Future Prospects for Computer-assisted Mathematics (CMS Notes 12/05)

UNIVERSITÉ LAVAL

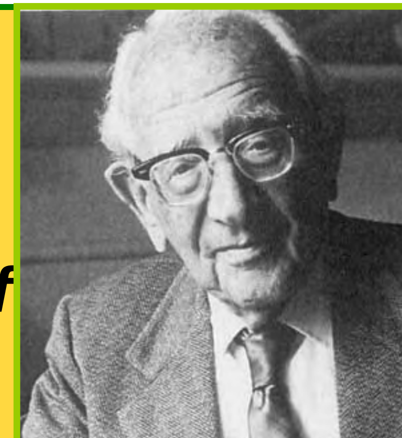
Mai 24

What is HIGH PERFORMANCE MATHEMATICS?

Jonathan Borwein, FRSC www.cs.dal.ca/~jborwein

Canada Research Chair in Collaborative Technology

"intuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication."



George Polya 1887-1987

ISM Institut des sciences mathématiques

DALHOUSIE UNIVERSITY *Inspiring Minds*

Revised 06/05/06

<u>2003: Me and my Avatar</u>
Designer now works for William Shatner ('Wild')

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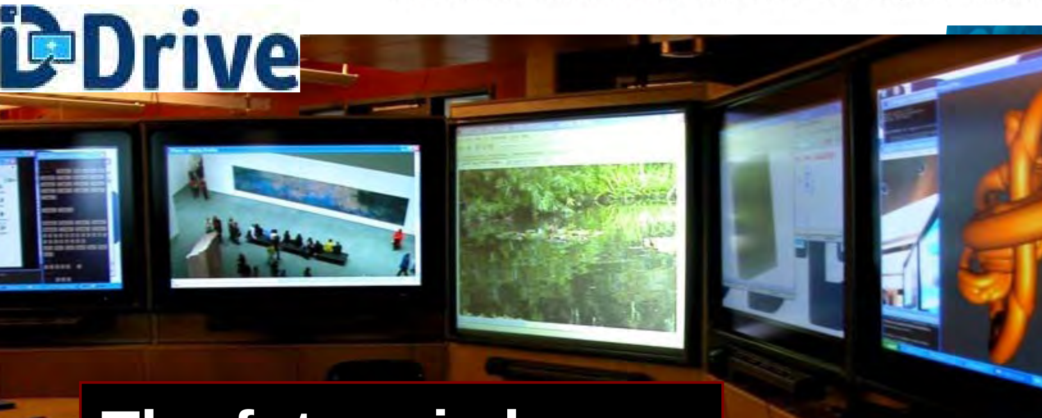
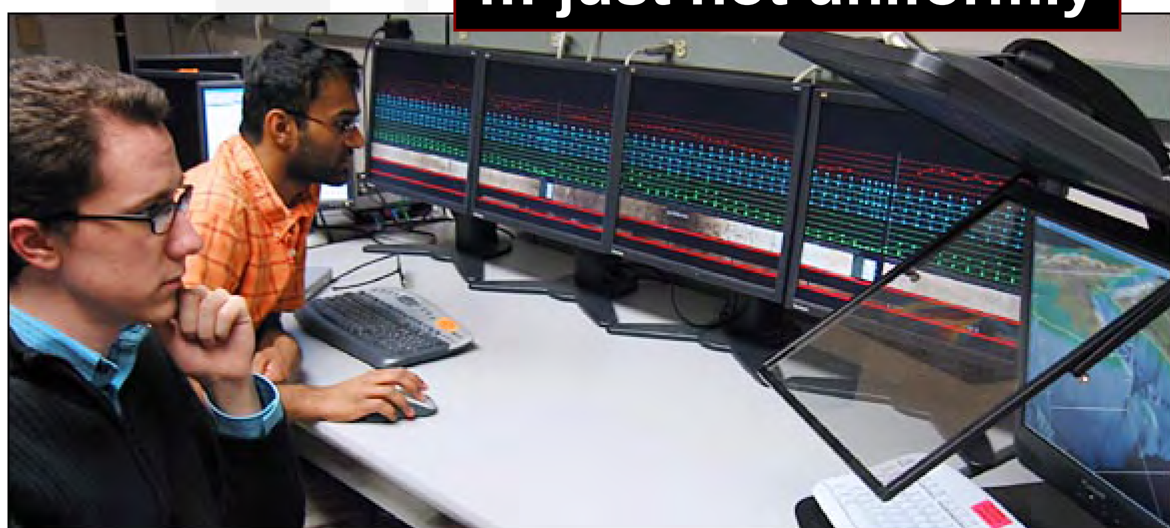


The future is here…

… just not uniformly

Remote Visualization via **Access Grid**

- The touch sensitive interactive **D-DRIVE**
- Immersion & Haptics
- and the 3D **GeoWall**



What is HIGH PERFORMANCE MATHEMATICS?

Some of my examples will be very high-tech
but most of the benefits can be had via

VOIP/SKYPE and a WEBCAM

MAPLE or MATLAB or …

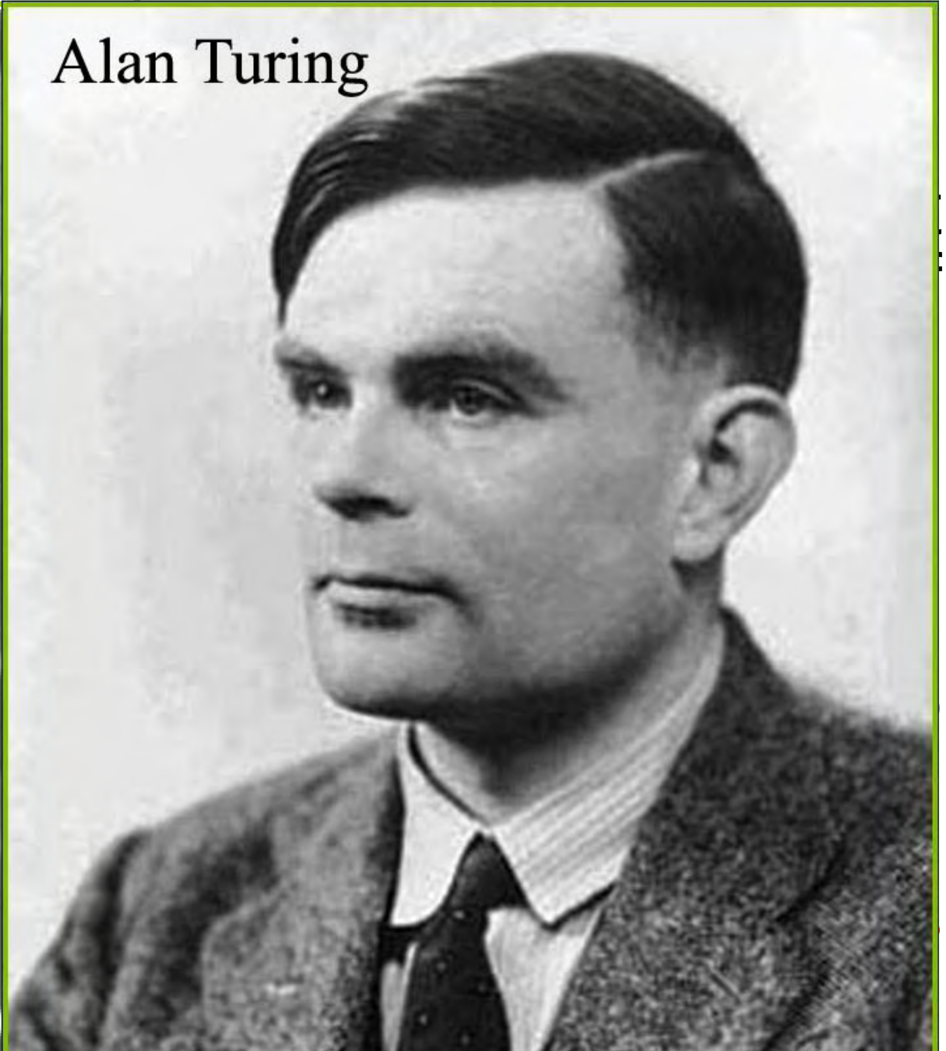
A REASONABLE LAPTOP

A SPIRIT OF ADVENTURE

in almost all areas of mathematics

ABSTRACT

"If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics." (**Kurt Godel,**1951)



We shall explore various tools available for deciding what to believe in mathematics, and, using accessible often visual examples, illustrate the rich experimental tool-box mathematicians now have access to.

To explain how mathematicians may use **High Performance Computation** (HPC) and what we have in common with other computational scientists I shall mention various **HPM** problems including:

$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right) dx \stackrel{?}{=} \frac{\pi}{8},$$

which is both numerically and symbolically quite challenging ....

and is answered at the end

From the Magazine | Innovators

It Doesn't Figure

The Omega Man

By MICHAEL D. LEMONICK

From the Sep. 12, 2005 issue
Posted Sunday, Sep. 04, 2005
and
Scientific American March 2006

TIME • Online Edition

Chaitin's universal halting constant

Ω=0.0000001000000100001000001000011101110011001001111000100100111 00

(Calude)

Over the past few decades, G
IBM's T.J. Watson Resea
N.Y., has been uncovering
higher math may be riddle
really a collection of random
reason. And rather than
principles, "I'm making the
done more like physics
experim
I'm dead

Chaitin's idea
complicated
help make o
math: Gödel'
system of m
particular co

Sounds
mathematica
night and wo

Pour voir comment la valeur du nombre Ω (oméga) est définie, voici un exemple simplifié. Supposons que l'ordinateur considéré n'ait que trois programmes qui s'arrêtent et qu'ils soient représentés par les chaînes de bits 110, 11100 et 11110. Ces programmes ont, respectivement, une longueur de 3, 5 et 5 bits. Si nous choisissons un programme par hasard en tirant à pile ou face chaque bit, la probabilité de trouver ces chaînes est respectivement de $1/2^3$, $1/2^5$ et $1/2^5$, la probabilité pour chaque bit étant 1/2. La valeur de Ω, la probabilité d'arrêt, pour cet ordinateur particulier, est donc donnée par :
$\Omega = 1/2^3 + 1/2^5 + 1/2^5 = 0{,}001 + 0{,}00001 + 0{,}00001 = 0{,}00110$ (en écriture binaire). Ce nombre binaire correspond à la probabilité d'obtenir l'un des trois arrêts par hasard – c'est la probabilité que notre ordinateur s'arrêtera.

Comme le programme 110 s'arrête, nous ne considérerons pas de programmes commençant par 110 et plus longs, par exemple 1100 ou 1101. Par conséquent, il n'y a pas à ajouter des termes de type 0,0001 pour chacun de ces programmes. On considère tous les programmes tels que 1100 et ainsi de suite comme décrits par le programme qui s'achève 110. Quand ces programmes s'arrêtent, ils arrêtent de réclamer des bits supplémentaires.

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1a. Communication, Collaboration and Computation.

1b. Visual Data Mining in Mathematics.

- ✓ **Fractals, Polynomials, Continued Fractions***
- ✓ **Pseudospectra and Code Optimization**

2. High Precision Mathematics.

The talk ends when I do

3. Integer Relation Methods.

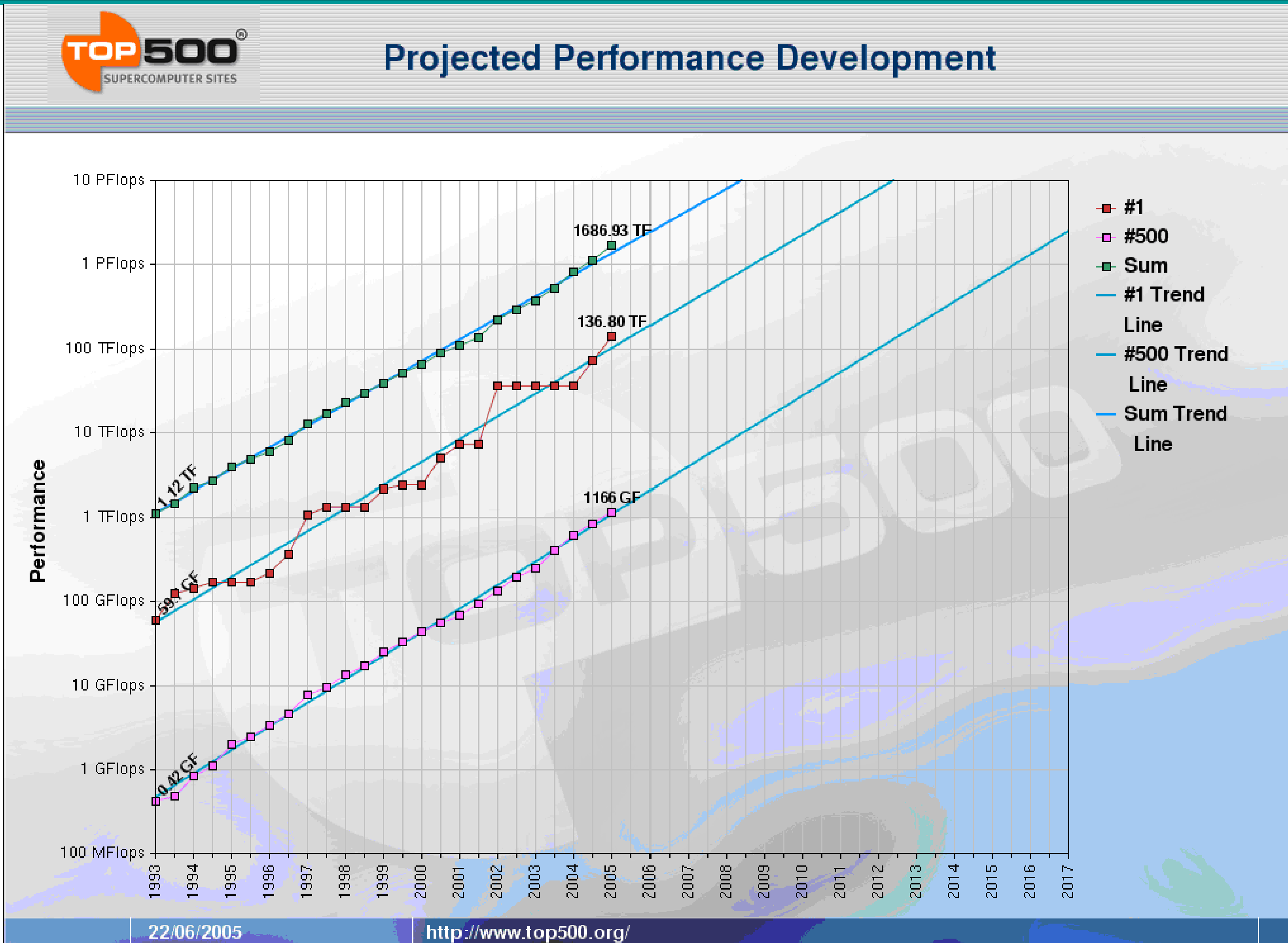
- ✓ **Chaos, Zeta* and the Riemann Hypothesis**
- ✓ **Hex-Pi and Normality**

4. Inverse Symbolic Computation.

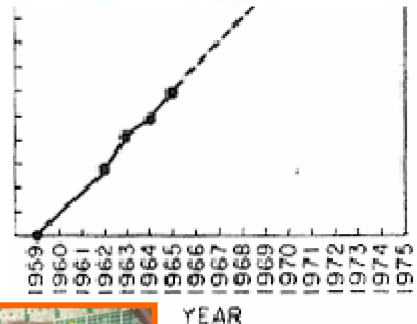
- ✓ **A problem of Knuth*, $\pi/8$, Extreme Quadrature**

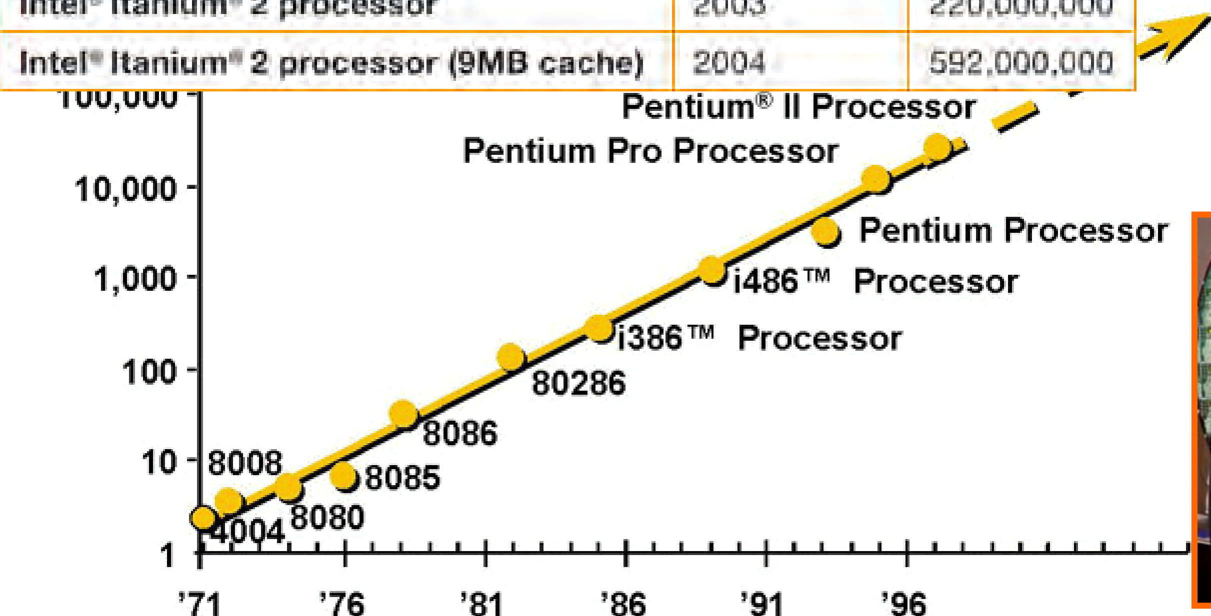
5. Demos and Conclusion.

Moore's 1965 Law continues:

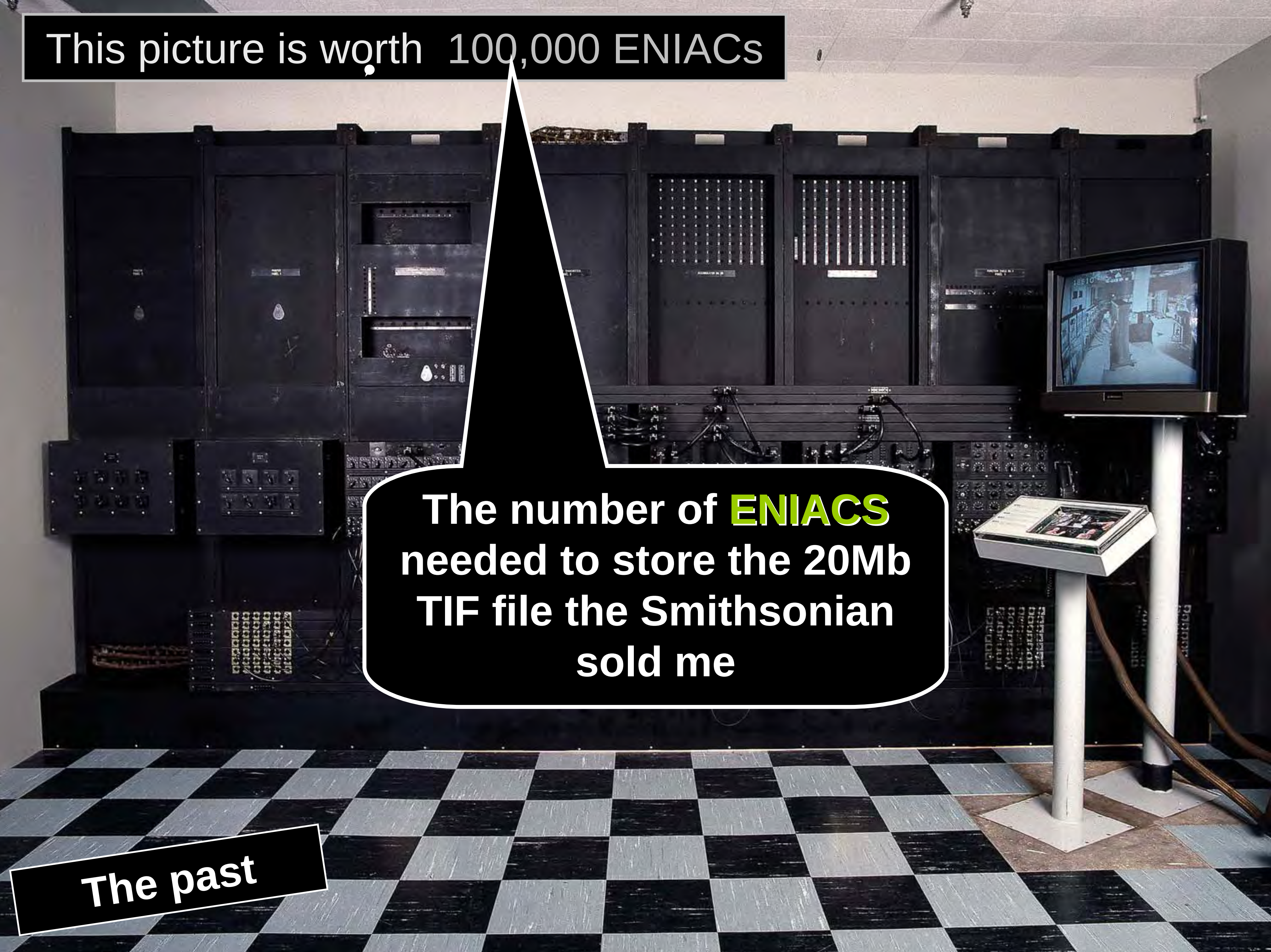


Microprocessor	Year of Introduction	Transistors
4004	1971	2,300
8008	1972	2,500
8080	1974	4,500
8086	1978	29,000
Intel286	1982	134,000
Intel386™ processor	1985	275,000
Intel486™ processor	1989	1,200,000
Intel® Pentium® processor	1993	3,100,000
Intel® Pentium® II processor	1997	7,500,000
Intel® Pentium® III processor	1999	9,500,000
Intel® Pentium® 4 processor	2000	42,000,000
Intel® Itanium® processor	2001	25,000,000
Intel® Itanium® 2 processor	2003	220,000,000
Intel® Itanium® 2 processor (9MB cache)	2004	592,000,000



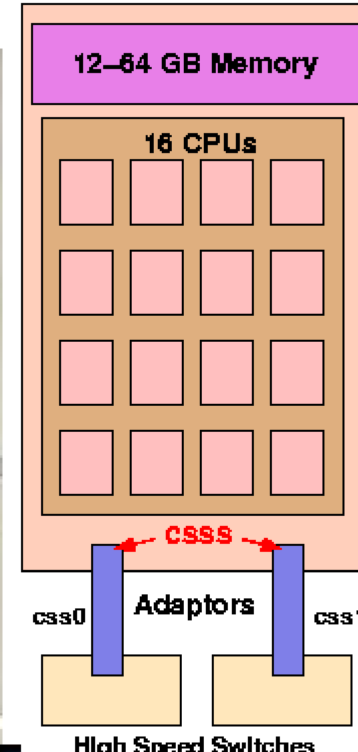


Moore's Law in 1965 and 2005


This picture is worth 100,000 ENIACs
The number of ENIACS needed to store the 20Mb TIF file the Smithsonian sold me
The past

NERSC's 6000 cpu Seaborg in 2004 (10Tflops/sec)

- we need new software paradigms for `bigga-scale' hardware



The present

Mathematical Immersive Reality
in Vancouver

IBM BlueGene/L system at LLNL

System
(64 cabinets, 64x32x32)

Supercomputer doubles own record

The Blue Gene/L supercomputer has broken its own record to achieve more than double the number of calculations it can do a second.

It reached 280.6 teraflops - that is 280.6 trillion calculations a second.

Blue Gene/L is the fastest computer in the world

5.6/11.2 GF/s
0.5 GB DDR

2.8/5.6 GF/s
4 MB

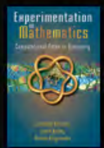
The future

2^{17} cpu's

Oct 2005 It has now run Linpack benchmark at over **280 Tflop /sec** **(4 x Canadian-REN)**

"It says it's sick of doing things like inventories and payrolls, and it wants to make some break-throughs in astrophysics."

Experiments in Mathematics



Jonathan M. Borwein
David H. Bailey
Roland Girgensohn
Produced with the assistance of Mason Ma

The reader who wants to get an introduction to this excitin approach to doing mathematics can do no better than the
—Notices of t

I do not think that I have had the good fortune to read tw entertaining and informative mathematics texts.
—Australian Mathematical Society

This *Experiments in Mathematics* CD contains the full text of b *matics by Experiment: Plausible Reasoning in the 21st Century* *mentation in Mathematics: Computational Paths to Discovery* i searchable form. The CD includes several "smart" enhancemen

- Hyperlinks for all cross references
- Hyperlinks for all Internet URLs
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- Enhanced search function, which assists one with a search particular mathematical formula or expression.

These enhancements significantly improve the usability of these reader's experience with the material.

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Experiments in Mathematics

Jonathan M. Borwein
David H. Bailey
Roland Girgensohn
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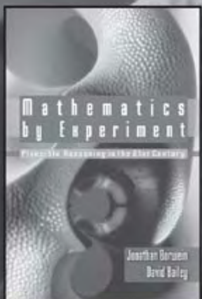
TICS

to read! . . .
how to become
atician."
Scientist Online

Jonathan M. Borwein, David H. Bailey, Roland Girgensohn
Produced with the assistance of Mason Macklem

"I do not think that I have had the good fortune to read two more entertaining and informative mathematics texts."

—*Gazette* of the Australian Mathematical Society



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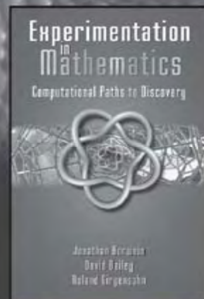

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xperiments in Mathematics

han M. Borwein, David H. Bailey, Roland Girger

short time since the first editions of *Mathematics by Experiment: Plausible Reaso* *1st Century* and *Experimentation in Mathematics: Computational Paths to Discovery* een a noticeable upsurge in interest in using computers to do real mathematics. The updated and enhanced the book files and have now made them available in PDF form OM. The CD includes several "smart" enhancements, including:

- Hyperlinks for all cross references (including theorems, figures, equations, etc.)
- Hyperlinks for all Internet URLs
- Hyperlinks for bibliographic references

gmented search facility assists one with a search for particular mathematical form ssions. These enhancements will significantly improve the usability of these files and t tself will enhance the reader's experience.

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Coming Soon!

Experimental Mathematics in Action

David H. Bailey, Jonathan M. Borwein, Neil Calkin, Roland Girgensohn, Russell Luke, Victor Moll

The emerging field of experimental mathematics has expanded to encompass a wide range of studies, all unified by the aggressive utilization of modern computer technology in mathematical research. This volume presents a number of case studies of experimental mathematics in action, together with some high level perspectives.

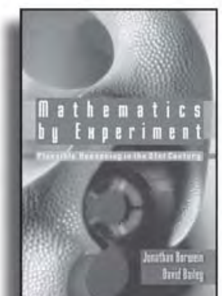
Specific case studies include:
-- analytic evaluation of integrals by means of symbolic and numeric computing techniques
-- evaluation of Apery-like summations
-- finding dependencies among high-dimension vectors (with applications to factoring large integers)
-- inverse scattering (reconstruction of physical objects based on electromagnetic or acoustic scattering)
-- investigation of continuous but nowhere differentiable functions.

In addition to these case studies, the book includes some background on the computational techniques used in these analyses.

September 2006; ISBN 1-56881-271-X; Hardcover; Approx. 200 pp.; $39.00

Mathematics by Experiment: Plausible Reasoning in the 21st Century

Jonathan Borwein, David Bailey



". . . experimental mathematics is here to stay. The reader who wants to get an introduction to this exciting approach to doing mathematics can do no better than [this book]."

— *Notices* of the AMS

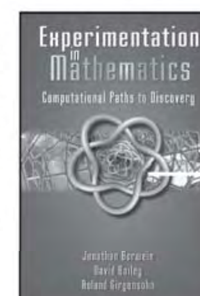
ISBN 1-56881-211-6; Hardcover; 298 pp.; $45.00

Experimentation in Mathematics: Computational Paths to Discovery

Jonathan Borwein, David Bailey, Roland Girgensohn

"These are such fun books to read! Actually, calling them books does not do them justice. They have the liveliness and feel of great Web sites, with their bite-size fascinating factoids and their many human- and math-interest stories and other gems. But do not be fooled by the lighthearted, immensely entertaining style. You are going to learn more math (experimental or otherwise) than you ever did from any two single volumes. Not only that, you will learn by osmosis how to become an experimental mathematician."

— *American Scientist*



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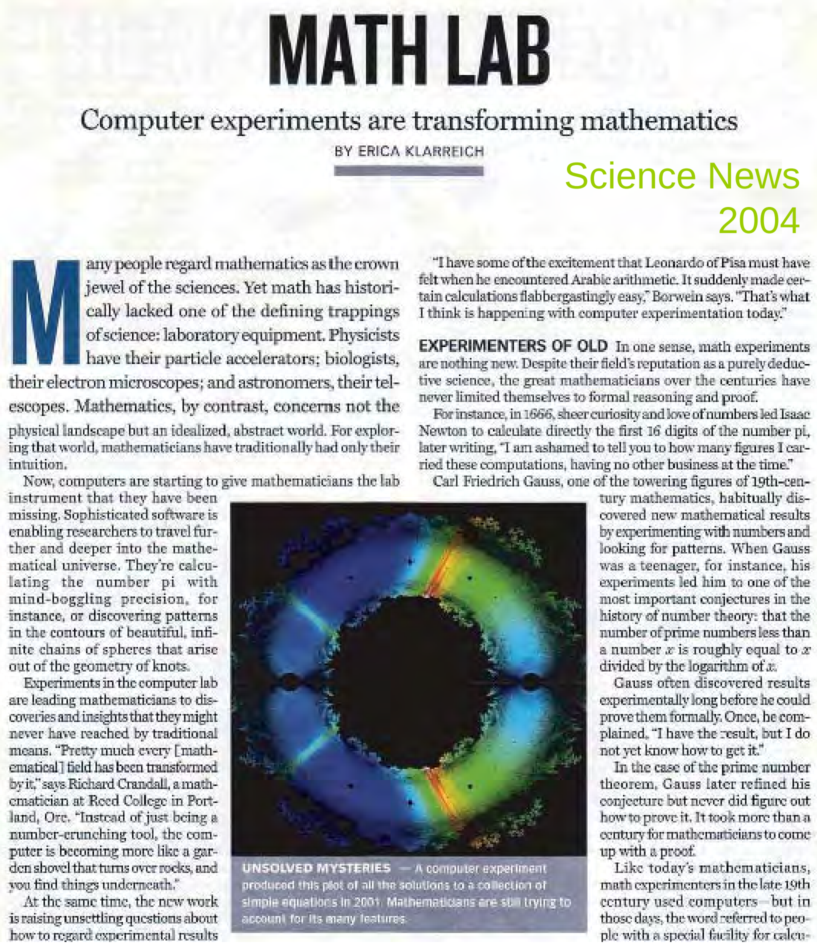
Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results



MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results

UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

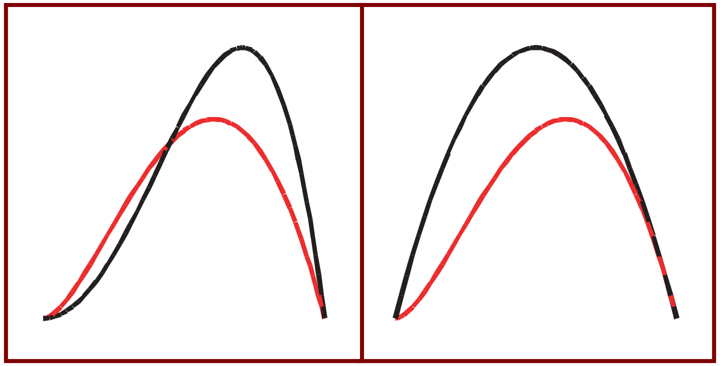
Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-

Science News 2004



Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

1 PARTIAL FRACTIONS and CONVEXITY

In a coupon collection thesis at SFU

- We consider a network *objective function* p_n given by

$$p_n(\vec{q}) = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} \frac{q_{\sigma(i)}}{\sum_{j=i}^{n} q_{\sigma(j)}}\right)\left(\sum_{i=1}^{n} \frac{1}{\sum_{j=i}^{n} q_{\sigma(j)}}\right)$$

summed over *all* $n!$ permutations; so a typical term is

$$\left(\prod_{i=1}^{n} \frac{q_i}{\sum_{j=i}^{n} q_j}\right)\left(\sum_{i=1}^{n} \frac{1}{\sum_{j=i}^{n} q_j}\right).$$

This looked pretty ugly

but Ian Affleck hoped p_n was convex !

◇ For $n = 3$ this is

6 TERMS LIKE

$$q_1 q_2 q_3 \left(\frac{1}{q_1 + q_2 + q_3}\right)\left(\frac{1}{q_2 + q_3}\right)\left(\frac{1}{q_3}\right)$$

$$\times\left(\frac{1}{q_1 + q_2 + q_3} + \frac{1}{q_2 + q_3} + \frac{1}{q_3}\right) .$$

• We wish to show p_n is *convex* on the positive orthant. First we try to simplify the expression for p_n.

COMPUTERS DO SOME THINGS BETTER THAN US

- The *partial fraction decomposition* gives:

$$
\begin{aligned}
p_1(x) &= \frac{1}{x}, \\
p_2(x_1, x_2) &= \frac{1}{x1} + \frac{1}{x2} - \frac{1}{x_1 + x_2}, \\
p_3(x_1, x_2, x_3) &= \frac{1}{x1} + \frac{1}{x2} + \frac{1}{x3} \\
&- \frac{1}{x_1 + x_2} - \frac{1}{x_2 + x_3} - \frac{1}{x_1 + x_3} \\
&+ \frac{1}{x_1 + x_2 + x_3}.
\end{aligned}
$$

So we predict the 'same' for $N = 4$ and

CHECK SYMBOLICALLY

CONJECTURE. For each $N \in \mathbb{N}$

$$p_N(x_1, \ldots, x_N) := \int_0^1 \left(1 - \prod_{i=1}^{N} (1 - t^{x_i})\right) \frac{dt}{t}$$

is convex, indeed 1/concave.

Non-convex integrand

- Check $N < 5$ via large symbolic Hessian

PROOF. A year later, *joint expectations* gave:

$$p_N(x) = \int_{\mathbb{R}^n_+} e^{-(y_1+\cdots+y_n)} \max\left(\frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n}\right) dy$$

[See *SIAM Electronic Problems and Solutions*.]

Convex integrand

Also in ToVA -- find a direct proof?

True, but why ?

The first series below was proven by Ramanujan. The next two were found & proven by Computer (Wilf-Zeilberger).

The candidates:

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} r_3(n)\,(42n+5)\left(\frac{1}{4^3}\right)^n$$

$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} r_5(n)\,(20n^2+8n+1)\left(\frac{-1}{4}\right)^n$$

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} r_5(n)\,(820n^2+180n+13)\left(\frac{-1}{4^5}\right)^n$$

$$\frac{32}{\pi^3} = \sum_{n=0}^{\infty} r_7(n)\,(168n^3+76n^2+14n+1)\left(\frac{1}{4^3}\right)^n$$

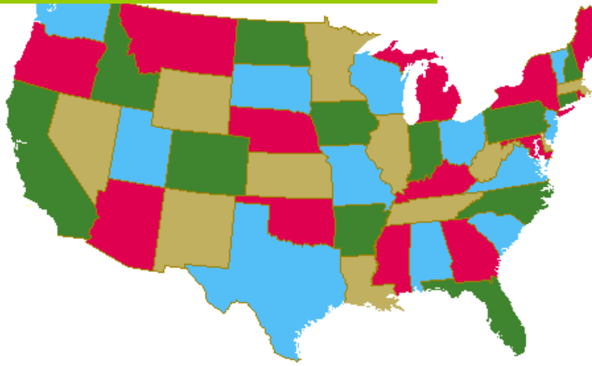
Here, in terms of factorials and rising factorials:

$$r_N(n) := \frac{\binom{2n}{n}^N}{4^{nN}} = \left(\frac{(1/2)_n}{n!}\right)^N.$$

The 4^{th} is only true

$$r_N(n) \sim_n \frac{1}{n^{N/2}}$$

Grand Challenges in Mathematics (CISE 2000)

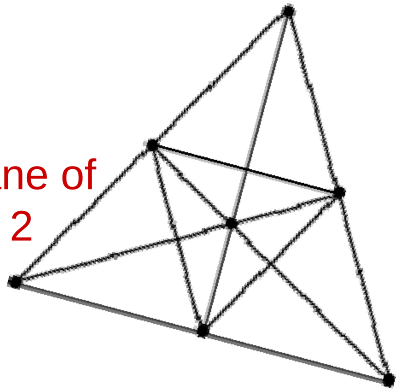


are few and far between

- **Four Colour Theorem** (1976,1997)
- **Kepler's problem** (Hales, 2004-12)

On an upcoming slide

- Nonexistence of **Projective Plane of Order 10**
  - **10^2+10+1** lines and points on each other (n+1 fold)
    - 2000 Cray hrs in 1990
    - next similar case:**18** needs10^{12} hours?
    - or a Quantum Computer



Fano plane of order 2

Fermat's Last Theorem (Wiles 1993, 1994)

- By contrast, any counterexample was too big to find (1985)

$$x^N + y^N = z^N, N > 2$$

has only trivial integer solutions

Cultural Maps in Mathematics

“Mathematicians are a kind of Frenchmen:

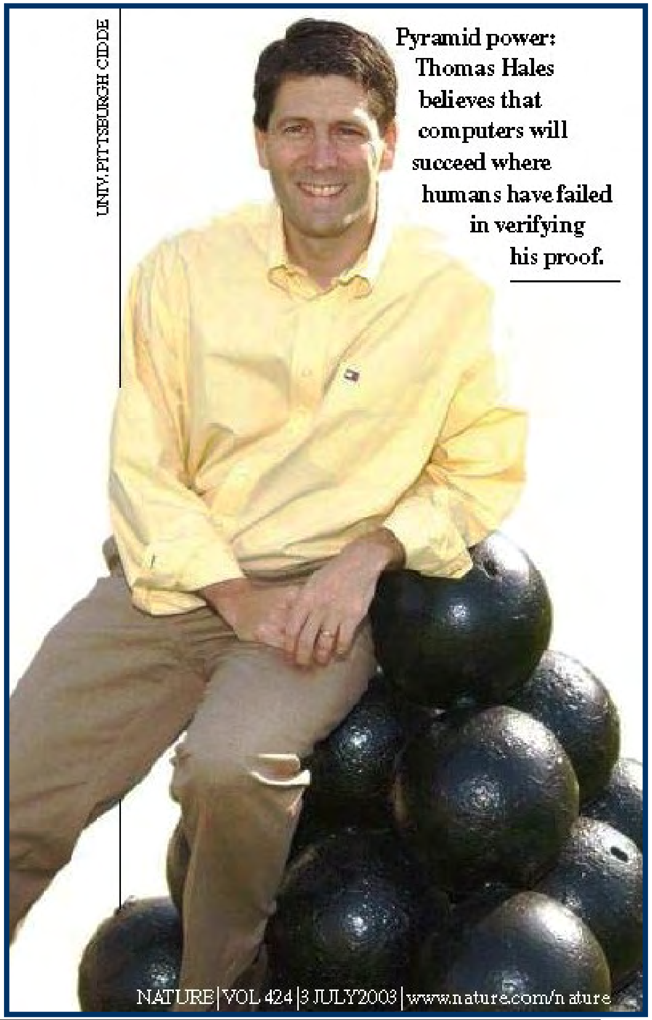
whatever you say to them they translate into their own language, and right away it is something entirely different.”

(Johann Wolfgang von Goethe)

Maximen und Reflexionen, no. 1279

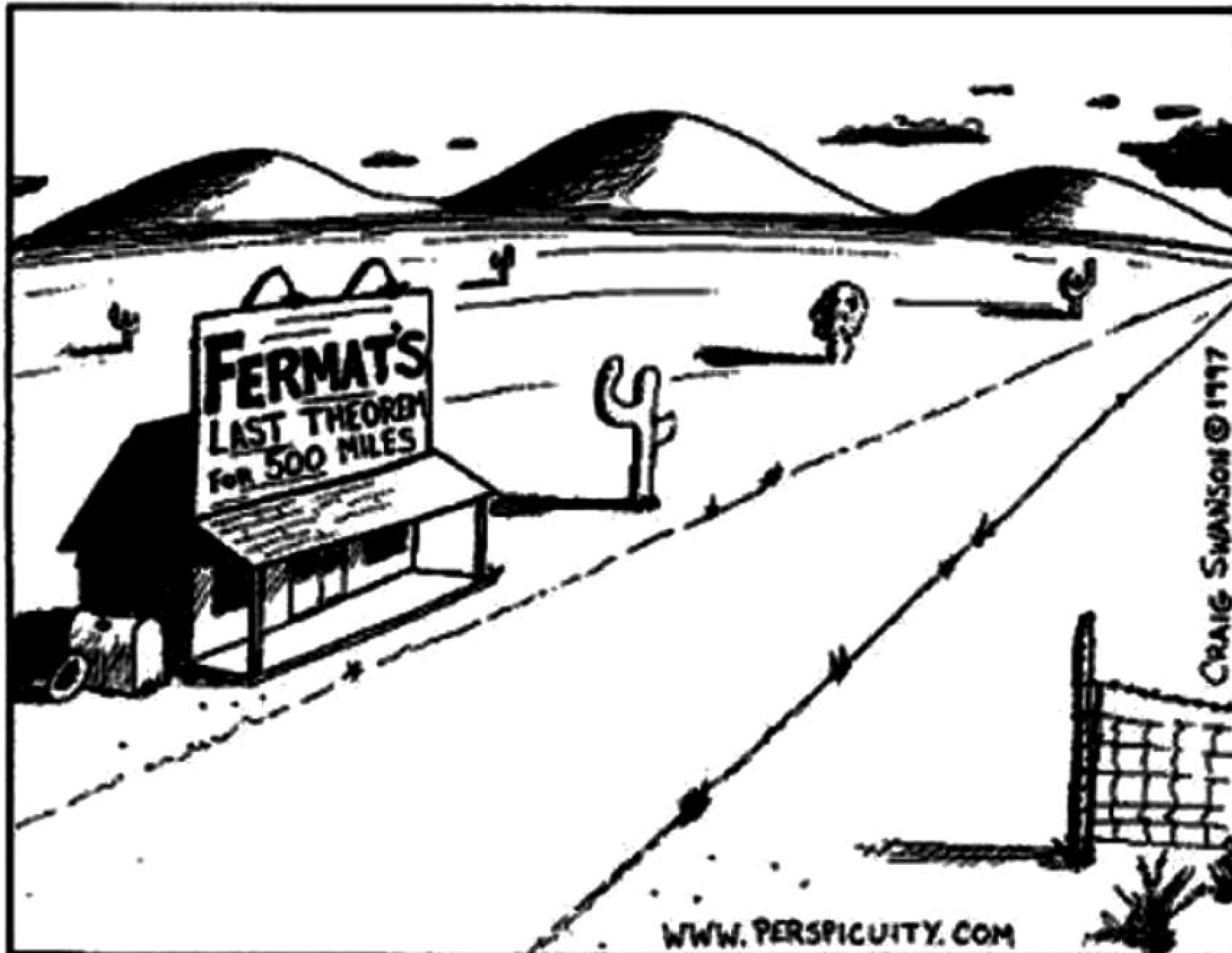
- **Kepler's** conjecture **the densest way to stack spheres is in a pyramid**
  - oldest problem in discrete geometry?
  - most interesting recent example of computer assisted proof
  - published in *Annals of Mathematics* with an "only 99% checked" disclaimer
  - Many varied reactions. *In Math, Computers Don't Lie. Or Do They?* (NYT, 6/4/04)

- **Famous earlier examples**: Four Color Theorem and Non-existence of a Projective Plane of Order 10.
  - the three raise quite distinct questions - both real and specious
  - as does status of classification of Finite Simple Groups



Formal Proof theory (code validation) has received an unexpected boost: automated proofs *may* now exist of the Four Color Theorem and Prime Number Theorem

• COQ: *When is a proof a proof* ? Economist, April 2005


FERMAT'S
LAST THEOREM
FOR 500 MILES
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WWW.PERSPICUITY.COM

Dalhousie Distributed Research Institute and Virtual Environment

East meets West: Collaboration goes National

Welcome to D-DRIVE whose mandate is to study and develop resources specific to distributed research in the sciences with first client groups being the following communities

- High Performance Computing
- Mathematical and Computational Science Research
- Science Outreach
  - ► Research
  - ► Education/TV

Atlantic Computational Excellence Network

ACEnet

AARMS

Dalhousie Distributed Research Institute and Virtual Environment

Coast to Coast Seminar Series

Tuesdays 3:30 – 4:30 pm Atlantic Time

http://projects.cs.dal.ca/ddrive/

Lead partners:

Dalhousie D-Drive – Halifax Nova Scotia

IRMACS – Burnaby, British Columbia

Other Participants so far:

University of British Columbia, University of Alberta, University of Alberta University of Saskatchewan, Lethbridge University, Acadia University, St Francis Xavier University, University of Western Michigan, MathResources Inc, University of North Carolina

The Experience

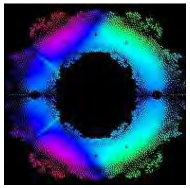
Fully Interactive multi-way audio and visual

Given good bandwidth audio is much harder

The closest thing to being in the same room

Shared Desktop for viewing presentations or sharing software

Dalhousie Distributed Research Institute and Virtual Environment



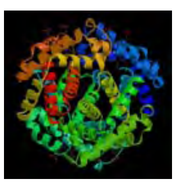
Jonathan Borwein, Dalhousie University
Mathematical Visualization

High Quality Presentations

Uwe Glaesser, Simon Fraser University
Semantic Blueprints of Discrete Dynamic Systems

Peter Borwein, IRMACS
The Riemann Hypothesis

Jonathan Schaeffer, University of Edmonton
Solving Checkers



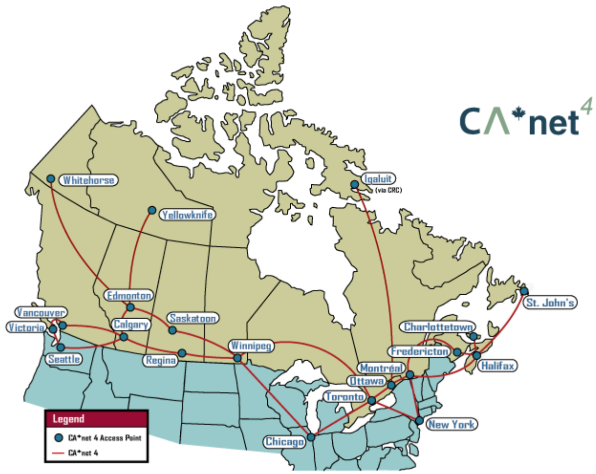
Arvind Gupta, MITACS
The Protein Folding Problem

Przemyslaw Prusinkiewicz, University of Calgary
Computational Biology of Plants

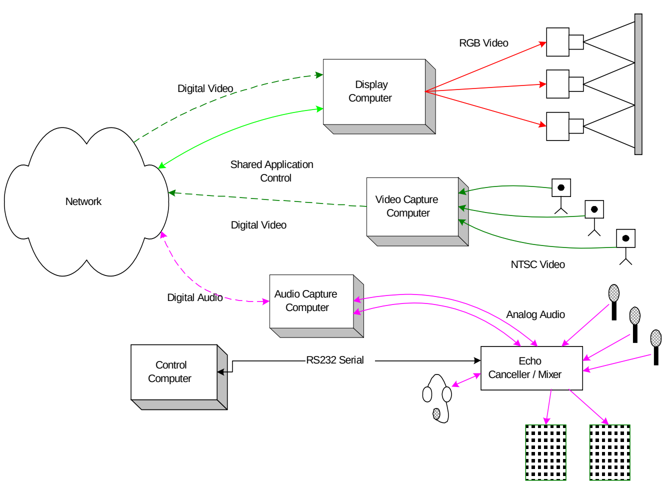
Karl Dilcher, Dalhousie University
Fermat Numbers, Wieferich and Wilson Primes

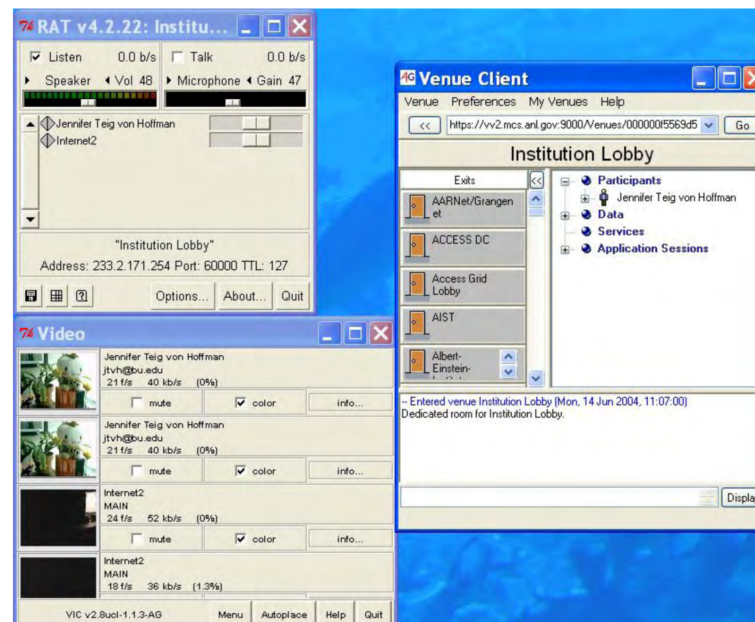
Dalhousie Distributed Research Institute and Virtual Environment

The Technology



High Bandwidth Connections (CA*net)
+
PC Workstations
+
Audio/Video Equipment
+
Open Source Software

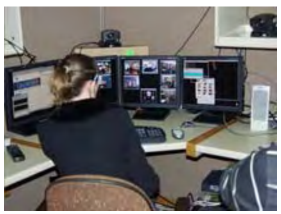


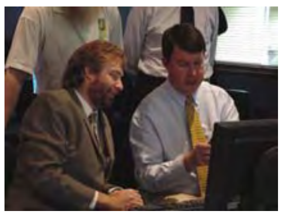


Dalhousie Distributed Research Institute and Virtual Environment

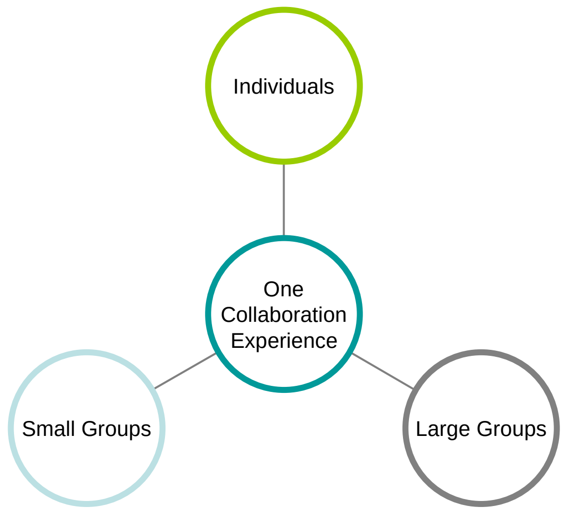
Institutional Requirements (Scalable Investment)

Personal Nodes (1-4 people)





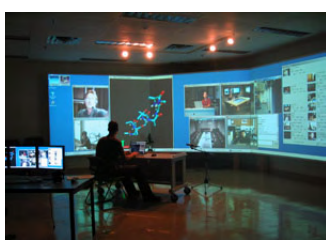
Cost: Less than $10,000 (CA)



Meeting Room Interactive Environment (2-20 people)

Cost: $150,000 (CA)

Small Group Projected Environment (2-10 people)



Cost: $25,000 - $100,000 (CA)

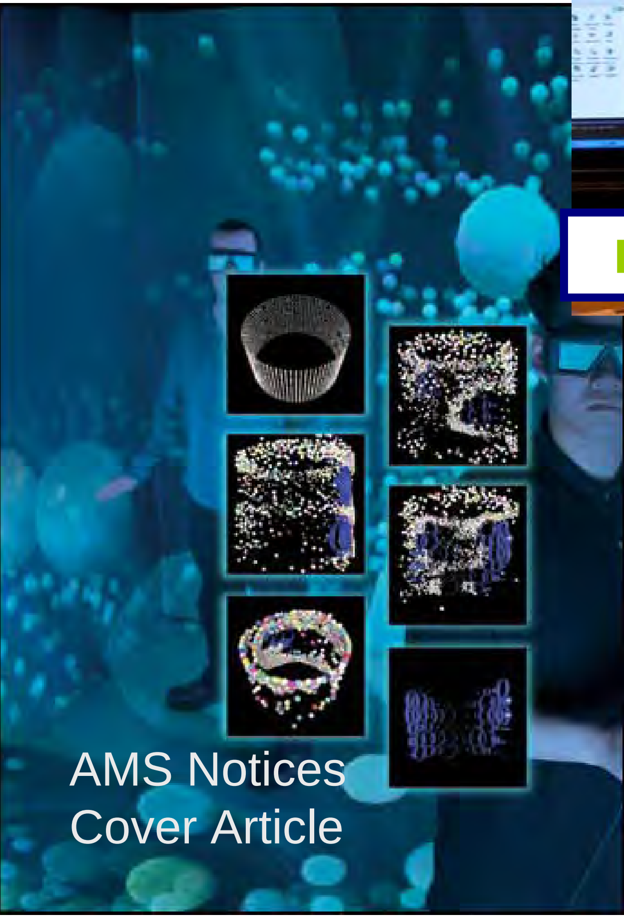
Visualization Auditorium

Cost: $500,000+ (CA)

Six degrees of net separation …

5 Smart Shared-Screens

Being emulated by the Canadian Kandahar mission



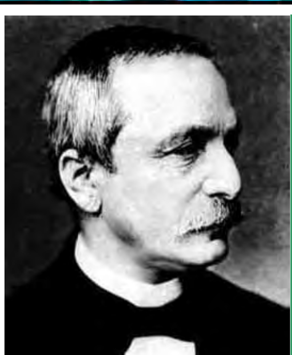
I shall now show a variety of uses of high performance computing and communicating as part of

Experimental Inductive Mathematics

Our web site:

www.experimentalmath.info

contains all links and references



“Elsewhere Kronecker said ``In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas." ... I would rather say ``computations" than ``formulas", but my view is essentially the same.”

Harold Edwards, *Essays in Constructive Mathematics*, 2004


Caveman Geometry
(2001)
Very cool for the one person with control

Trans-Canada 'C2C' Seminar
Tuesdays PST 11.30 MST 12.30 AST 3.30 and even 7.30 GMT
[Sept 28 - PBB on RH]

The 2,500 sq-metre IRMACS research centre

SFU building is a also a 190cpu G5 Grid

At the official April opening, I gave one of the four presentations from D-DRIVE

"What I appreciate even more than its remarkable speed and accuracy are the words of understanding and compassion I get from it."

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1b. Visual Data Mining in Mathematics.

- ✓ Fractals, Polynomials, Continued Fractions*
- ✓ Pseudospectra and Code Optimization

2. High Precision Mathematics.

3. Integer Relation Methods.

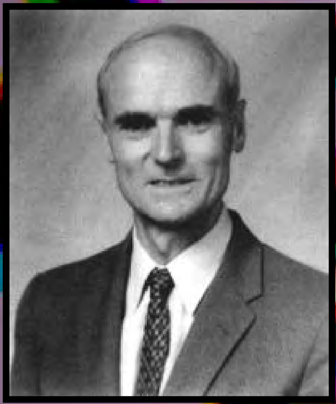
- ✓ Chaos, Zeta* and the Riemann Hypothesis
- ✓ Hex-Pi and Normality

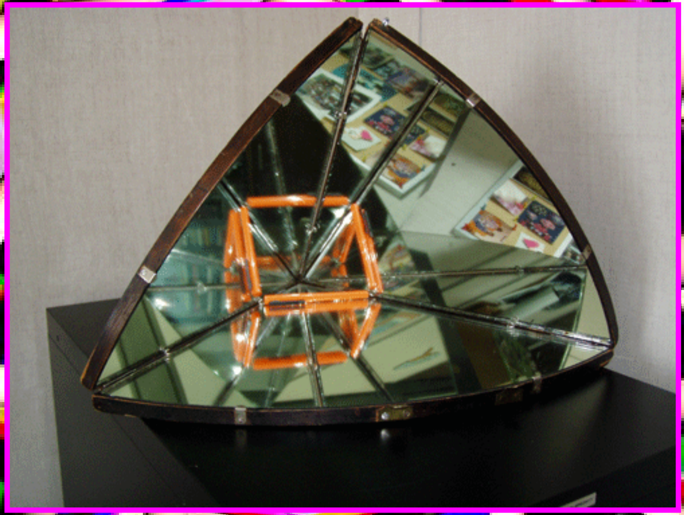
4. Inverse Symbolic Computation.

- ✓ A problem of Knuth*, $\pi/8$, Extreme Quadrature

5. Demos and Conclusion.

COXETER'S (1927) Kaleidescope

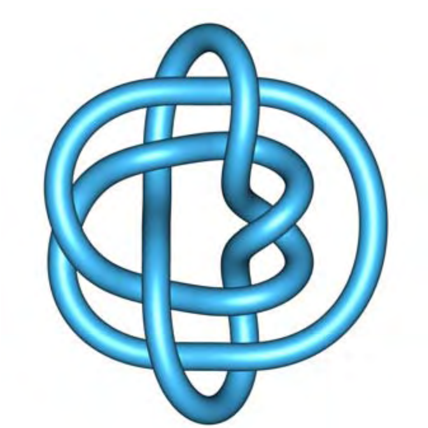


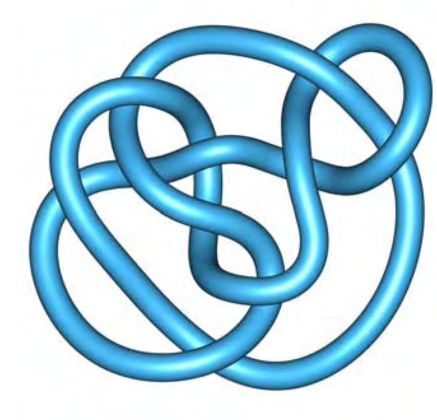


Interactive Proofs

The Perko Pair 10_{161} and 10_{162}

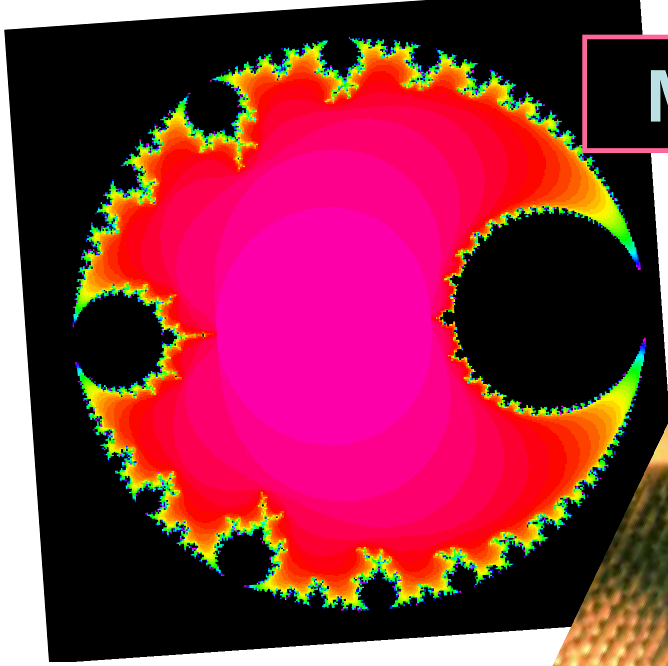
are two adjacent 10-crossing knots (1900)



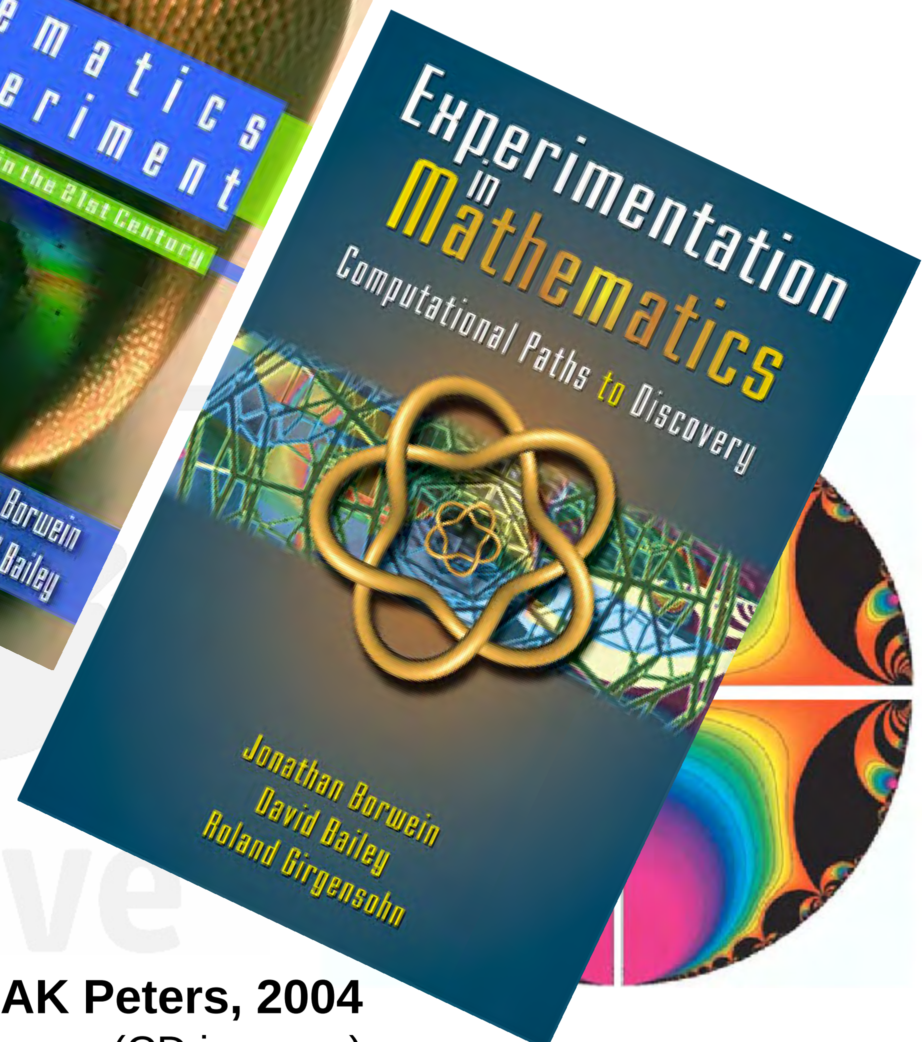


- **first shown to be the same by Ken Perko in 1974**
- **and beautifully made dynamic in [KnotPlot] (open source)**

More Mathematical Data Mining



An unusual Mandelbrot parameterization



Various visual examples follow

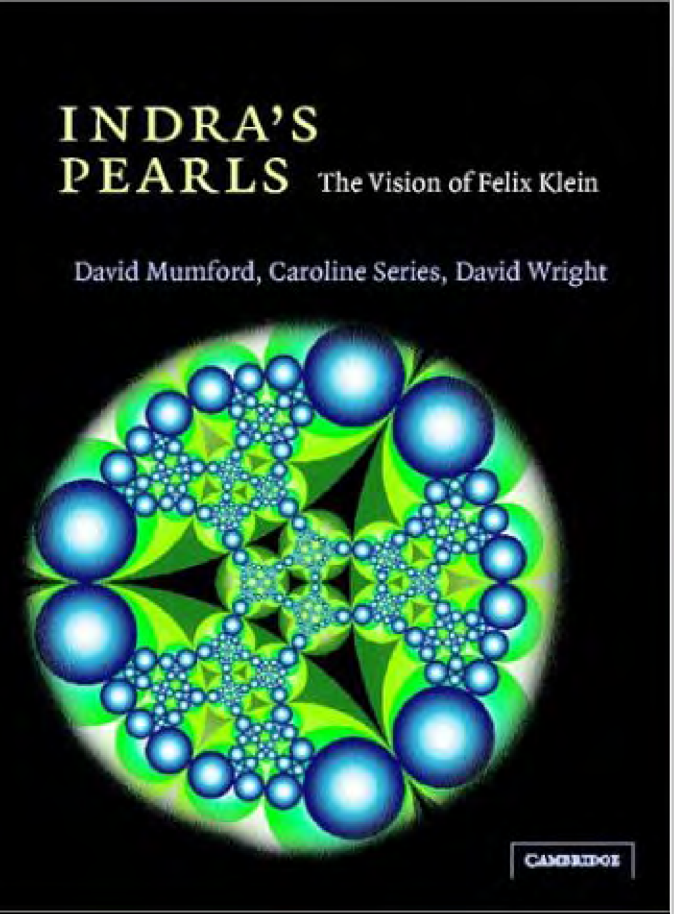
- Indra’s pearls
- Roots of `1/-1’ polynomials
- Ramanujan’s fraction
- Sparsity and Pseudospectra

AK Peters, 2004

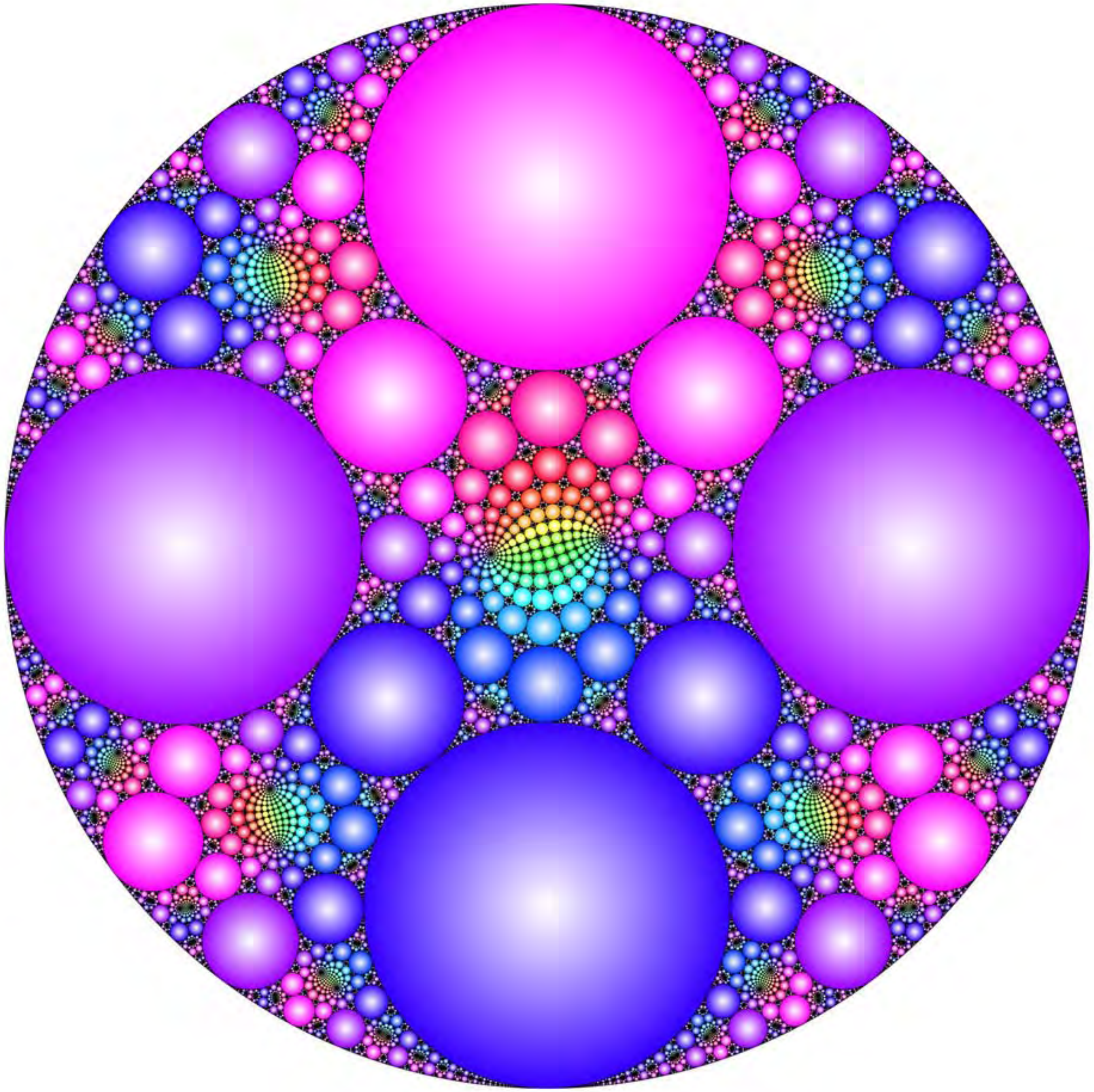
(CD in press)

Indra's Pearls

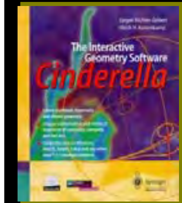
A merging of 19th and 21st Centuries



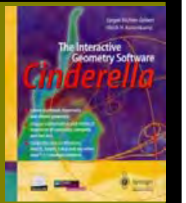
Double cusp group

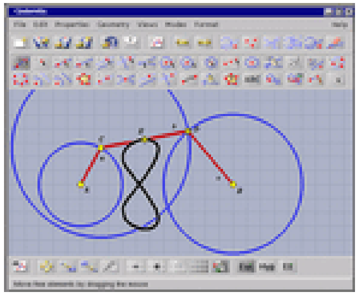


2002: http://klein.math.okstate.edu/IndrasPearls/

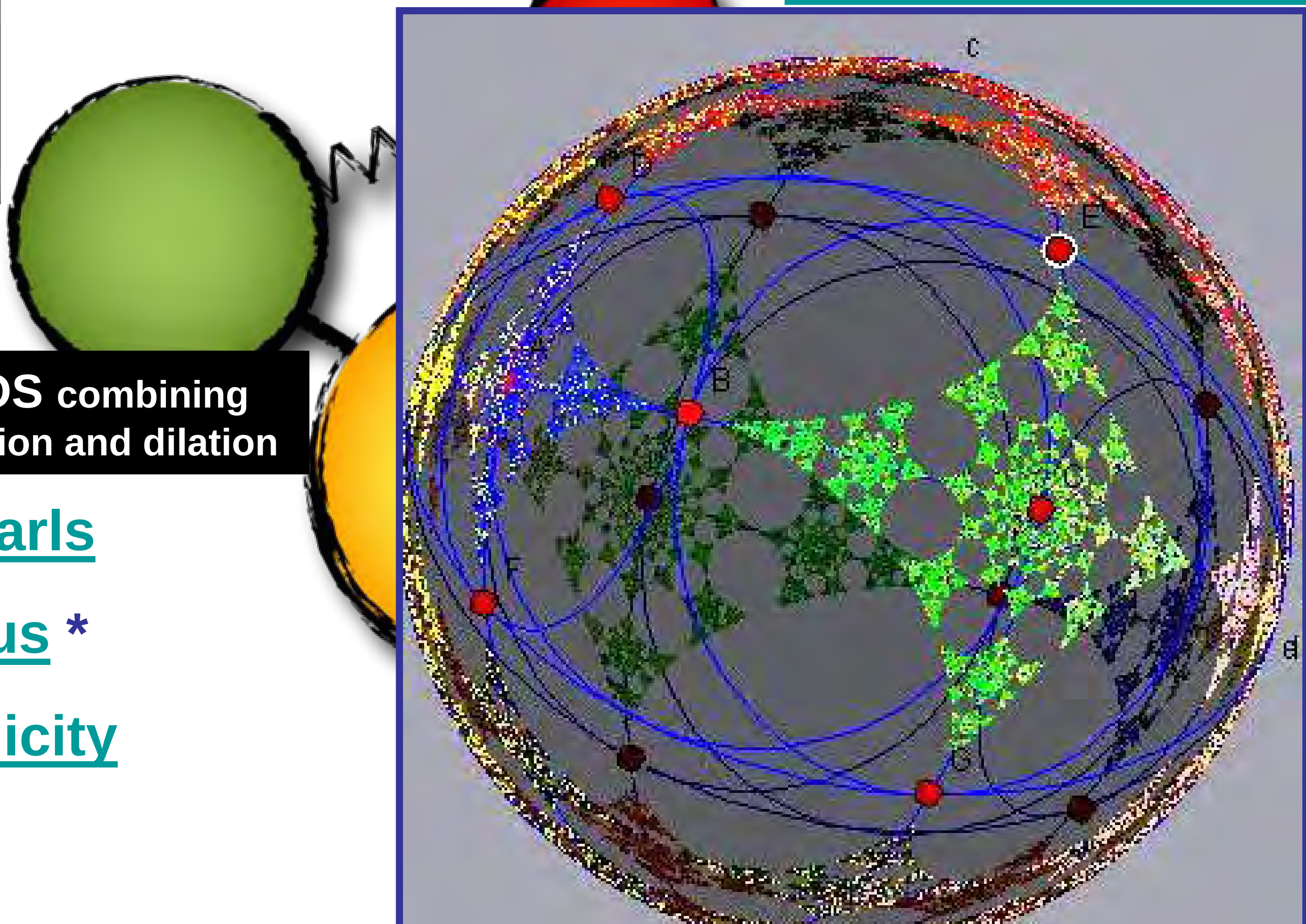


CINDERELLA's dynamic geometry



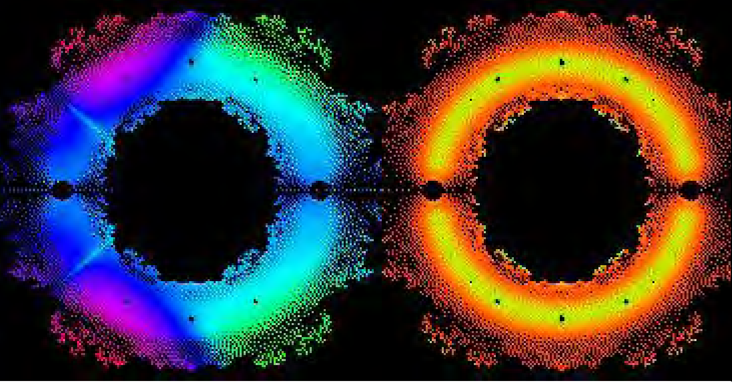


www.cinderella.de



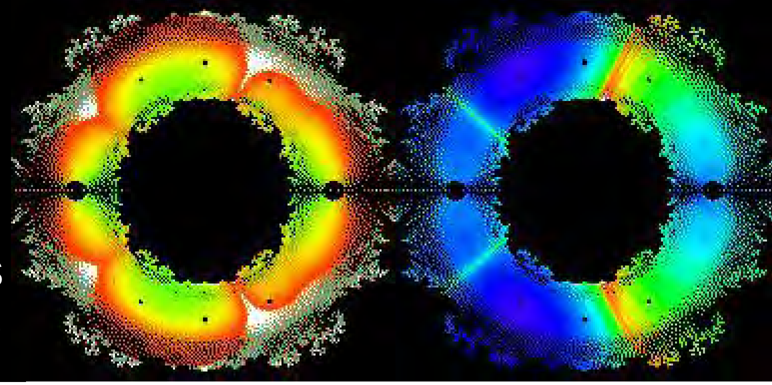
FOUR DEMOS combining inversion, reflection and dilation

1. Indraspearls
2. Apollonius *
3. Hyperbolicity
4. Gasket



Roots of Zeros

What you draw is
what you see
("visible structures
in number theory")



Striking fractal patterns formed by plotting complex zeros for all polynomials in powers of x with coefficients 1 and -1 to degree 18

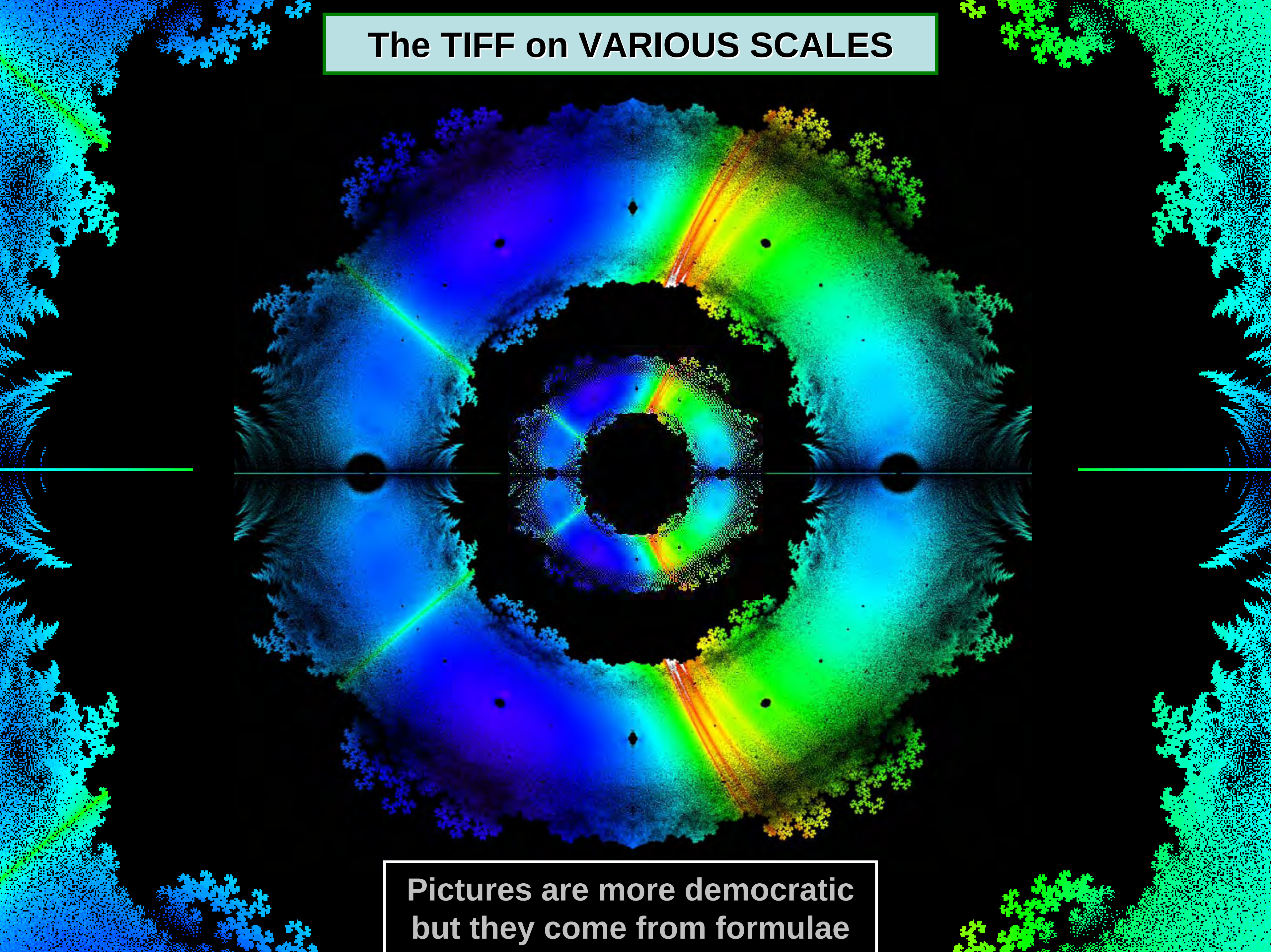
Coloration is by sensitivity of polynomials to slight variation around the values of the zeros. **The color scale represents a normalized sensitivity** to the range of values; red is insensitive to violet which is strongly sensitive.

- All zeros are pictured (at **3600 dpi**)
- Figure 1b is colored by their local density
- Figure 1d shows sensitivity relative to the $\mathbf{x^9}$ term
- **The white and orange striations are not understood**

A wide variety of patterns and features become visible, leading researchers to totally unexpected mathematical results

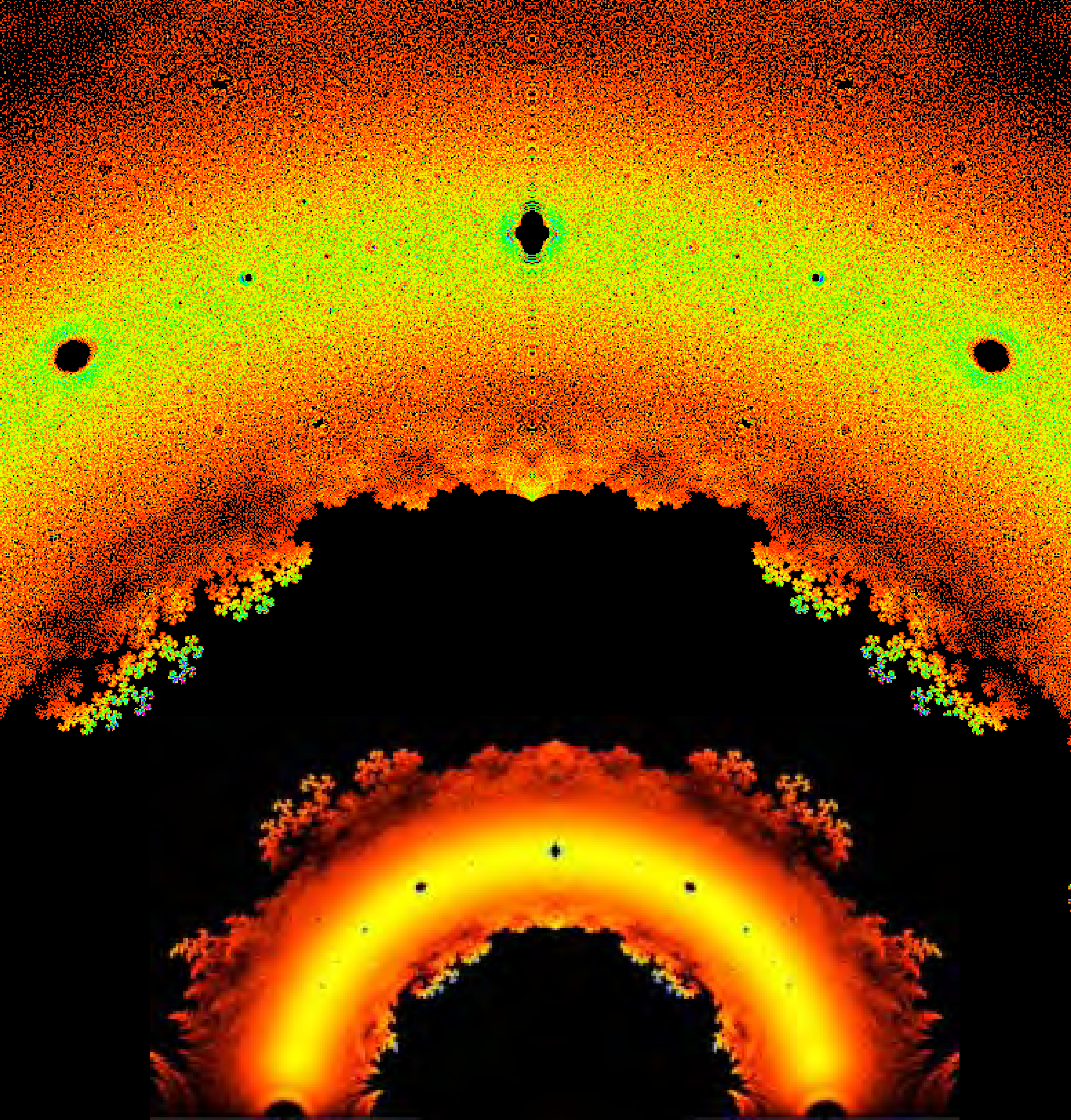
"The idea that we could make biology mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe mathematics will become biological!" Greg Chaitin, Interview, 2000.

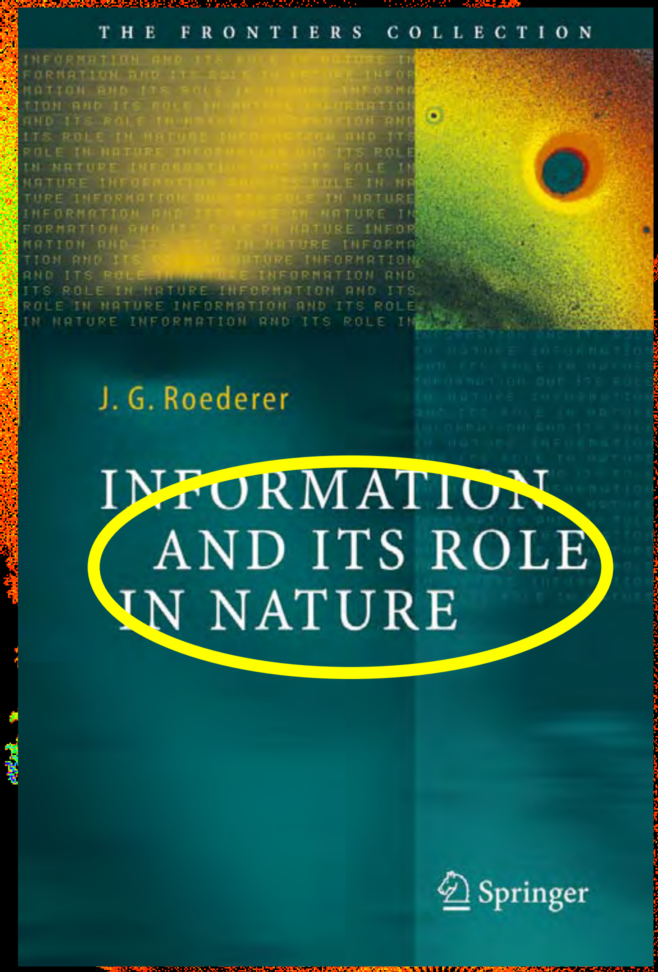
The TIFF on VARIOUS SCALES

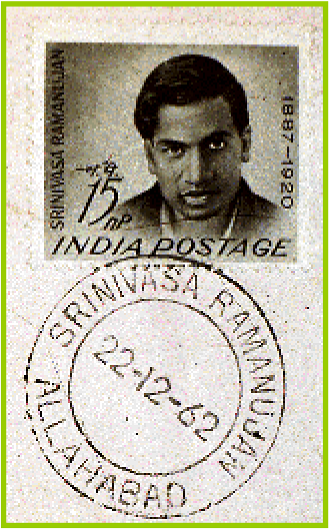


Pictures are more democratic
but they come from formulae

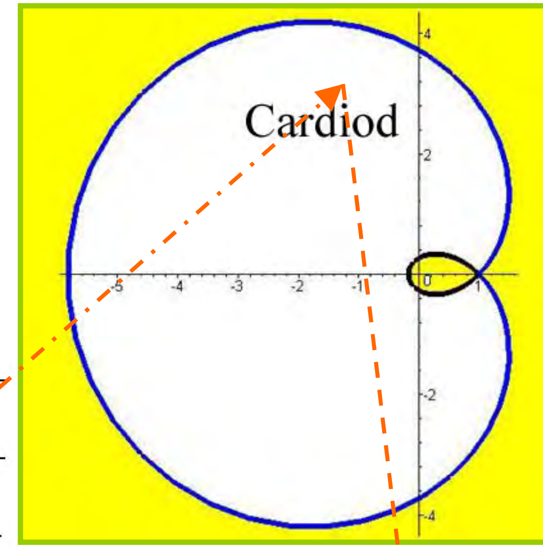
Roots in the most stable colouring







Ramanujan's Arithmetic-Geometric Continued fraction (CF)



$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ldots}}}}$$

For a,b>0 the CF satisfies a lovely symmetrization

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2}$$

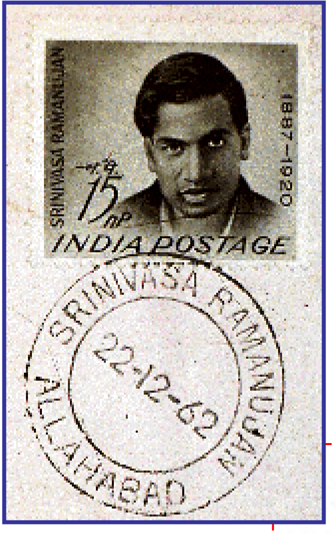
Computing directly was too hard; even 4 places of $\mathcal{R}_1(1,1) = \log 2$ **?**

We wished to know for which a/b in C this all held

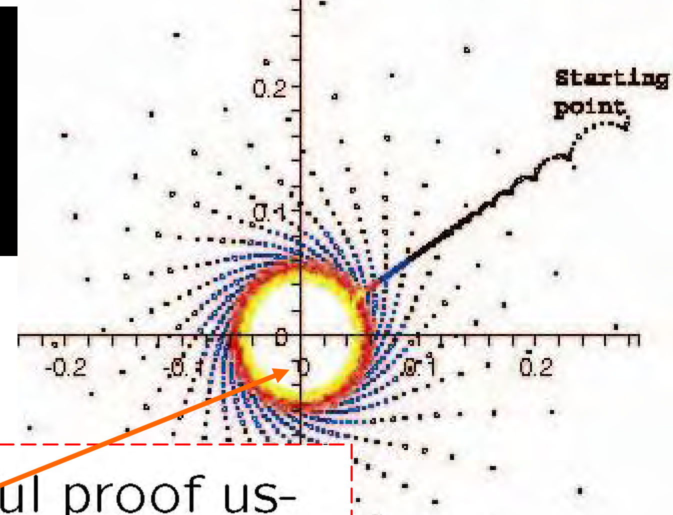
A scatterplot revealed a precise cardioid where r=a/b.

Which discovery it remained to prove?

$$\left|\frac{a+b}{2}\right| \geq \sqrt{|ab|}$$



Ramanujan's Arithmetic-Geometric Continued fraction



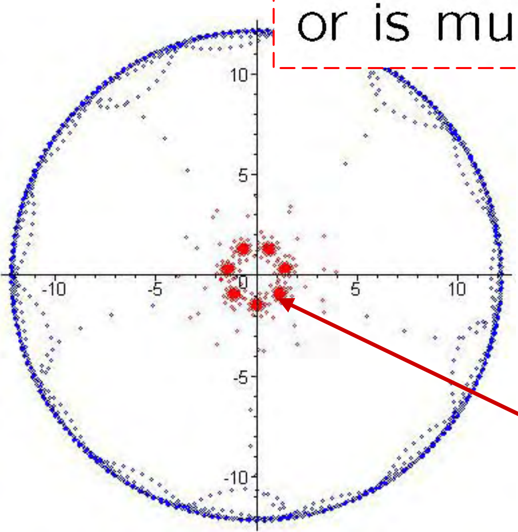
1. The Blackbox

2. Seeing convergence

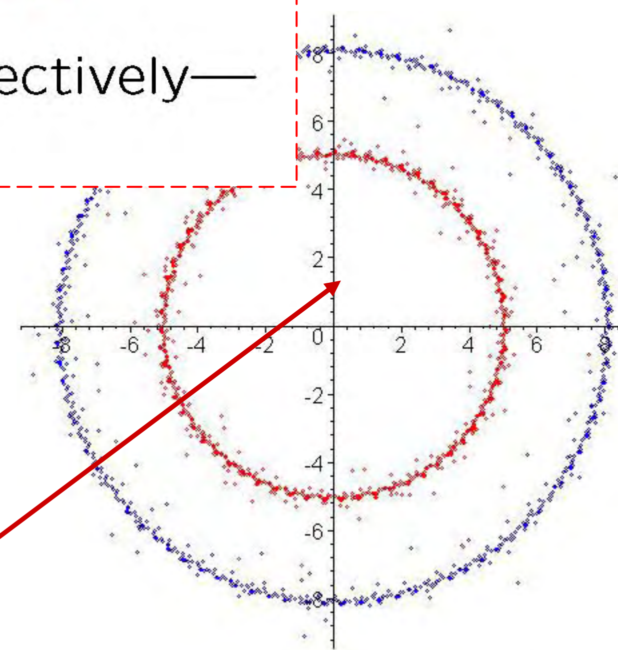
Six months later we had a beautiful proof using genuinely new dynamical results. Starting from the *dynamical system* $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1}\left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd respectively— or is much more general.*



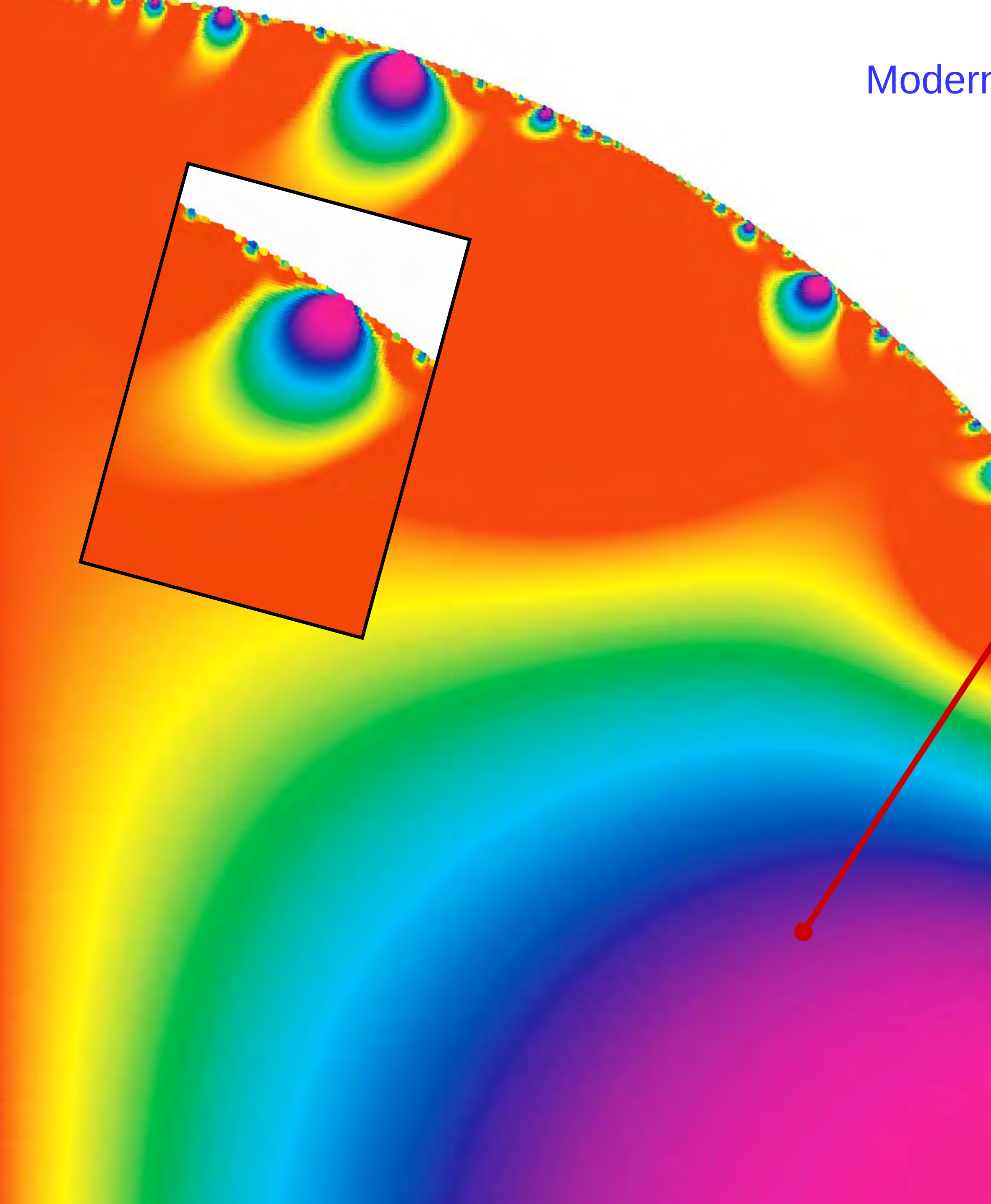
3. Attractors. *Normalizing* by $n^{1/2}$ three cases appear

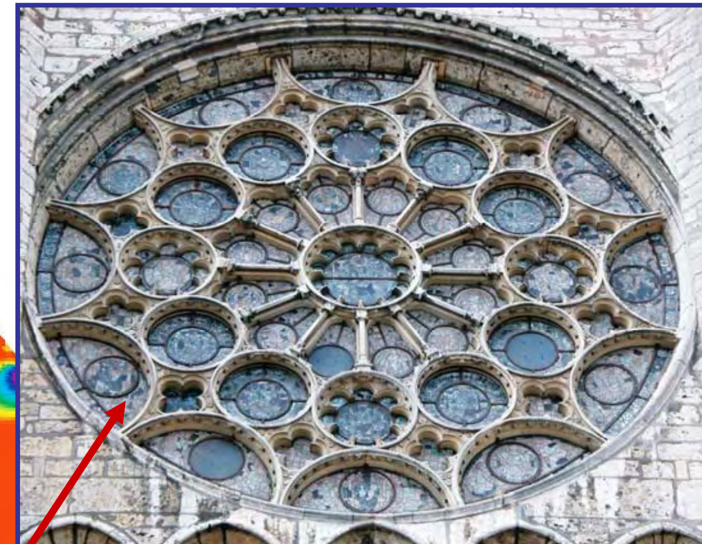


Mathematics and the aesthetic

Modern approaches to an ancient affinity

(CMS-Springer, 2005)



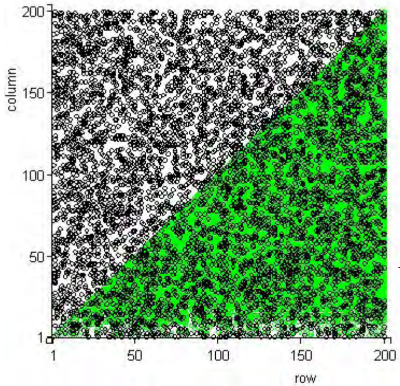


Why should I refuse a good dinner simply because I don't understand the digestive processes involved?

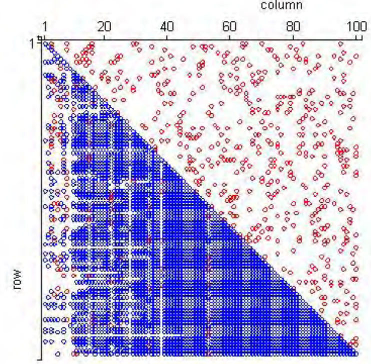
Oliver Heaviside (1850 - 1925)

✓ **when criticized for his daring use of operators before they could be justified formally**

Pseudospectra or Stabilizing Eigenvalues



Gaussian elimination of random sparse (10%-15%) matrices

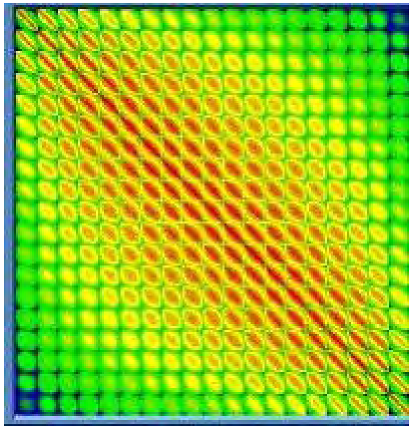


‘Large’ (10^5 to 10^8) **Matrices must be seen**

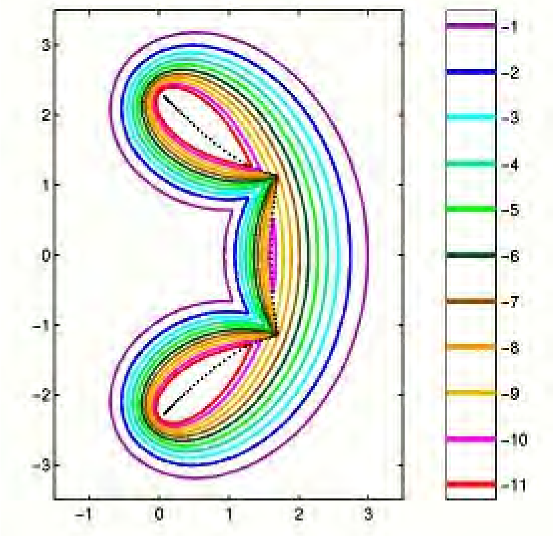
- ✓ conditioning and ill-conditioning
- ✓ sparsity and its preservation

- ✓ eigenvalues
- ✓ singular values (helping Google work)

A dense inverse



Pseudospectrum of a banded matrix

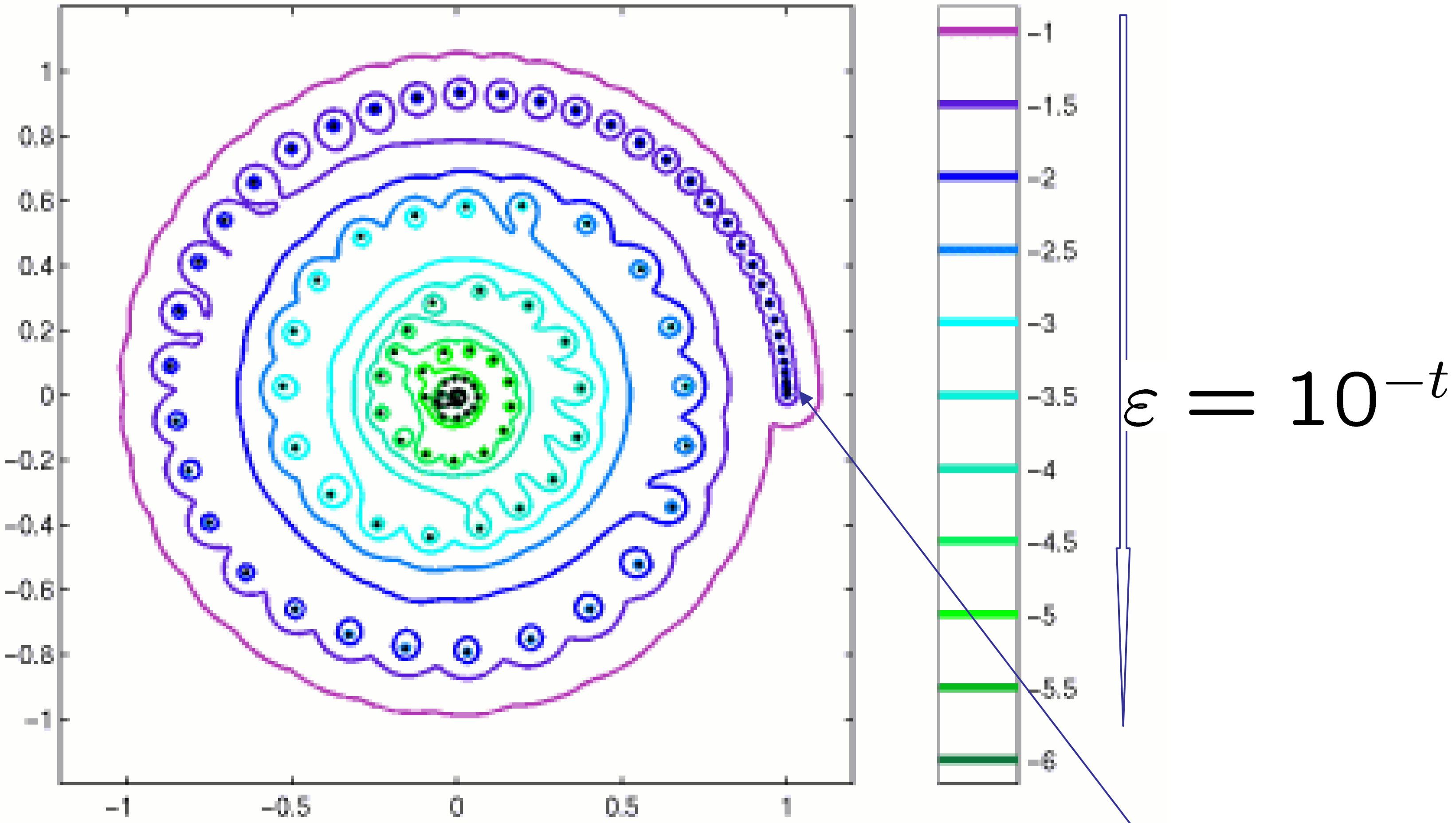


The pseudo spectrum of A: for $\varepsilon > 0$

$$\sigma_\varepsilon(A) = \{\lambda : \inf \|Ax - \lambda x\| \le \varepsilon\}$$

http://web.comlab.ox.ac.uk/projects/pseudospectra

An Early Use of Pseudospectra (Landau, 1977)



An infinite dimensional integral equation in laser theory

✓discretized to a matrix of dimension **600**

✓projected onto a well chosen invariant subspace of dimension **109**

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1. Visual Data Mining in Mathematics.
   - ✓ Fractals, Polynomials, Continued Fractions*
   - ✓ Pseudospectra and Code Optimization
2. High Precision Mathematics.
3. Integer Relation Methods.
   - ✓ Chaos, Zeta* and the Riemann Hypothesis
   - ✓ Hex-Pi and Normality
4. Inverse Symbolic Computation.
   - ✓ A problem of Knuth*, $\pi/8$, Extreme Quadrature
5. Demos and Conclusion.

A WARMUP Computational Proof

Suppose we know that 1<N<10 and that N is an integer

- **computing N to 1 significant place with a certificate** will prove the value of N. *Euclid's method* is basic to such ideas.

Likewise, suppose we know α is *algebraic of degree d and length λ*

(coefficient sum in absolute value)

If P is polynomial of degree D & length L **EITHER** P(α) = 0 **OR**

$$|P(\alpha)| \geq \frac{1}{L^{d-1}\lambda^D}$$

Example (MAA, April 2005). Prove that

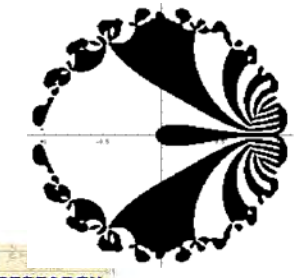
$$\int_{-\infty}^{\infty} \frac{y^2}{1+4\,y+y^6-2\,y^4-4\,y^3+2\,y^5+3\,y^2}\,dy = \pi$$

Proof. Purely **qualitative analysis** with partial fractions and arctans shows the integral is $\pi\ \beta$ where β is algebraic of degree *much* less than **100 (actually 6)**, length *much* less than **100,000,000**.With **P(x)=x-1** (D=1,L=2, d=6, λ=?), this means *checking* the identity to **100** places is plenty of **PROOF.**

A fully symbolic Maple proof followed. **QED**

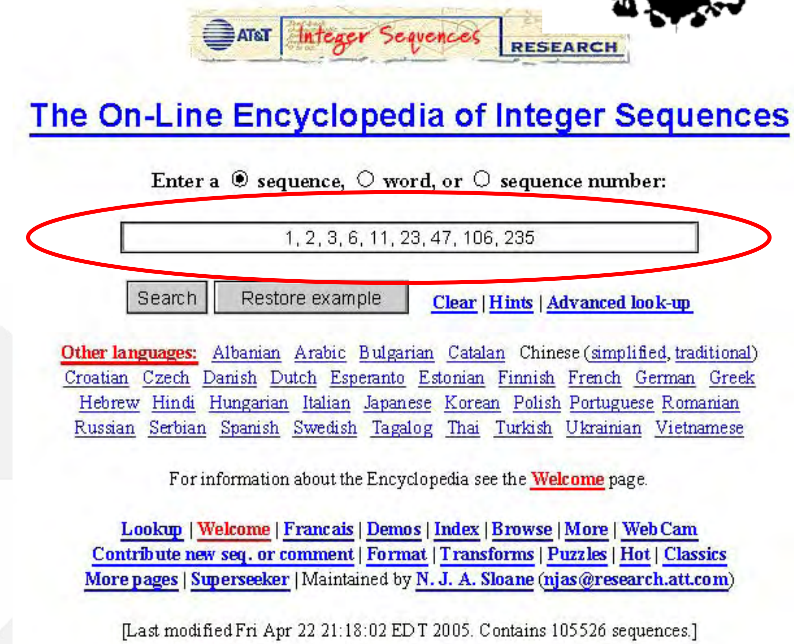
$$|\beta - 1| < 1/(32\lambda) \mapsto \beta = 1$$

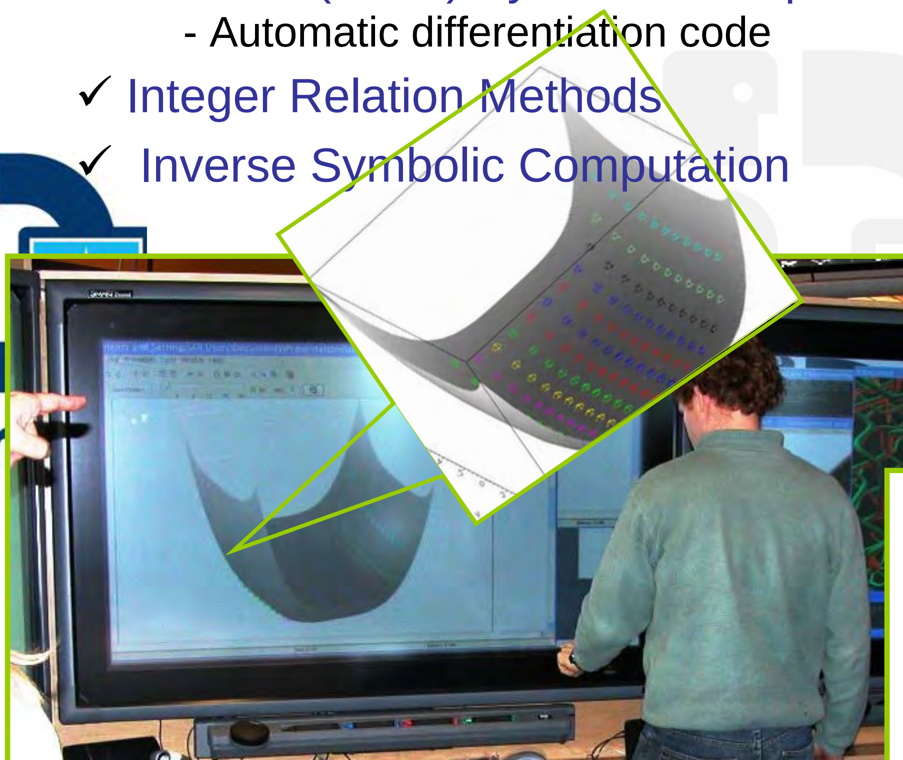
Numeric and Symbolic Computation



❑ Central to my work - with Dave Bailey - meshed with visualization, randomized checks, many web interfaces and

- ✓ Massive (serial) Symbolic Computation
  - Automatic differentiation code
- ✓ Integer Relation Methods
- ✓ Inverse Symbolic Computation





*Parallel derivative free optimization in **Maple***

Other useful tools : Parallel Maple

- Sloane's online sequence database
- Salvy and Zimmerman's generating function package '*gfun*'
- Automatic identity proving: Wilf-Zeilberger method for hypergeometric functions

Greetings from the On-Line Encyclopedia of Integer Sequences!

AT&T Integer Sequences RESEARCH

Matches (up to a limit of 30) found for 1 2 3 6 11 23 47 106 235 :

[It may take a few minutes to search the whole database, depending on how many matches are found (the second and later look are faster)]

An Exemplary Database

ID Number: A000055 (Formerly M0791 and N0299)

URL: http://www.research.att.com/projects/OEIS?Anum=A000055

Sequence: 1,1,1,1,2,3,6,11,23,47,106,235,551,1301,3159,7741,19320, 48629,123867,317955,823065,2144505,5623756,14828074, 39299897,104636890,279793450,751065460,2023443032, 5469566585,14830871802,40330829030,109972410221

Name: Number of trees with n unlabeled nodes.

Comments: Also, number of unlabeled 2-gonal 2-trees with n 2-gons.

References
F. Bergeron, G. Labelle and P. Leroux, Combinatorial Species and Tree-Like Structures, Camb. 1998, p. 279.
N. L. Biggs et al., Graph Theory 1736-1936, Oxford, 1976, p. 49.
S. R. Finch, Mathematical Constants, Cambridge, 2003, pp. 295-316.
D. D. Grant, The stability index of graphs, pp. 29-52 of Combinatorial Mathematics (Proceedings 2nd Australian Conf.), Lect. Notes Math. 403, 1974.
F. Harary, Graph Theory. Addison-Wesley, Reading, MA, 1969, p. 232.
F. Harary and E. M. Palmer, Graphical Enumeration, Academic Press, NY, 1973, p. 58 and 244.
D. E. Knuth, Fundamental Algorithms, 3d Ed. 1997, pp. 386-88.
R. C. Read and R. J. Wilson, An Atlas of Graphs, Oxford, 1998.
J. Riordan, An Introduction to Combinatorial Analysis, Wiley, 1958, p. 138.

Integrated real time use

- moderated
- 120,000 entries
- grows daily
- AP book had 5,000

Links:
P. J. Cameron, Sequences realized by oligomorphic permutation groups J. Integ. Seqs. Vo
Steven Finch, Otter's Tree Enumeration Constants
E. M. Rains and N. J. A. Sloane, On Cayley's Enumeration of Alkanes (or 4-Valent Trees),
N. J. A. Sloane, Illustration of initial terms
E. W. Weisstein, Link to a section of The World of Mathematics.
Index entries for sequences related to trees
Index entries for "core" sequences
G. Labelle, C. Lamathe and P. Leroux, Labeled and unlabeled enumeration of k-gonal 2-tr

Formula: G.f.: A(x) = 1 + T(x)-T^2(x)/2+T(x^2)/2, where T(x) = x + x^2 + 2*x^3 + ...

Fast Arithmetic
(Complexity Reduction in Action)

Multiplication

- Karatsuba multiplication (200 digits +) or Fast Fourier Transform (FFT)

… in ranges from 100 to 1,000,000,000,000 digits

- The other operations $\times, \div, \sqrt{\cdot}$
  via Newton's method
- Elementary and special functions
  via Elliptic integrals and Gauss AGM

$O\left(n^{\log_2(3)}\right)$

For example:

$$\left(a + c \cdot 10^N\right) \times \left(b + d \cdot 10^N\right)$$
$$= ab + (ad + bc) \cdot 10^N + cd \cdot 10^{2N}$$
$$= ab + \underbrace{\{(a+c)(b+d) - ab - cd\}}_{\text{three multiplications}} \cdot 10^N + cd \cdot 10^{2N}$$

Karatsuba replaces one 'times' by many 'plus'

FFT multiplication of multi-billion digit numbers reduces centuries to minutes. Trillions must be done with Karatsuba!

Ising Integrals (Jan 2006)

The following integrals arise in Ising theory of mathematical physics:

$$C_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

Richard Crandall showed that this can be transformed to a 1-D integral:

$$C_n = \frac{2^n}{n!}\int_0^\infty t K_0^n(t)\, dt$$

where K_0 is a modified Bessel function. We then computed 400-digit numerical values, from which these results were found (and proven):

$$C_3 = \mathrm{L}_{-3}(2) = \sum_{n \ge 0}\left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2}\right)$$

$$C_4 = 14\zeta(3)$$

$$\lim_{n\to\infty} C_n = 2e^{-2\gamma}$$

- via **PSLQ** and the **Inverse Calculator** to which we now turn

"What it comes down to is our software
is too hard and our hardware is too soft."

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1. Visual Data Mining in Mathematics.
   - ✓ Fractals, Polynomials, Continued Fractions*
   - ✓ Pseudospectra and Code Optimization
2. High Precision Mathematics.
3. Integer Relation Methods.
   - ✓ Chaos, Zeta and the Riemann Hypothesis
   - ✓ Hex-Pi and Normality
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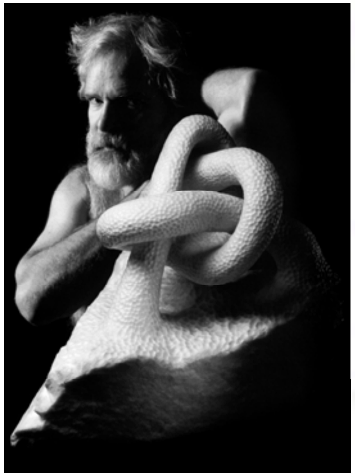
   - ✓ A problem of Knuth*, $\pi/8$, Extreme Quadrature
5. Demos and Conclusion.

Integer Relation Methods

The PSLQ Integer Relation Algorithm

Let (x_n) be a vector of real numbers. An integer relation algorithm finds integers (a_n) such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$



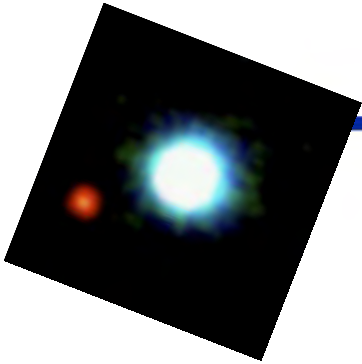
- At the present time, the PSLQ algorithm of mathematician-sculptor Helaman Ferguson is the best-known integer relation algorithm.
- PSLQ was named one of ten "algorithms of the century" by *Computing in Science and Engineering*.
- High precision arithmetic software is required: at least d x n digits, where d is the size (in digits) of the largest of the integers a_k.

An Immediate Use

To see if a is algebraic of degree N, consider $(1,a,a^2,\ldots,a^N)$

Combinatorial optimization for pure mathematics (also LLL)

Application of PSLQ: Bifurcation Points in Chaos Theory



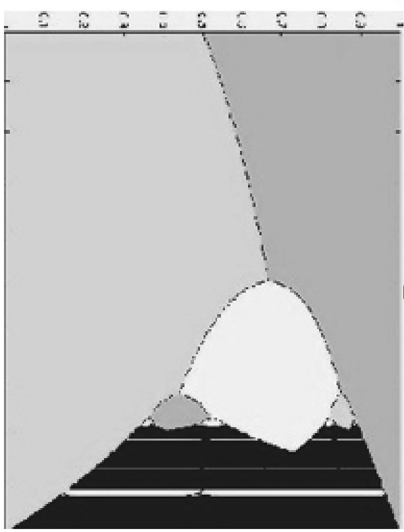
$B_3 = 3.54409035955\ldots$ is third bifurcation point of the logistic iteration of chaos theory:

$$x_{n+1} = r x_n (1 - x_n)$$

i.e., B_3 is the smallest r such that the iteration exhibits 8-way periodicity instead of 4-way periodicity.

In 1990, a predecessor to PSLQ found that B_3 is a root of the polynomial

$$\begin{aligned} 0 = {} & 4913 + 2108t^2 - 604t^3 - 977t^4 + 8t^5 + 44t^6 + 392t^7 \\ & - 193t^8 - 40t^9 + 48t^{10} - 12t^{11} + t^{12} \end{aligned}$$



Recently B_4 was identified as the root of a 256-degree polynomial by a much more challenging computation.

These results have subsequently been proven formally.

- The proofs use **Groebner basis** techniques
- Another useful part of the HPM toolkit

PSLQ and Zeta

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Euler
(1707-73)

1. via PSLQ to 50,000 digits
(250 terms)

$$= \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \ldots$$

2005 Bailey, Bradley & JMB *discovered and proved* - in Maple - three *equivalent* binomial identities

1

$$\mathcal{Z}(x) = 3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2}$$

$$= \sum_{k=0}^{\infty} \zeta(2k+2) x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}$$

$$= \frac{1 - \pi x \cot(\pi x)}{2x^2}$$

2. reduced as hoped

2

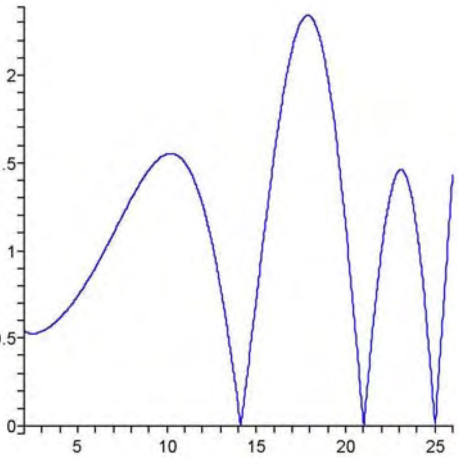
$$3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2 - m^2}{n^2 - m^2}}{\binom{2k}{k}(k^2 - n^2)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}$$

3

$${}_3F_2\left(\begin{matrix} 3n, n+1, -n \\ 2n+1, n+1/2 \end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}$$

3. was easily **computer proven**
(Wilf-Zeilberger)
MAA: human proof?

Visualizing the Riemann Hypothesis
(A Millennium Problem)



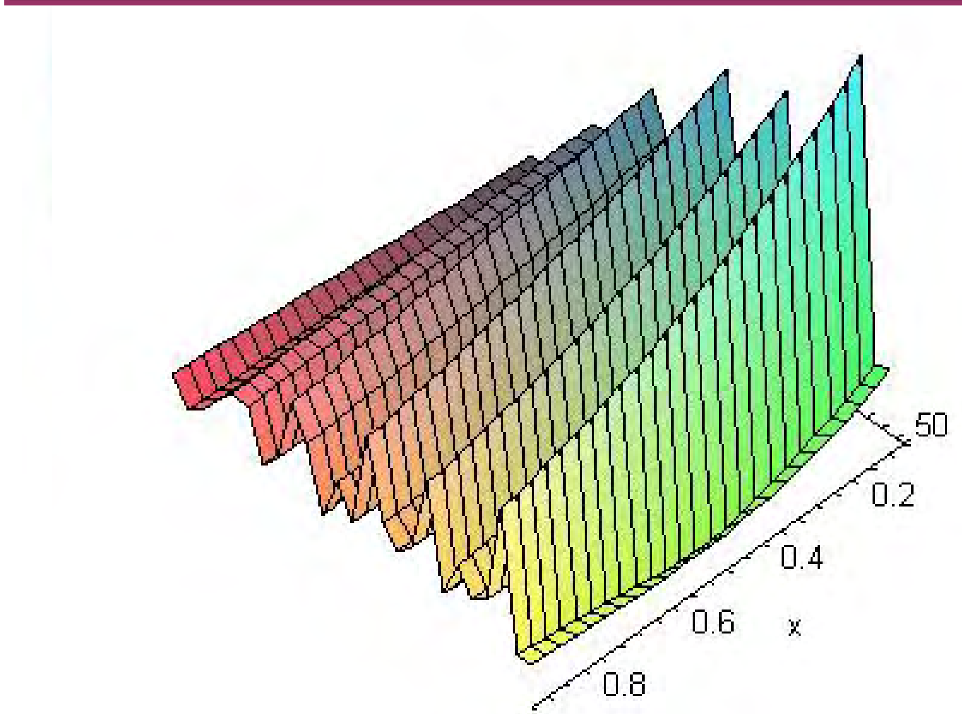
The imaginary parts of first 4 zeroes are:

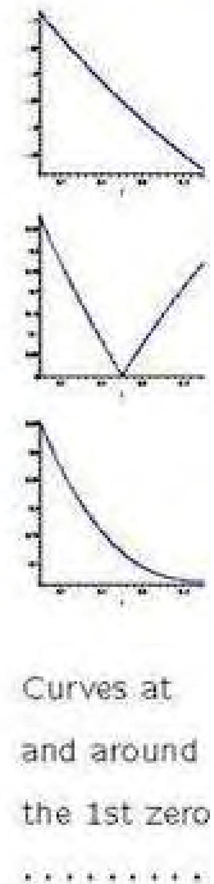
14.134725142
21.022039639

25.010857580
30.424876126

The first 1.5 billion are on the *critical line*

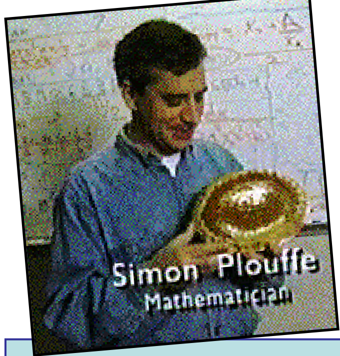
Yet at 10^{22} the ***"Law of small numbers"*** still rules (Odlyzko)





‘All non-real zeros have real part one-half’
(The Riemann Hypothesis)

Note the **monotonicity** of **$x \mapsto |\zeta(x+iy)|$** is **equivalent to RH** discovered in a Calgary class in 2002
by Zvengrowski and Saidak



PSLQ and Hex Digits of Pi

Finalist for the $100K Edge of Computation Prize won by David Deutsch

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{k\, 2^k}$$

My brother made the observation that this log formula allows one to compute binary digits of log 2 *without*

Edge The Third Culture

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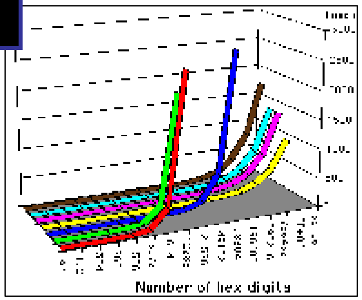
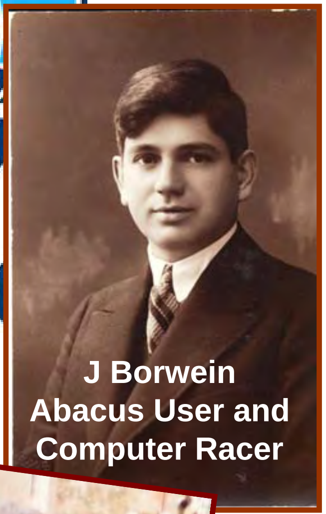
THE $100,000 EDGE OF COMPUTATION SCIENCE PRIZE

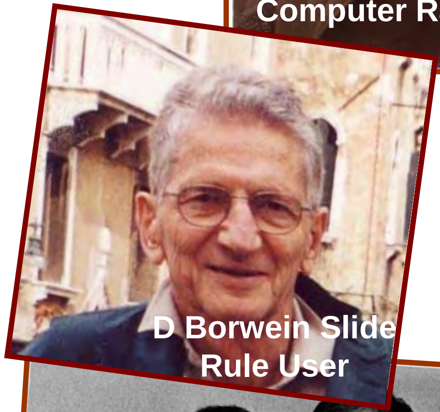
For individual scientific work, extending the computational idea, performed, published, or newly applied within the past ten years.

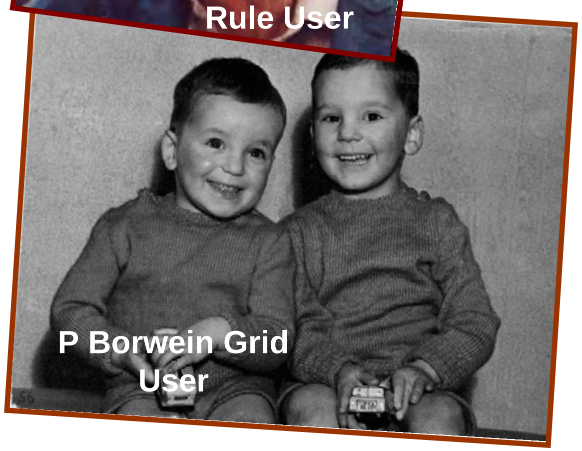
The Edge of Computation Science Prize, established by Edge Foundation, Inc., is a $100,000 prize initiated and funded by science philanthropist Jeffrey Epstein.

The pre-designed Algorithm ran the next day




J Borwein
Abacus User and
Computer Racer


D Borwein Slide
Rule User


P Borwein Grid
User

ALGORITHMIC PROPERTIES

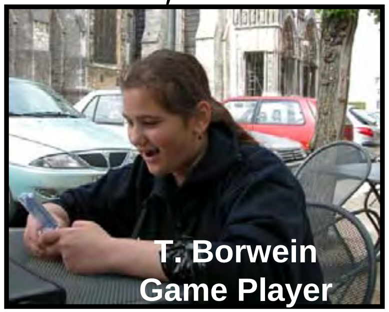
(1) produces a modest-length string hex or binary digits of π, beginning at an arbitrary position, using no prior bits;

Now built into some compilers!

(2) is implementable on any modern computer;

(3) requires no multiple precision software;

(4) requires very little memory; and


T. Borwein
Game Player

(5) has a computational cost growing only slightly faster than the digit position.

PiHex– A distributed effort to calculate Pi

http://www.cecm.sfu.ca/projects/pihex/

Getting Started Latest Headlines www.icbc.ca

- Join PiHex
  - Download
  - Source Code
- About
  - Credits
  - Status
  - Top Producers
- What's New?
- Other Projects
- Who am I?
- Email me!

174962 hits since the counter last reset.

PiHex

A distributed effort to calculate Pi.

Percival 2004

The Quadrillionth Bit of Pi is '0'!
The Forty Trillionth Bit of Pi is '0'!
The Five Trillionth Bit of Pi is '0'!

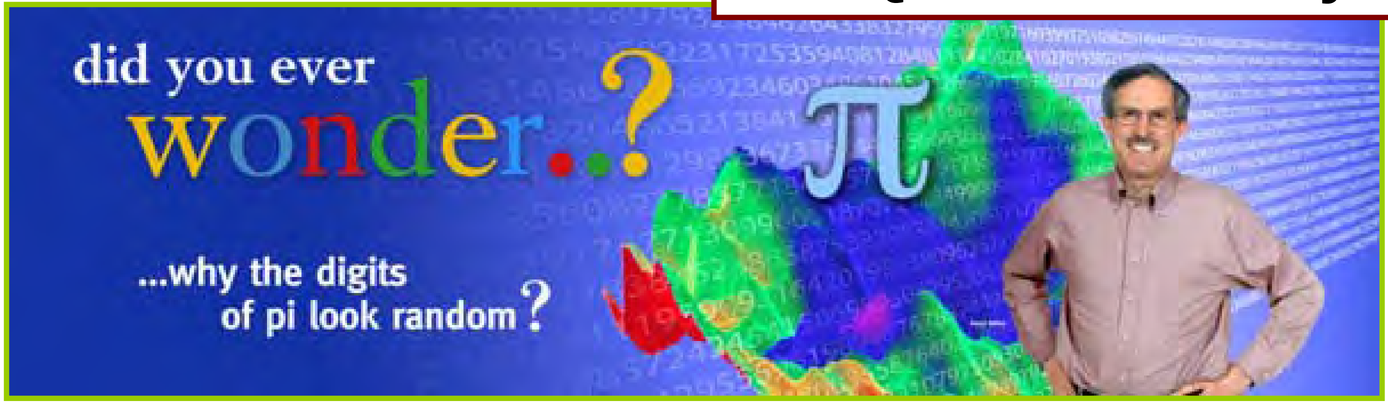
PiHex was a distributed computing project which used idle computing power to set three records for calculating specific bits of Pi. PiHex has now finished.

Position	Hex Digits Beginning At This Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B
2.5×10^{14}	E6216B069CB6C1

Undergraduate **Colin Percival's** grid computation **PiHex** rivaled **Finding Nemo**

1999 on 1736 PCS in 56 countries using 1.2 million Pentium2 cpu-hours

PSLQ and Normality of Digits



Bailey and Crandall observed that BBP numbers most probably are normal and make it precise with a hypothesis on the behaviour of a dynamical system.

- For example Pi is normal in Hexadecimal if the iteration below, starting at zero, is uniformly distributed in [0,1]

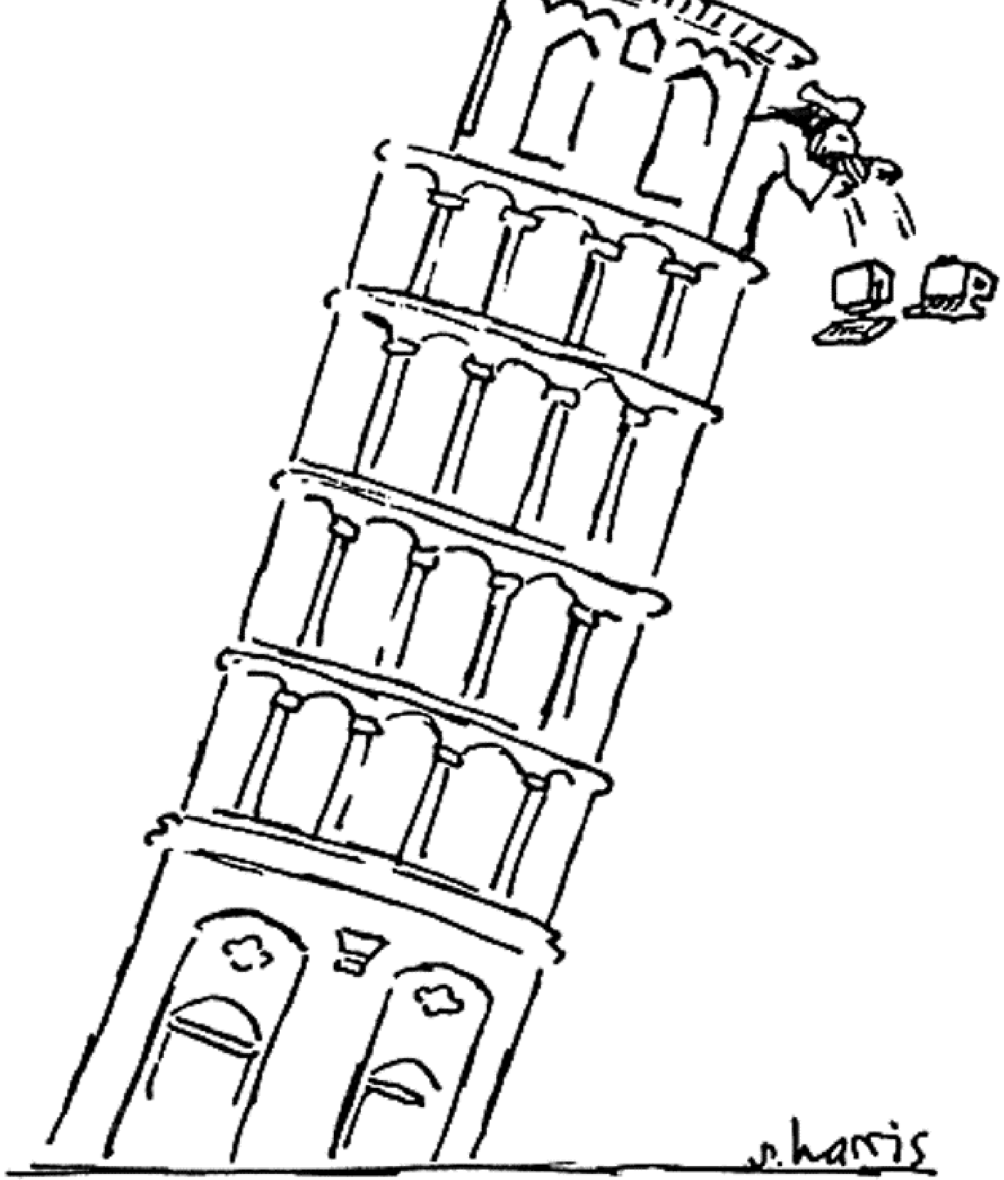
$$x_n = \left\{16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\}$$

Consider the hex digit stream:

$$d_n = \lfloor 16 x_n \rfloor$$

We have checked this gives first million hex-digits of Pi

Is this always the case? The weak Law of Large Numbers implies this is **very probably true**!



IF THERE WERE COMPUTERS IN GALILEO'S TIME

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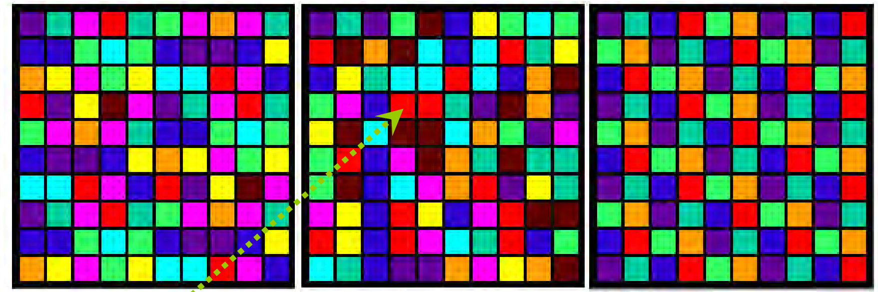
A Colour and an Inverse Calculator (1995)

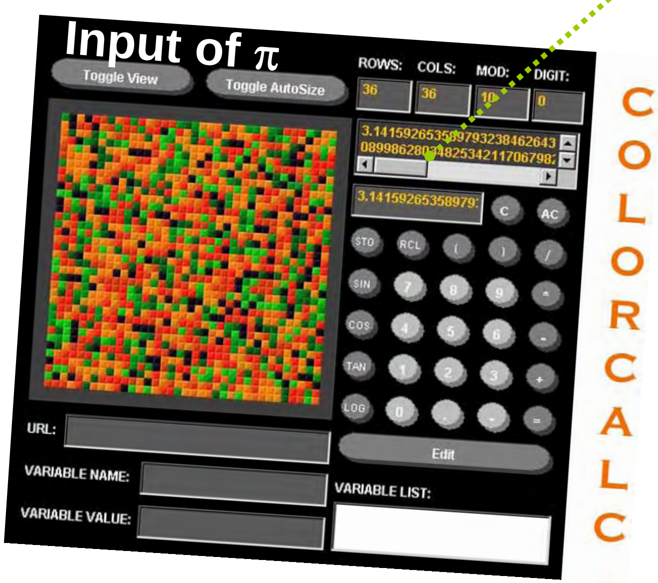
Inverse Symbolic Computation



Archimedes: $223/71 < \pi < 22/7$

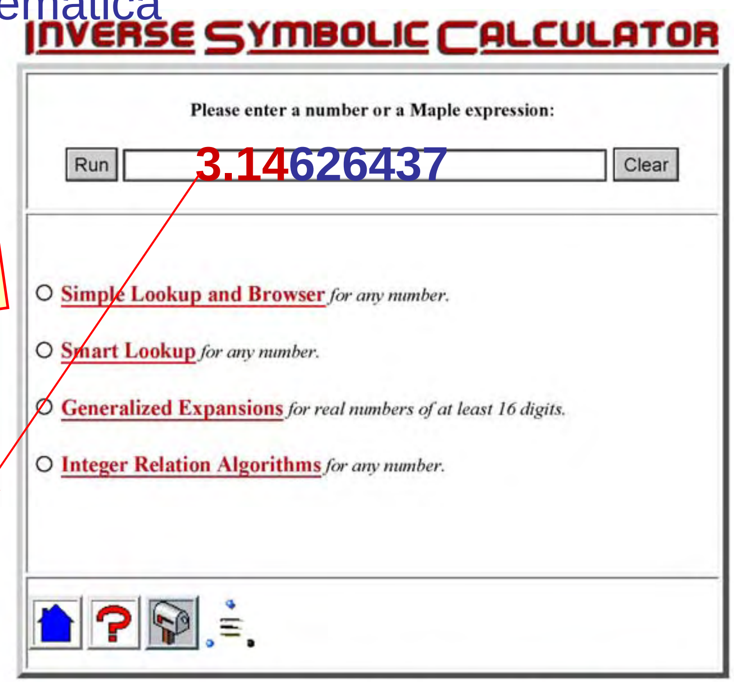
Inferring mathematical structure from numerical data

- Mixes *large table lookup*, integer relation methods and intelligent preprocessing – needs ***micro-parallelism***
- It faces the “curse of exponentiality”
- Implemented as **Recognize** in Mathematica and **identify** in Maple



identify(sqrt(2.)+sqrt(3.))

$$\sqrt{2}+\sqrt{3}$$



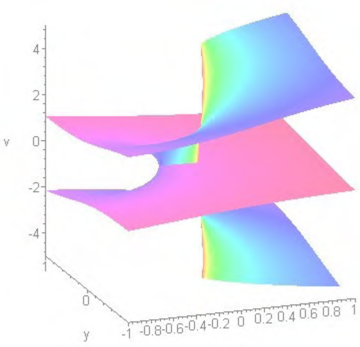
Expressions that are **not** numeric like ln(Pi*sqrt(2)) are evaluated in Maple in symbolic form first, followed by a floating point evaluation followed by a lookup.

Knuth's Problem

A guided proof followed on **asking why** Maple could compute the answer so fast.

The answer is Gonnet's **Lambert's W** which solves

W exp(W) = x

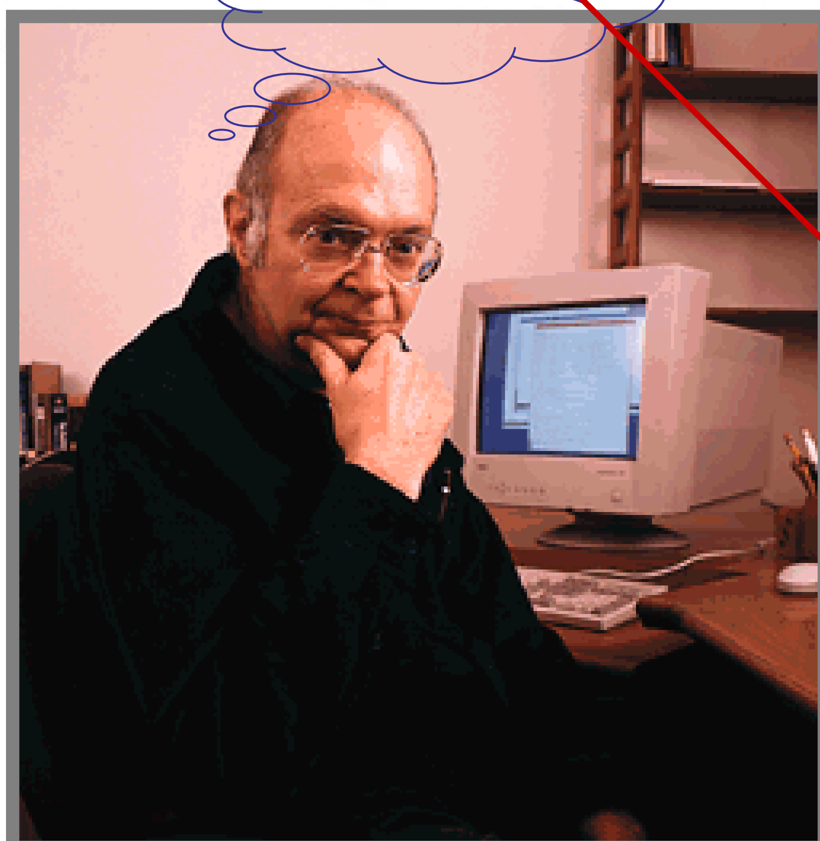


W's **Riemann** surface

Donald Knuth* asked for a closed form evaluation of:

$$\sum_{k=1}^{\infty} \left\{ \frac{k^k}{k!\, e^k} - \frac{1}{\sqrt{2\,\pi\, k}} \right\} = -0.084069508727655 \ldots .$$

"instrumentation"



te 20 or 200 digits

shown on next slide

in the *Inverse Sym-*

turns

$$\approx \frac{2}{3} + \frac{\zeta(1/2)}{\sqrt{2\,\pi}}.$$

ich *Maple* 9.5 on a

in under 6 seconds

*** ARGUABLY WE ARE DONE**

ENTERING

evalf(Sum(k^k/k!/exp(k)-1/sqrt(2*Pi*k),k=1..infinity),16)

'Simple Lookup' fails;
'Smart Look up' gives:

INVERSE SYMBOLIC CALCULATOR

The ISC is the **Inverse Symbolic Calculator**, a set of programs and specialized tables of mathematical constants dedicated to the identification of real numbers. It also serves as a way to produce identities with functions and real numbers. It is one of the main ongoing projects at the Centre for Experimental and Constructive Mathematics (CECM).

Results of the search:

Maple output:

.08406950872765600

.84069508727656006e-1

Value to be looked up: .84069508727656006e-1 = K

```
Performing a smart lookup on .8406950872765600e-1:
```

Function	Result	Precision	Matches
K-2/3	.58259715793901066666666666	16	1

```
579390106 was probably generated by one
s or found in one of the given tables.
Answers are given from shortest to longest description

Mixed constants with 5 operations
5825971579390106 = Zeta(1/2)/sr(2)/sr(Pi)
```

Browse around .5825971579390106.

Quadrature I. Hyperbolic Knots

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt \stackrel{?}{=} L_{-7}(2) \qquad (@)$$

where

$$L_{-7}(s) = \sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^s} + \frac{1}{(7n+2)^s} - \frac{1}{(7n+3)^s} + \frac{1}{(7n+4)^s} - \frac{1}{(7n+5)^s} - \frac{1}{(7n+6)^s} \right].$$

"Identity" (@) has been verified to 20,000 places. I have *no idea* of how to prove it.

The easiest of 998 empirical results (PSLQ, PARI, SnapPea) linking physics/topology (LHS) to number theory (RHS).
[JMB-Broadhurst, 1996]

We have certain knowledge without proof

Extreme Quadrature …
20,000 Digits (50 Certified)
on 1024 CPUs

Ш. The integral was split at the nasty interior singularity
Ш. The sum was `easy'.
Ш. All fast arithmetic & function evaluation ideas used

Run-times and speedup ratios on the Virginia Tech G5 Cluster

CPUs	Init	Integral #1	Integral #2	Total	Speedup
1	*190013	*1534652	*1026692	*2751357	1.00
16	12266	101647	64720	178633	15.40
64	3022	24771	16586	44379	62.00
256	770	6333	4194	11297	243.55
1024	199	1536	1034	2769	993.63

Parallel run times (in seconds) and speedup ratios for the 20,000-digit problem

Expected and unexpected scientific spinoffs

- **1986-1996**. Cray used quartic-Pi to check machines in factory
- **1986**. Complex FFT sped up by factor of two
- **2002**. Kanada used hex-pi (20hrs not 300hrs to check computation)
- **2005**. Virginia Tech (this integral pushed the limits)
- **2006**. A 3D Ising integral took 18.2 hrs on 256 cpus (for 500 places)
- **1995-** Math Resources (another lecture)

Quadrature II. Ising Susceptibility Integrals

Bailey, Crandall and I are currently studying:

$$D_n \;:=\; \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2}{\left(\sum_{j=1}^n (u_j+1/u_j)\right)^2}\frac{du_1}{u_1}\cdots\frac{du_n}{u_n}.$$

The first few values are known: $D_1=2$, $D_2=2/3$, while

$$D_3 = 8+\frac{4}{3}\pi^2 - 27\ \mathrm{L}_{-3}(2)$$

and

$$D_4 = \frac{7}{12}\zeta(3) = \frac{4}{9}\pi^2 - \frac{1}{6} - \frac{7}{2}\zeta(3)$$

- ✓ Computer Algebra Systems can (with help) find the first 3
- ✓ **D_4 is a remarkable 1997 result due to McCoy--Tracy--Wu**

An Extreme Ising Quadrature

Recently Tracy asked for help 'experimentally' evaluating D_5

Using `PSLQ` this entails being able to evaluate a five dimensional integral to at least 50 or 250 places so that one can search for combinations of 6 to 15 constants

✓ Monte Carlo methods can certainly not do this

✓ We are able to reduce D_5 to a horrifying several-page-long 3-D symbolic integral !

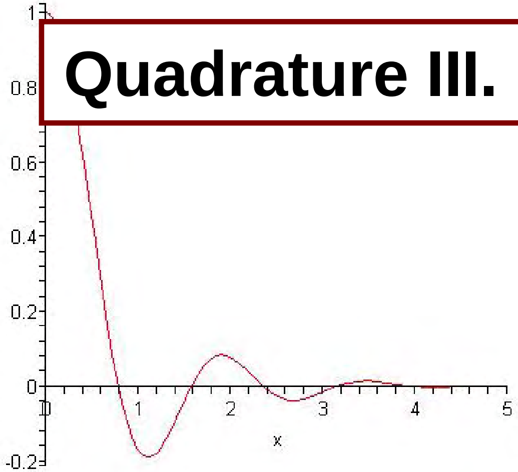
✓**A 256 cpu `tanh-sinh' computation at LBNL provided 500 digits in 18.2 hours on ``Bassi", an IBM Power5 system:**

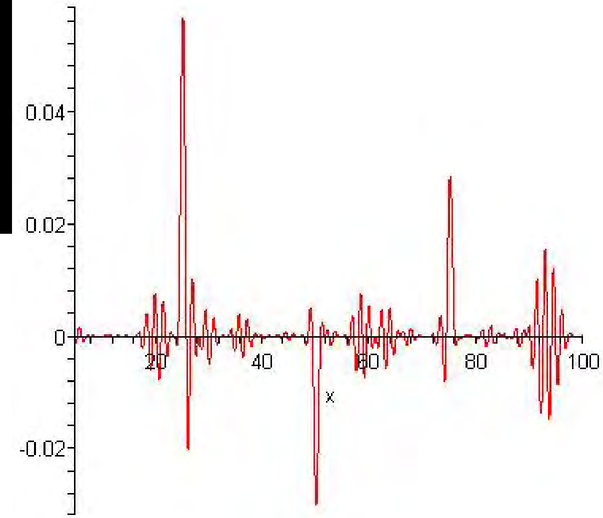
A FIRST

0.00248460576234031547995050915390974963506067764248751615870769
216182213785691543575379268994872451201870687211063925205118620
699449975422656562646708538284124500116682230004545703268769738
489615198247961303552525851510715438638113696174922429855780762
804289477702787109211981116063406312541360385984019828078640186
930726810988548230378878848758305835125785523641996948691463140
911273630946052409340088716283870643642186120450902997335663411
372761220240883454631501711354084419784092245668504608184468...

Quadrature III. Pi/8?

A numerically challenging integral tamed





$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right)\, dx \stackrel{?}{=} \frac{\pi}{8}.$$

Now **π/8** equals

0.392699081698724154807830422909937860524645434

while the **integral** is

0.392699081698724154807830422909937860524646174

A careful *tanh-sinh quadrature* **proves** this difference after **43 correct digits**

Fourier analysis explains this happens when a hyperplane meets a hypercube (LP)

Before and After

REFERENCES

Pla … *ics by Experiment:* … A.K. Peters, 2003.

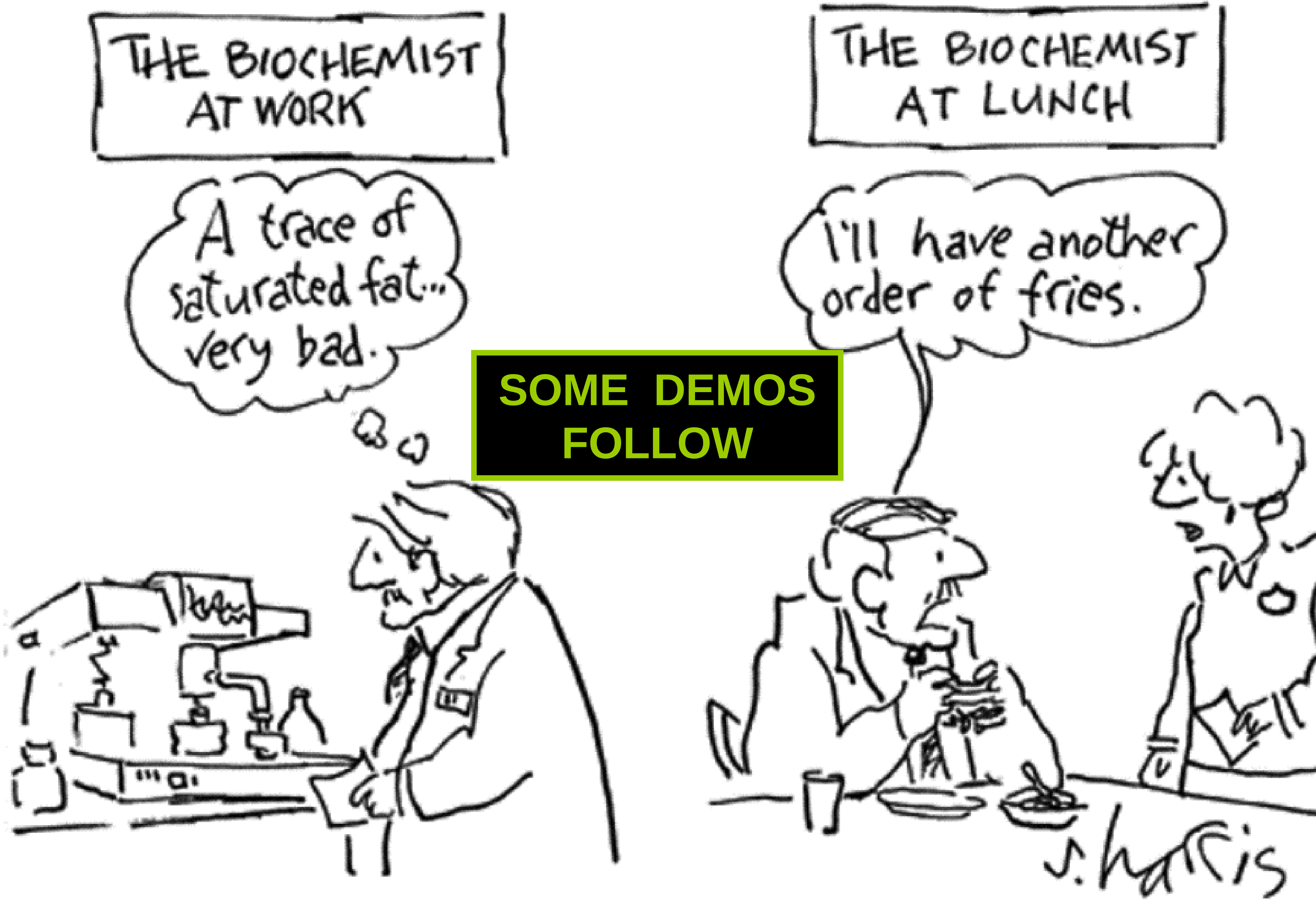
J.M … sohn,
Ex … *tational Paths to*
Dis … **Active CDs 2006]**

D.H … ntal Mathematics:
Ex … *otices Amer. Math.*
So

Enigma

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

- J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.



SOME DEMOS FOLLOW

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5. **Demos and Conclusion.**

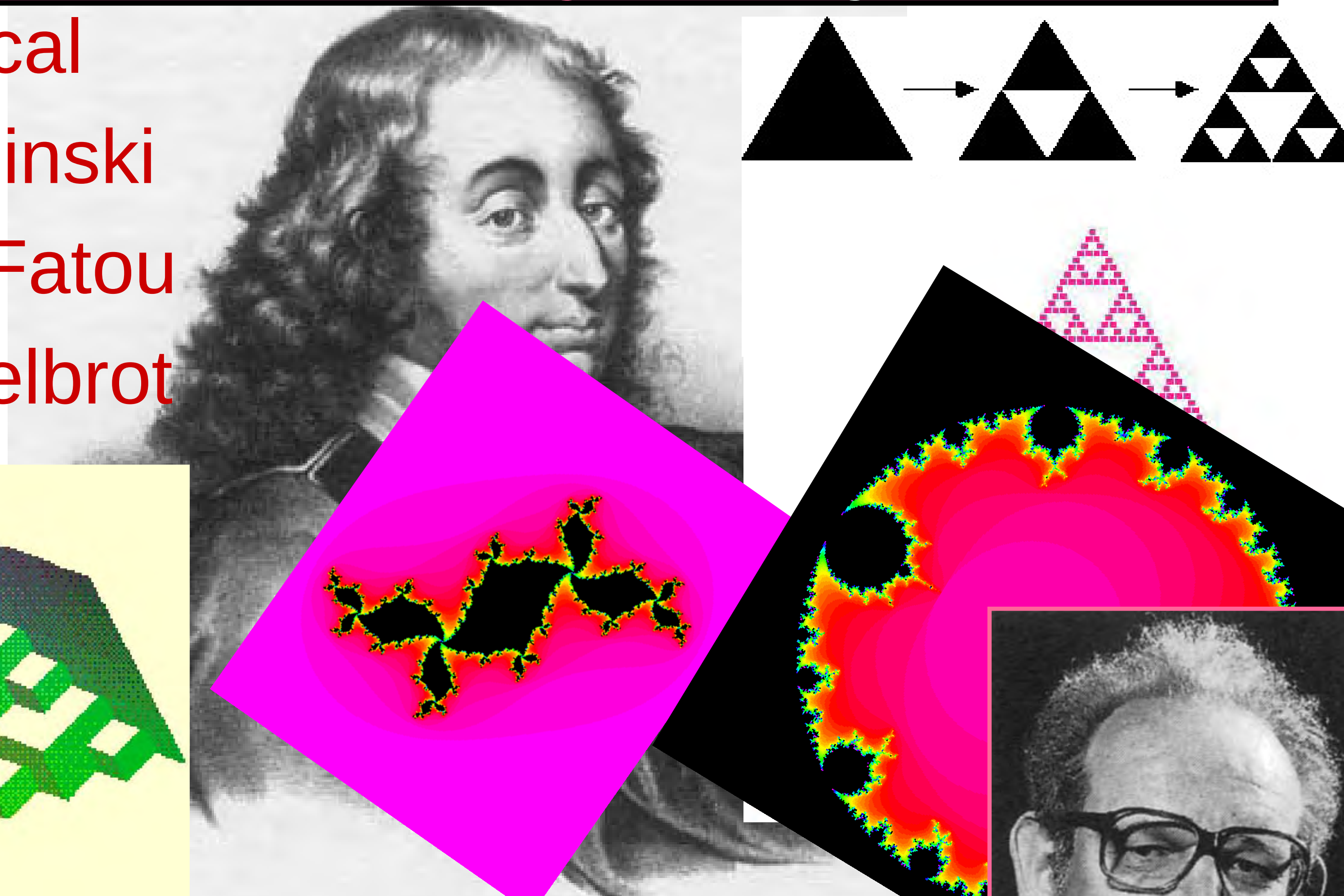
Inverse Systems and Self-Similarity everywhere

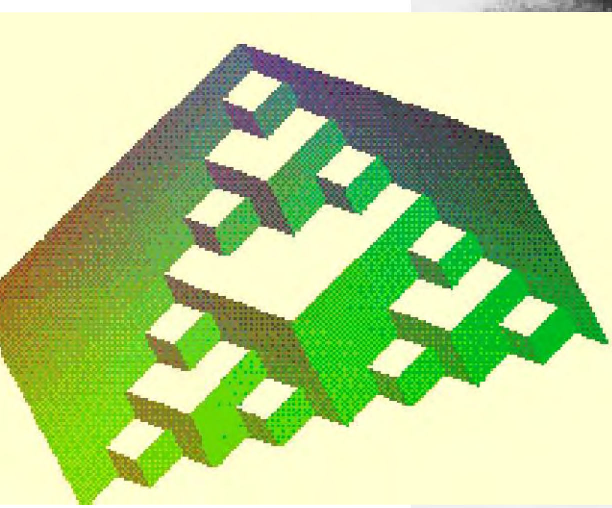
From Pascal

and Sierpinski

to Julia, Fatou

& Mandelbrot





'cut and fold'

Truly modern mathematics in nature, art and applications

Pascal's Triangle Interface

INSTRUCTIONS www.cecm.sfu.ca/interfaces

Output Image

Rows (max 100): 30

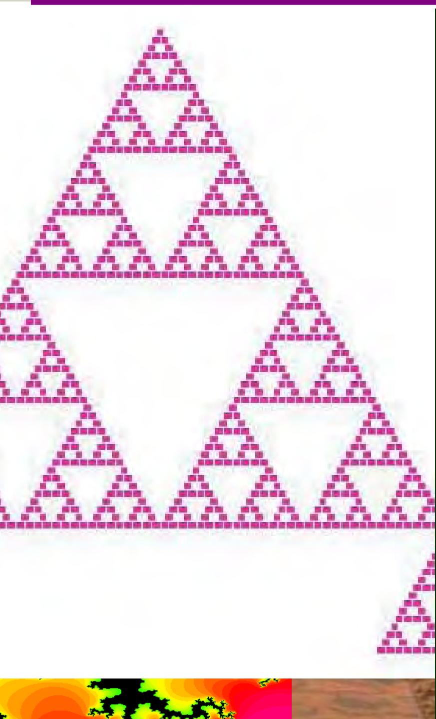
Modulus (2 to 16): 5

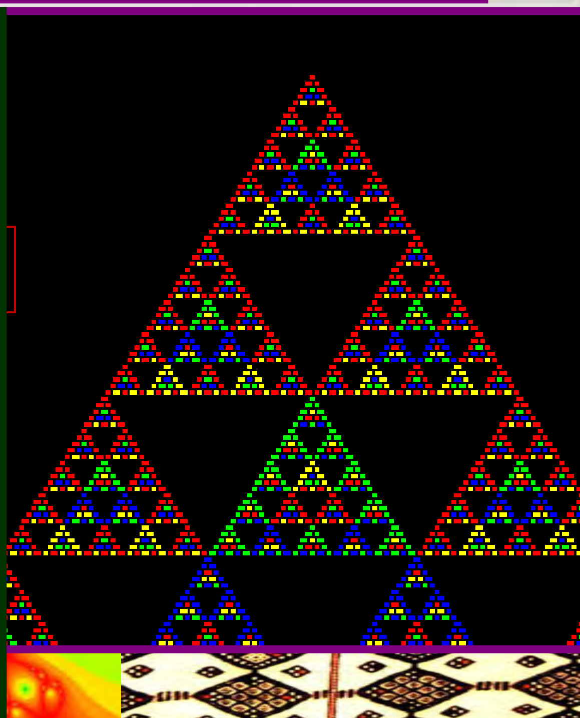
Image size: 300

Deterministic and Random

1 1 1 1 2 1 1 3 3 1 1 4 6 4 1 1 5 10 10 5 1

1 6 15 20 15 6 1 1 7 21 35 21 7 1



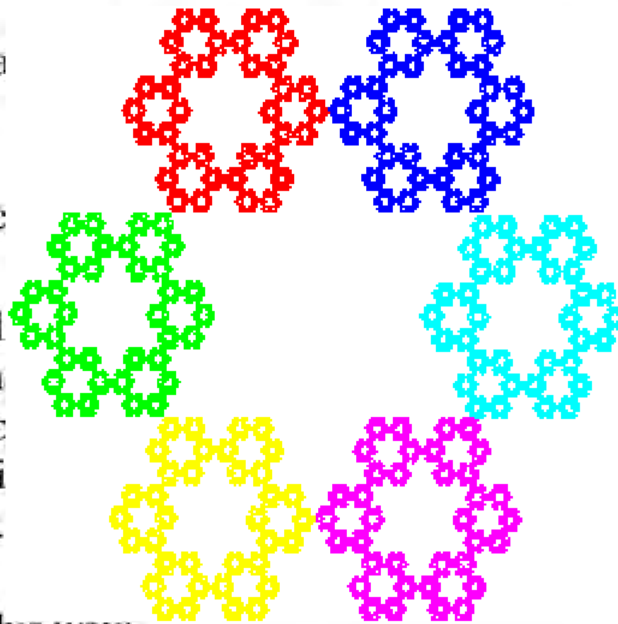


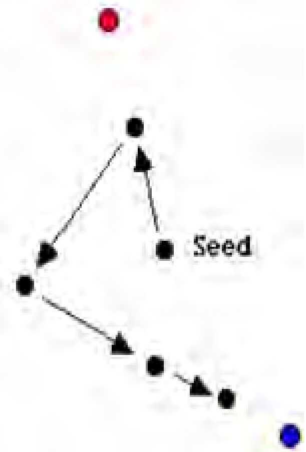
FRACTALINA

About Fractalina

Fractalina allows the input and itera ms. play "the chaos game".

To see it in action, you can go direc



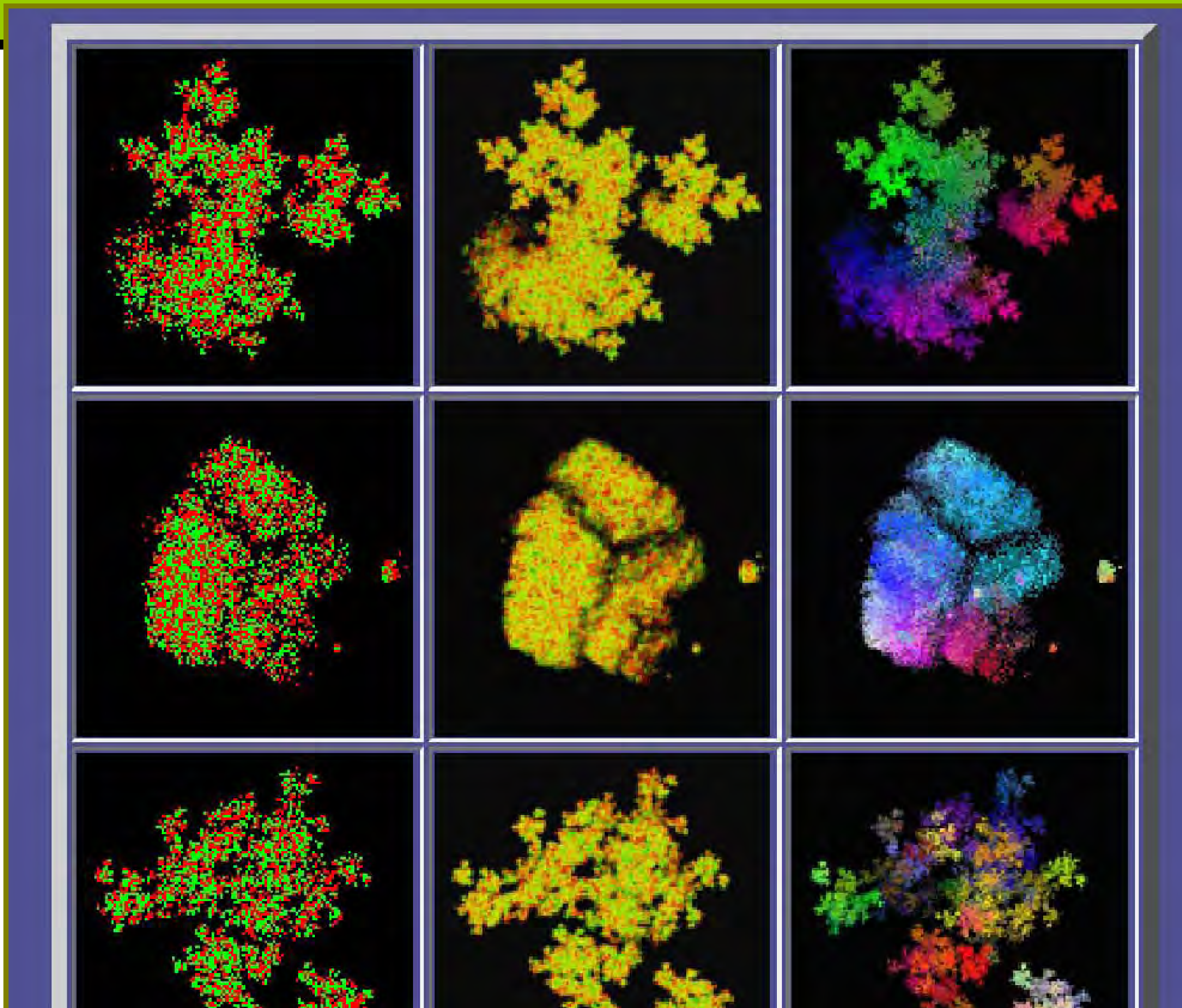


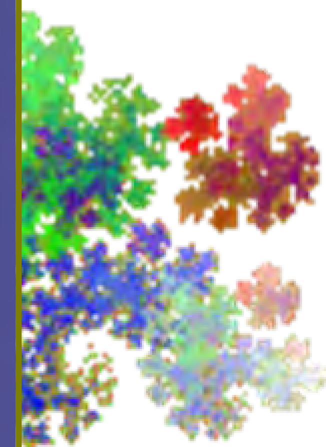
The chaos game begins with the sel e transformations are of a special kind. Each transformation h rotate or compress. We sometimes informally think of the p n lives, so we can say things like "go halfway to the transformati , compression, and position values that determine how it works.

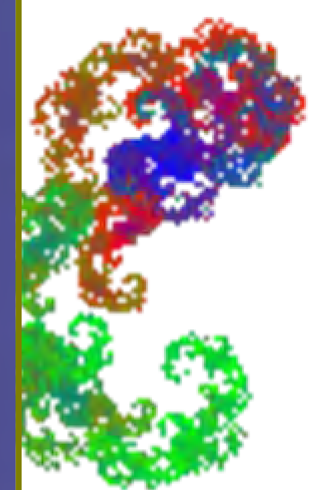
The chaos game can be explained this way:

1. Starting at any point, randomly choose one of the transformations.
2. Go part of the way towards the center point of that transformation and rotate part way around it.
3. Repeat the process from the resulting point.

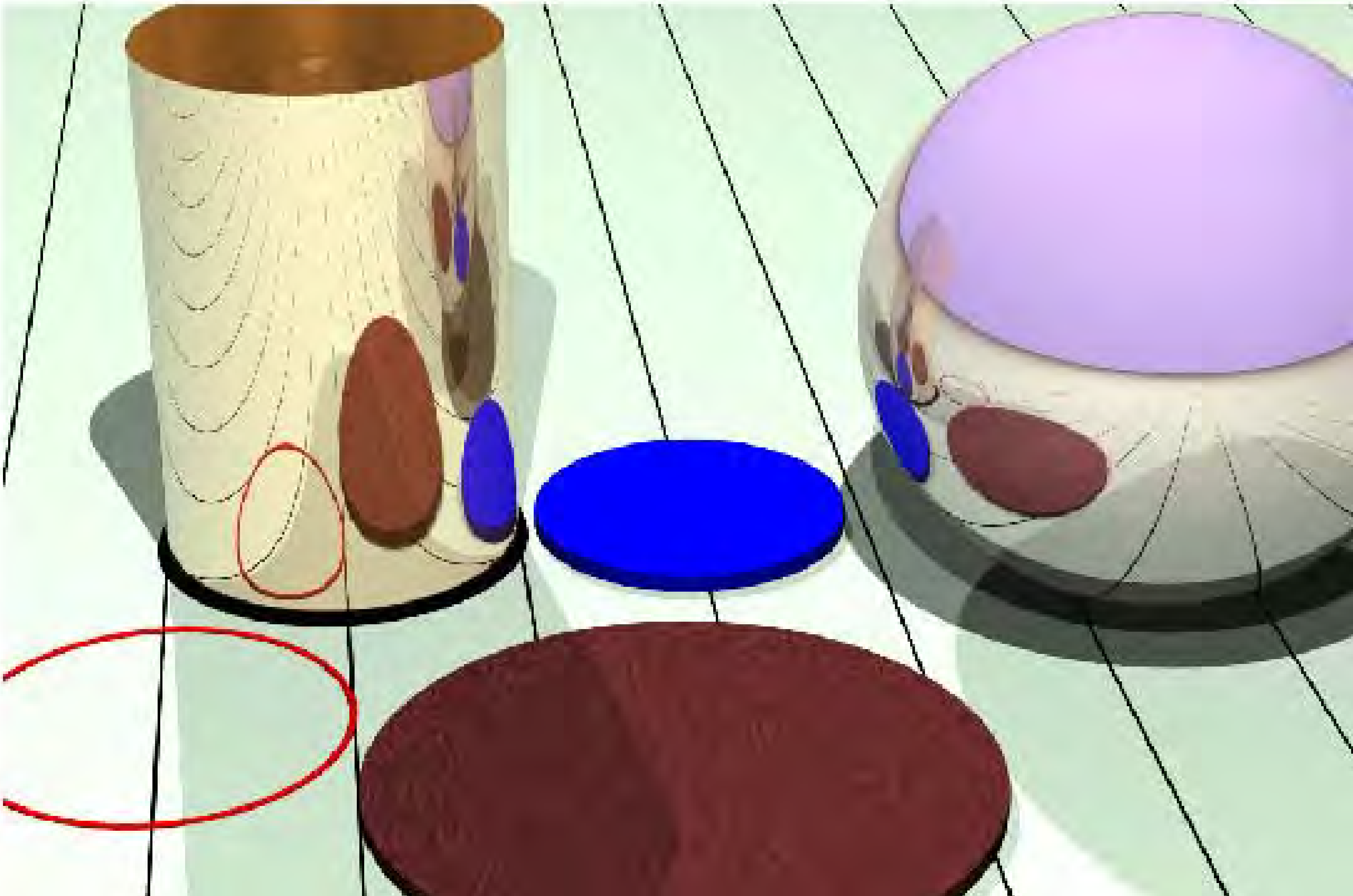
Chaos Games in Genetics

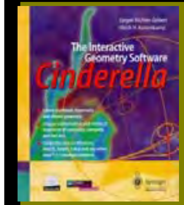




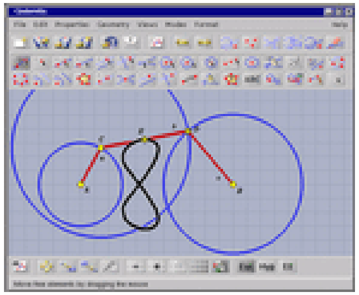


(Euclidean) Reflection in a Circle:

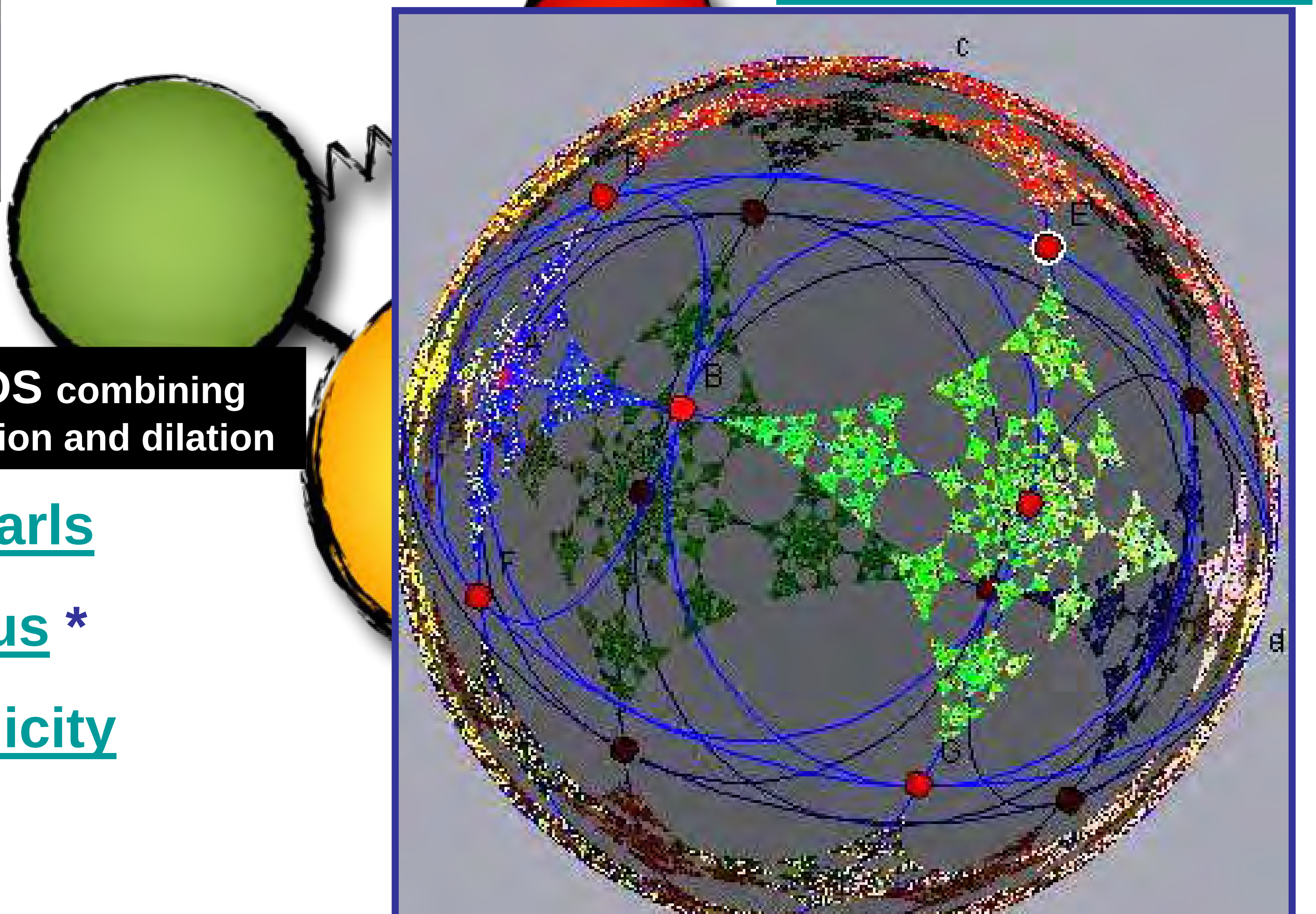




CINDERELLA's dynamic geometry



www.cinderella.de



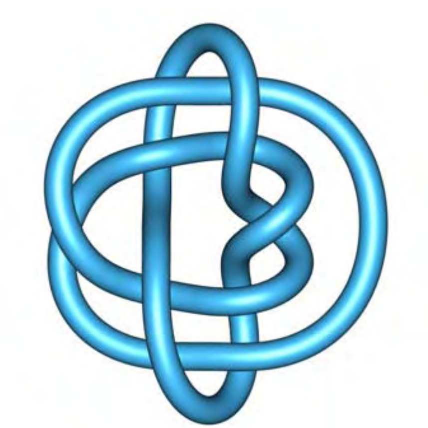
FOUR DEMOS combining inversion, reflection and dilation

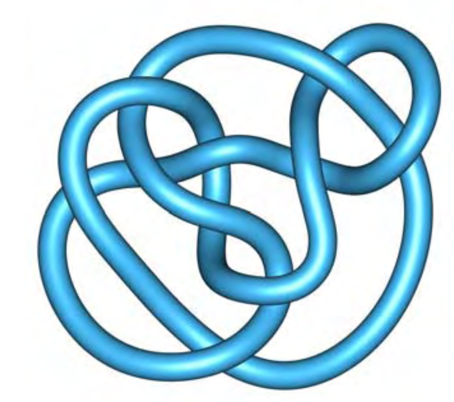
1. Indraspearls
2. Apollonius *
3. Hyperbolicity
4. Gasket

KnotPlot's Interactive Proofs

The Perko Pair 10_{161} and 10_{162}

are two adjacent 10-crossing knots (1900)





First shown to be the same by Ken Perko in 1974

and beautifully made dynamic in KnotPlot (open source)


THEORIES
PROOFS
S. Harris

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1. Visual Data Mining in Mathematics.
   - ✓ Fractals, Polynomials, Continued Fractions*
   - ✓ Pseudospectra and Code Optimization
2. High Precision Mathematics.
3. Integer Relation Methods.
   - ✓ Chaos, Zeta* and the Riemann Hypothesis,
   - ✓ Hex-Pi and Normality
4. Inverse Symbolic Computation.
   - ✓ A problem of Knuth*, $\pi/8$, Extreme Quadrature
5. Conclusion.

CONCLUSION

ENGINES OF DISCOVERY: The 21st Century Revolution

The Long Range Plan for High Performance Computing in Canada

The LRP tells a Story

- The Story
- Executive Summary
- Main Chapters
  - Technology
  - Operations
  - HQP
  - Budget
- 25 Case Studies
  - many sidebars

One Day ...

High-performance computing (HPC) affects the lives of Canadians every day. We can best explain this by telling you a story. It's about an ordinary family on an ordinary day, Russ, Susan, and Kerri Sheppard. They live on a farm 15 kilometres outside Wyoming, Ontario. The land first produced oil, and now it yields milk; and that's just fine locally.

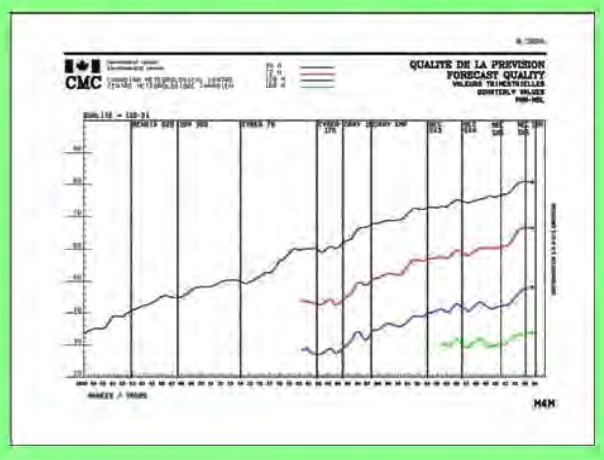
Their day, Thursday, May 29, 2003, begins at 4:30 am when the alarm goes off. A busy day, Susan Zhong-Sheppard will fly to Toronto to see her father, Wu Zhong, at Toronto General Hospital; he's very sick from a stroke. She takes a quick shower and packs a day bag for her 6 am flight from Sarnia's Chris Hadfield airport. Russ Sheppard will stay home at their dairy farm, but his day always starts early. Their young daughter Kerri can sleep three more hours until school.

Waiting, Russ looks outside and thinks, *It's been a dryish spring. Where's the rain?*

In their farmhouse kitchen on a family-sized table sits a PC with a high-speed Internet line. He logs on and finds the Farmer Daily site. He then chooses the Environment Canada link, clicks on Ontario, and then scans down for Sarnia-Lambton.

WEATHER PREDICTION

The "quality" of a five-day forecast in the year 2003 was equivalent to that of a 36-hour forecast in 1963 [REF 1]. The quality of daily forecasts has risen sharply by roughly one day per decade of research and HPC progress. Accurate forecasts transform into billions of dollars saved annually in agriculture and in natural disasters. Using a model developed at Dalhousie University (Prof. Keith Thompson), the Meteorological Service of Canada has recently been able to predict coastal flooding in Atlantic Canada early enough for the residents to take preventative action.



The backbone that makes so much of our Canadian science possible