz,\nu) - e^{i\nu\pi/2}\mathcal{I}(-iz,\nu)\right) - \frac{1}{\pi}\int_0^\infty \cdots$$

$$I_\nu(z) = \frac{1}{2}\Big(\mathcal{I}(z,\nu) + e^{i\nu\pi}\mathcal{I}(-z,\nu,0,\pi/2) + e^{-i\nu\pi}\mathcal{I}(-z,-\nu,0,\pi/2)\Big)$$
$$- \frac{\sin\nu\pi}{\pi}\int_0^\infty \cdots$$
$$= \frac{1}{2\pi}\left(\mathcal{I}(z,\nu) + \cos\nu\pi\mathcal{I}(-z,\nu) - \sin\nu\pi\mathcal{I}^*(-z,\nu)\right)$$
$$- \frac{\sin\nu\pi}{\pi}\int_0^\infty \cdots,$$

where

$$\mathcal{I}^*(z,\nu) = \frac{2e^z}{\nu}\sum_{n=0}^{\infty}\frac{r_{2n+2}(2i\nu)}{(2n+1)!}B_{n+\frac{1}{2}}(z).$$

To get the generalizations we want, we basically just need to evaluate the infinite integrals.

Let us look at the integrals in the J and Y cases. A change of variables plus integration by parts gives us

$$\int_0^\infty e^{-\nu t - z\sinh t}\,dt = \frac{1}{\nu} - \frac{z}{\nu}\int_0^\infty e^{-zs} e^{-\nu \operatorname{arcsinh} s}\,ds.$$

The expansion of $e^{-\nu \operatorname{arcsinh} s}$ about $s = 0$, used in the finite case to obtain the series, is only valid on $[0, 1)$.

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For large s, it makes sense to expand about infinity!

The series, valid on $(1, \infty)$, is

$$s^{\nu} e^{-\nu \operatorname{arcsinh} s} = \sum_{n=0}^{\infty} \frac{A_n(\nu)}{s^{2n}},$$

where $A_0(\nu) = 2^{-\nu}$ and for $n \geq 1$,

$$A_n = -\frac{(\nu + 2n - 2)(\nu + 2n - 1)}{4n(n+\nu)} A_{n-1},$$

from which we easily obtain

$$A_n(\nu) = \frac{(-1)^n \nu 2^{-\nu} (\nu + n + 1)_{n-1}}{2^{2n} n!}.$$

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Note that when ν is a negative integer, we have problems with the recurrence.

When $n = \lfloor (1-\nu)/2 \rfloor$, the numerator is 0. When $n = -\nu$, the denominator is zero.

In this case, $A_n(\nu) = (-1)^{\nu+1} A_{n+\nu}(-\nu)$ for $n \geq -\nu$

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If we only used the expansions at 0 and ∞, we could get a series; but there are issues with interchanging summations and integration, since we are integrating up to the boundary of the interval of convergence.

Even after justifying the interchange, the resulting series is very slow due to the “bad” approximation by the series near the boundary.

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Localize!

For fixed k, $f_k(s) := e^{-\nu \operatorname{arcsinh}(k+s)}$ satisfies the second order differential equation

$$f_k''(s) = \frac{1}{k^2+1+2ks+s^2}\left(\nu^2 f_k(s) - (k+s)f_k'(s)\right).$$

So if we set

$$e^{-\nu \operatorname{arcsinh}(k+s)} = \sum_{n=0}^{\infty} \frac{a_n(k,\nu)}{n!} s^n,$$

then we have the recurrence relation

$$a_{n+2} = \frac{1}{k^2+1}\left((\nu^2-n^2)a_n - k(2n+1)a_{n+1}\right),$$

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Localize!

For fixed k, $f_k(s) := e^{-\nu \operatorname{arcsinh}(k+s)}$ satisfies the second order differential equation

$$f_k''(s) = \frac{1}{k^2 + 1 + 2ks + s^2} \left(\nu^2 f_k(s) - (k+s) f_k'(s) \right).$$

So if we set

$$e^{-\nu \operatorname{arcsinh}(k+s)} = \sum_{n=0}^{\infty} \frac{a_n(k,\nu)}{n!} s^n,$$

then we have the recurrence relation

$$a_{n+2} = \frac{1}{k^2+1} \left((\nu^2 - n^2) a_n - k(2n+1) a_{n+1} \right),$$

with

$$a_0 = (k + \sqrt{k^2+1})^{-\nu}, \qquad a_1 = -\frac{\nu a_0}{\sqrt{k^2+1}}.$$

We can subdivide $[0, \infty)$ into the intervals $[0, 1/2], [1/2, 3/2], \ldots, [N - 1/2, N + 1/2], [N + 1/2, \infty)$ and on each interval expand $e^{-\nu \operatorname{arcsinh} s}$ at k, the centre of the interval.

Each of these series has radius of convergence $\sqrt{k^2 + 1}$ and so we may interchange summation and integration.

For the infinite interval at the end, we use the expansion about infinity.

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Thus for any positive integer N, we have

$$\int_0^\infty e^{-zs} e^{-\nu \operatorname{arcsinh} s} ds =$$

$$\sum_{n=0}^{\infty} \left(\frac{a_n(0,\nu)}{n!} \alpha_n(z) + \beta_n(z) \sum_{k=1}^{N} e^{-kz} \frac{a_n(k,\nu)}{n!} \right.$$

$$\left. + A_n(\nu) G_n(N + \tfrac{1}{2}, z, \nu) \right),$$

where

$$\alpha_n(z) := \int_0^{1/2} e^{-zs} s^n ds = -\frac{e^{-z/2}}{2^n z} + \frac{n}{z} \alpha_{n-1}(z),$$

$$\beta_n(z) := \int_{-1/2}^{1/2} e^{-zs} s^n ds = \frac{e^{z/2}}{(-2)^n z} - \frac{e^{-z/2}}{2^n z} + \frac{n}{z} \beta_{n-1}(z),$$

and

$$G_n(\theta, z, \nu) := \frac{e^{-\theta z}}{\theta^{2n+\nu-1}} \int_0^\infty e^{-\theta z s}(1+s)^{-2n-\nu} ds$$

$$= \frac{1}{(\nu+2n-1)(\nu+2n-2)} \times \left(\frac{e^{-\theta z}(\nu+2n-2-\theta z)}{\theta^{2n+\nu-1}} + z^2 G_{n-1}(\theta, z, \nu) \right).$$

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So we have found a representation for the Bessel functions in terms of several sums:

Sums involving $\mathcal{I}$ from the integral on $[0, \pi]$, where each summand looks like

$$\frac{r_{n+1}(2\nu)}{n!} B_{n(+1/2)}(z),$$

sums from the subdivisions of the real line on the infinite integral, where a typical summand is

$$\frac{a_n(k, \nu)}{n!} \beta_n(z) e^{-kz},$$

and the sum from the tail, where each summand is

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Let us first look at

$$\frac{r_{n+1}(2\nu)}{n!}.$$

For simplicity we consider the case n even, $n = 2m$. Then this is

$$\prod_{j=1}^{m}\left(1 - \frac{1}{2j} - \frac{4\nu^2}{(2j-1)(2j)}\right),$$

which is bounded and decreasing for $m > 2|\nu|^2$. Similarly for odd n.

Also, (for *arbitrary n*)

$$B_n(z) = \frac{1}{2^{n+3/2}}\int_0^1 e^{-zu}u^{n-1/2}du$$

so it is bounded by

$$|B_n(z)| \leq \frac{\max(1, e^{-\operatorname{Re}(z)})}{2^{n+3/2}}.$$

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Thus the terms of type

$$\frac{r_{n+1}(2\nu)}{n!} B_n(z) = O_{\nu,z}(2^{-n}),$$

where the big-O constant can be explicitly computed.

For terms of the type

$$\frac{a_n(k,\nu)}{n!}\beta_n(z)e^{-kz},$$

note that $a_n(k,\nu)/n!$ are the Taylor coefficients, and so they are $O\left(\frac{1}{(k^2+1)^{n/2}}\right)$ from the radius of convergence. We can fairly easily get a weaker but explicit geometric bound using the recurrence relation for $a_n(k,\nu)$.

$\beta_n(z)$ is the n-th moment of the exponential, and can be explicitly computed. A simple estimate yields

$$|\beta_n(z)e^{-kz}| \leq \frac{e^{-(k-1/2)\,\mathrm{Re}(z)}}{2^n}.$$

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For terms of the type

$$A_n(\nu)G_n(N+\tfrac{1}{2},z,\nu),$$

we can get a bound

$$|A_n(\nu)| \le \frac{|\nu 2^{\lceil|\nu|\rceil-\nu-1}|}{n}$$

from the explicit formula,

and use bounds for the incomplete gamma function to get explicit big-O constants for the bound

$$G_n(N+\tfrac{1}{2},z,\nu) = O_{\nu,z}((N+1/2)^{-\mathrm{Re}(\nu)-2n}).$$

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Putting it all together, we see that the (slowest) sums converge like 2^{-n}, and with explicit big-O constants we may determine how many terms are needed for a specific accuracy.

Other features to note:

- For each type of sum, the summands are all computable via recursion.
- The most difficult computation involved are the computation of B_0 and G_0, each of which involves an incomplete gamma evaluation. It should be noted that this can be done via continued fractions, so this scheme can be thought of as a continued fraction evaluation scheme for Bessel functions.
- The sum involving $A_n G_n$ is bounded like $O_\nu(e^{-z(N+1/2)})$ by estimating the integral of the tail. So one can avoid the computation of G_0 altogether by choosing a large enough N.

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- Along the same lines, one does not need to compute all of the sums involving β_n for large k unless one needs accuracy beyond about $e^{-(k-1/2)\,\mathrm{Re}(z)}$.
- In addition to choosing an optimal N, one can also adjust the intervals in dividing the integral on $[0,\infty)$. In particular, the sum arising out of an interval on (a,b) expanded at k converges like $O\left((b-a)e^{-a\,\mathrm{Re}(z)}\frac{\max(|k-a|^n,|b-k|^n)}{(k^2+1)^{n/2}}\right)$.

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Our computation scheme has some advantages over the traditional ascending-asymptotic switching scheme:

- Our series are all uniformly geometrically convergent, whereas some asymptotic formulas are divergent series, and some are only algebraically convergent (i.e., like $n^{-\alpha}$ rather than 2^{-n}).
- Each summand in our series is a product of functions that depend only on ν or only on z, and thus these values can be stored and recycled for one-ν-many-z or one-z-many-ν computations. Note also that each of these functions is eventually decreasing.

The following table compares the performance between the ascending series, the standard divergent asymptotic series, and our series for J_ν with the choice $N = 1$.

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Table: Comparison between various series for $J_\nu(z)$.

		Absolute value of the difference between the true value and		
(ν, z)	M	Ascending Series	Asymptotic Series	Exp-arc Series
$\nu = 6.2$ $z = 100$	10	10^{22}	10^{-32}	10^{-5}
	50	10^{41}	10^{-76}	10^{-18}
	100	10^{22}	10^{-89}	10^{-33}
	150	10^{-19}	10^{-79}	10^{-49}
	200	10^{-75}	10^{-55}	10^{-64}
$\nu = 12.3$ $z = 50$	10	10^{18}	10^{-23}	10^{2}
	30	10^{17}	10^{-41}	10^{-10}
	50	10^{6}	10^{-45}	10^{-17}
	70	10^{-11}	10^{-42}	10^{-23}
	100	10^{-45}	10^{-28}	10^{-33}
$\nu = 12.3$ $z = 75 + 57i$	10	10^{27}	10^{-4}	10^{13}
	50	10^{38}	10^{-48}	10^{-17}
	100	10^{14}	10^{-59}	10^{-33}
	120	10^{-2}	10^{-56}	10^{-39}
	150	10^{-31}	10^{-47}	10^{-48}
	200	10^{-89}	10^{-20}	10^{-64}

- Thank you for your attention!
- The paper is in press in **JMAA**.
- A preprint is available at the AARMS docserver
  `http://locutus.cs.dal.ca:8088/archive/00000371/`