Invited Paper

Structure Theory for Maximally Monotone Operators with Points of Continuity

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Abstract

In this paper, we consider the structure of maximally monotone operators in Banach space whose domains have nonempty interior and we present new and explicit structure formulas for such operators. Along the way, we provide new proofs of norm-to-weak* closedness and of property (Q) for these operators (as recently proven by Voisei). Various applications and limiting examples are given.

Keywords: Local boundedness, maximally monotone operator, monotone operator, norm-weak* graph closedness, property (Q).

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1 Introduction

Monotone operators have frequently proven to be a key class of objects in both modern Optimization and Analysis; see, e.g., [1–3], the books [4–14] and the references given therein.

In this paper, we consider the structure of maximally monotone operators in Banach space whose domains have nonempty interior—which as we shall see implies the existence of points with various continuity properties—and we present new and explicit structure formulas for such operators. Along the way, we give new proofs of several of Voisei's recent results: norm-to-weak* closedness and property (Q) for these operators. We also revisit one more-classical result due to Auslender. Various applications and limiting examples are given.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic notations and background in Monotone Operator Theory. In Section 3, we collect preliminary results for future reference and the reader's convenience. In Section 4, we study local boundedness properties of monotone operators and also give a somewhat simpler proof of a recent result of Voisei [15]. The main result (Theorem 5.2) is proved in Section 5, and we also present a new proof of a result of Auslender (Theorem 5.1). A second structure theorem — which yields a strong version of property (Q) for maximally monotone operators

(Theorem 5.3) — is also provided. In Section 6 we present a few extra illustrative examples. Finally, we list some open questions raised from our paper and the two most central open problems in Monotone Operator Theory in Section 7.

2 Preliminaries

We assume throughout that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. The *closed unit ball* in X is denoted by $B_X := \big\{x \in X \mid \|x\| \leq 1\big\}$, $B_\delta(x) := x + \delta B_X$ (where $\delta > 0$ and $x \in X$) and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as a relation, point-to-set mapping or multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A := \big\{(x, x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of A. The *domain* of A is $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$ and $\operatorname{ran} A := A(X)$ is the *range* of A.

Recall that A is *monotone* iff

$$\langle x - y, x^* - y^*\rangle \geq 0, \quad \forall (x, x^*) \in \operatorname{gra} A \;\forall (y, y^*) \in \operatorname{gra} A, \tag{1}$$

and *maximally monotone* iff A is monotone and A has no proper monotone extension (in the sense of graph inclusion). Let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$. We say (x, x^*) is *monotonically related to* $\operatorname{gra} A$ iff

$$\langle x - y, x^* - y^*\rangle \geq 0, \quad \forall (y, y^*) \in \operatorname{gra} A.$$

As much as possible we adopt standard convex analysis notation. Given a subset C of X, $\operatorname{int} C$ is the *interior* of C and $\overline{C}$ is the *norm closure* of C. For the set $D \subseteq X^*$, $\overline{D}^{\mathrm{w}^*}$ is the weak* closure of D, and the norm $\times$ weak* closure of $C \times D$ is $\overline{C \times D}^{\|\cdot\| \times \mathrm{w}^*}$. The *indicator function* of C, written as ι_C, is defined at $x \in X$ by

$$\iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

For every $x \in X$, the *normal cone* operator of C at x is defined by $N_C(x) := \big\{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leq 0\big\}$, if $x \in C$; and $N_C(x) := \varnothing$, if $x \notin C$; the *tangent cone* operator of C at x is defined by $T_C(x) := \big\{y \in X \mid \sup_{x^* \in N_C(x)} \langle y, x^* \rangle \leq 0\big\}$, if $x \in C$; and $T_C(x) := \varnothing$, if $x \notin C$. The *hypertangent cone* of C at x, $H_C(x)$, coincides with the interior of $T_C(x)$ (see [16, 17]).

Let $f \colon X \to \left]-\infty, +\infty\right]$. Then $\operatorname{dom} f := f^{-1}(\mathbb{R})$ is the *domain* of f. We say f is proper iff $\operatorname{dom} f \neq \varnothing$. Let f be proper. The *subdifferential* of f is defined by

$$\partial f \colon X \rightrightarrows X^* \colon x \mapsto \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^* \rangle + f(x) \leq f(y)\}.$$

Let $g \colon X \to \left]-\infty, +\infty\right]$. Then the *inf-convolution* $f \Box g$ is the function defined on X by

$$f \Box g \colon x \mapsto \inf_{y \in X} \left[f(y) + g(x - y))\right].$$

We say a net $(a_\alpha)_{\alpha \in \Gamma}$ in X is *eventually bounded* iff there exist $\alpha_0 \in \Gamma$ and $M \geq 0$ such that

$$\|a_\alpha\| \leq M, \quad \forall \alpha \succeq_\Gamma \alpha_0.$$

We denote by $\longrightarrow$ and $\rightharpoonup_{\mathrm{w}^*}$ respectively, the norm convergence and weak* convergence of nets.

Let $A : X \rightrightarrows X^*$ be monotone with $\operatorname{dom} A \neq \varnothing$ and consider a set $S \subseteq \operatorname{dom} A$. We define $A_S : X \rightrightarrows X^*$ by

$$\operatorname{gra} A_S = \overline{\operatorname{gra} A \cap (S \times X^*)}^{\|\cdot\| \times \mathrm{w}^*}$$

$$= \big\{(x, x^*) \mid \exists \text{ a net } (x_\alpha, x^*_\alpha)_{\alpha\in\Gamma} \text{ in } \operatorname{gra} A \cap (S \times X^*) \text{ such that } x_\alpha \longrightarrow x,$$

$$x^*_\alpha \rightharpoonup_{\mathrm{w}^*} x^*\big\}. \quad (3)$$

If $\operatorname{int}\operatorname{dom} A \neq \varnothing$, we denote by $A_{\mathrm{int}} := A_{\operatorname{int}\operatorname{dom} A}$. We note that $\operatorname{gra} A_{\operatorname{dom} A} = \overline{\operatorname{gra} A}^{\|\cdot\| \times \mathrm{w}^*} \supseteq \operatorname{gra} A$ while $\operatorname{gra} A_S \subseteq \operatorname{gra} A_T$ for $S \subseteq T$.

Let $A : X \rightrightarrows X^*$. Following [18], we say A has the upper-semicontinuity property *property (Q)* iff for every net $(x_\alpha)_{\alpha\in J}$ in X such that $x_\alpha \longrightarrow x$, we have

$$\bigcap_{\alpha\in J} \overline{\operatorname{conv}\left[\bigcup_{\beta \succeq_J \alpha} A(x_\beta)\right]}^{\mathrm{w}^*} \subseteq Ax. \quad (4)$$

3 Preliminary Results

We start with a classic compactness theorem.

Fact 3.1 [Banach–Alaoglu](See [19, Theorem 2.6.18] or [20, Theorem 3.15].) The closed unit ball B_{X^*} in X^* is weak* compact.

Fact 3.2 [Rockafellar](See [21, Theorem A], [12, Theorem 3.2.8], [11, Theorem 18.7] or [5, Theorem 9.2.1].) Let $f : X \to \left]-\infty, +\infty\right]$ be a proper lower semicontinuous convex function. Then ∂f is maximally monotone.

The prior result can fail in both incomplete normed spaces and in complete metrizable locally convex spaces [5]. The next two important central results now has many proofs (see also [5, Ch. 8]).

Fact 3.3 [Rockafellar](See [22, Theorem 1] or [7, Theorem 2.28].) Let $A : X \rightrightarrows X^*$ be monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then A is locally bounded at $x \in \operatorname{int}\operatorname{dom} A$, i.e., there exist $\delta > 0$ and $K > 0$ such that

$$\sup_{y^* \in Ay} \|y^*\| \leq K, \quad \forall y \in (x + \delta B_X) \cap \operatorname{dom} A.$$

Fact 3.4 [Rockafellar] (See [22, Theorem 1] or [11, Theorem 27.1 and Theorem 27.3].) Let $A : X \rightrightarrows X^*$ be maximal monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then $\operatorname{int}\operatorname{dom} A = \operatorname{int}\overline{\operatorname{dom} A}$ and $\overline{\operatorname{dom} A}$ is convex.

The final two results we give are elementary.

Fact 3.5 ([23, Section 2, page 539].) Let $A : X \rightrightarrows X^*$ be maximally monotone and a net $(a_\alpha, a^*_\alpha)_{\alpha \in \Gamma}$ in $\operatorname{gra} A$. Assume that $(a_\alpha, a^*_\alpha)_{\alpha \in \Gamma}$ norm $\times$ weak* converges to (x, x^*) and $(a^*_\alpha)_{\alpha \in \Gamma}$ is eventually bounded. Then $(x, x^*) \in \operatorname{gra} A$.

Fact 3.6 (See [24, Proposition 4.1.7].) Let C be a convex subset of X with $\operatorname{int} C \neq \varnothing$. Then for every $x \in C$, $\operatorname{int} T_C(x) = \bigcup_{\lambda > 0} \lambda\,[\operatorname{int} C - x]$.

4 Local Boundedness Properties

The following result is extracted from part of the proof of [25, Proposition 3.1]. For the reader's convenience, we repeat the proof here.

Fact 4.1 [Boundedness below] Let $A : X \rightrightarrows X^*$ be monotone and $x \in \operatorname{int}\operatorname{dom} A$. Then there exist $\delta > 0$ and $M > 0$ such that $x + \delta B_X \subseteq \operatorname{dom} A$ and $\sup_{a \in x+\delta B_X} \|Aa\| \leq M$. Assume that (z, z^*) is monotonically related to $\operatorname{gra} A$. Then

$$\langle z - x, z^* \rangle \geq \delta \|z^*\| - (\|z - x\| + \delta) M. \tag{5}$$

Proof. Since $x \in \operatorname{int}\operatorname{dom} A$, using Fact 3.3, there exist $\delta > 0$ and $M > 0$ such that

$$Aa \neq \varnothing \quad \text{and} \quad \sup_{a^* \in Aa} \|a^*\| \leq M, \quad \forall a \in (x + \delta B_X). \tag{6}$$

Then we have

$$\begin{aligned}
&\langle z - x - b, z^* - b^* \rangle \geq 0, \quad \forall b \in \delta B_X, b^* \in A(x + b) \\
&\Rightarrow \langle z - x, z^* \rangle - \langle b, z^* \rangle + \langle z - x - b, -b^* \rangle \geq 0, \quad \forall b \in \delta B_X, b^* \in A(x + b) \\
&\Rightarrow \langle z - x, z^* \rangle - \langle b, z^* \rangle \geq \langle z - x - b, b^* \rangle, \quad \forall b \in \delta B_X, b^* \in A(x + b) \\
&\Rightarrow \langle z - x, z^* \rangle - \langle b, z^* \rangle \geq -(\|z - x\| + \delta) M, \quad \forall b \in \delta B_X \quad \text{(by (6))} \\
&\Rightarrow \langle z - x, z^* \rangle \geq \langle b, z^* \rangle - (\|z - x\| + \delta) M, \quad \forall b \in \delta B_X. \qquad (7)
\end{aligned}$$

Hence we have $\langle z - x, z^* \rangle \geq \delta \|z^*\| - (\|z - x\| + \delta) M$. ☐

Fact 4.1 leads naturally to the following result which has many precursors [11, 15].

Lemma 4.1 [Strong directional boundedness] Let $A : X \rightrightarrows X^*$ be monotone and $x \in \operatorname{int}\operatorname{dom} A$. Then there exist $\delta > 0$ and $M > 0$ such that $x + 2\delta B_X \subseteq \operatorname{dom} A$ and $\sup_{a \in x+2\delta B_X} \|Aa\| \leq M$. Assume also that (x_0, x_0^*) is monotonically related to $\operatorname{gra} A$. Then

$$\sup_{a\in[x+\delta B_X,\, x_0[,\, a^*\in Aa} \|a^*\| \leq \frac{1}{\delta}\left(\|x_0-x\|+1\right)\left(\|x_0^*\|+2M\right),$$

where $[x+\delta B_X,\, x_0[:= \big\{(1-t)y+tx_0 \mid 0\leq t<1, y\in x+\delta B_X\big\}$.

Proof. Since $x \in \operatorname{int}\operatorname{dom} A$, by Fact 3.3, there exist $\delta > 0$ and $M > 0$ such that

$$x+2\delta B_X \subseteq \operatorname{dom} A \quad \text{and} \quad \sup_{a^*\in Aa} \|a^*\| \leq M, \quad \forall a \in (x+2\delta B_X). \tag{8}$$

Let $y \in x+\delta B_X$. Then by (8),

$$y+\delta B_X \subseteq \operatorname{dom} A \quad \text{and} \quad \sup_{a^*\in Aa} \|a^*\| \leq M, \quad \forall a \in (y+\delta B_X). \tag{9}$$

Let $t \in [0,1[$ and $a^* \in A((1-t)y+tx_0)$. By the assumption that (x_0, x_0^*) is monotonically related to $\operatorname{gra} A$, we have

$$\big\langle a^*-x_0^*, (1-t)(y-x_0)\big\rangle = \big\langle a^*-x_0^*, (1-t)y+tx_0-x_0\big\rangle \geq 0.$$

Thus

$$\langle a^*, x_0-y\rangle \leq \langle x_0-y, x_0^*\rangle. \tag{10}$$

By Fact 4.1 and (9),

$$\begin{aligned}
\delta\|a^*\| &\leq \big\langle (1-t)y+tx_0-y, a^*\big\rangle + \big(\|(1-t)y+tx_0-y\|+\delta\big)M \\
&\leq \big\langle t(x_0-y), a^*\big\rangle + \big(\|x_0-y\|+\delta\big)M \\
&\leq \big\langle t(x_0-y), a^*\big\rangle + \big(\|x_0-x\|+2\delta\big)M \quad (\text{since } y\in x+\delta B_X).
\end{aligned} \tag{11}$$

Then by (11) and (10),

$$\begin{aligned}
\|a^*\| &\leq \frac{1}{\delta} t\langle x_0 - y, x_0^*\rangle + \frac{M}{\delta}\|x_0 - x\| + 2M \leq \frac{1}{\delta}\|x_0 - y\| \cdot \|x_0^*\| + \frac{M}{\delta}\|x_0 - x\| + 2M \\
&\leq \frac{1}{\delta}\big(\|x_0 - x\| + \delta\big)\|x_0^*\| + \frac{M}{\delta}\|x_0 - x\| + 2M \quad (\text{since } y \in x + \delta B_X) \\
&\leq \frac{1}{\delta}\|x_0 - x\| \cdot \|x_0^*\| + \|x_0^*\| + \frac{M}{\delta}\|x_0 - x\| + 2M \\
&= \frac{1}{\delta}\|x_0 - x\|\big(\|x_0^*\| + M\big) + \|x_0^*\| + 2M \\
&\leq \frac{1}{\delta}\big(\|x_0 - x\| + 1\big)\big(\|x_0^*\| + 2M\big).
\end{aligned}$$

Hence

$$\sup_{a \in [x + \delta B_X, x_0[, a^* \in Aa} \|a^*\| \leq \frac{1}{\delta}\left(\|x_0 - x\| + 1\right)\left(\|x_0^*\| + 2M\right).$$

We now have the required estimate. $\square$

The following result — originally conjectured by the first author in [26] — was established by Voisei in [15, Theorem 37] as part of a more complex set of results (See [15] for more general results.). We next give a somewhat simpler proof by applying a similar technique to that used in the proof of [25, Prop 3.1, subcase 2].

Theorem 4.1 [Eventual boundedness] Let $A : X \rightrightarrows X^*$ be monotone such that $\operatorname{int} \operatorname{dom} A \neq \varnothing$. Then every norm $\times$ weak* convergent net in $\operatorname{gra} A$ is eventually bounded.

Proof. As the result and hypotheses are again invariant under translation, we can and do suppose that $0 \in \operatorname{int} \operatorname{dom} A$. Let $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ in $\operatorname{gra} A$ be such that

$$(a_\alpha, a_\alpha^*) \text{ norm} \times \text{weak}^* \text{ converges to } (x, x^*). \tag{12}$$

Clearly, it suffices to show that $(a_\alpha^*)_{\alpha\in\Gamma}$ is eventually bounded. Suppose to the contrary that $(a_\alpha^*)_{\alpha\in\Gamma}$ is not eventually bounded. Then there exists a subnet of $(a_\alpha^*)_{\alpha\in\Gamma}$, for convenience, still denoted by $(a_\alpha^*)_{\alpha\in\Gamma}$, such that

$$\lim_\alpha \|a_\alpha^*\| = +\infty. \tag{13}$$

We can and do suppose that $a_\alpha^* \neq 0, \forall \alpha \in \Gamma$. By Fact 4.1, there exist $\delta > 0$ and $M > 0$ such that

$$\langle a_\alpha, a_\alpha^* \rangle \geq \delta \|a_\alpha^*\| - (\|a_\alpha\| + \delta)M, \quad \forall \alpha \in \Gamma. \tag{14}$$

Then we have

$$\langle a_\alpha, \frac{a_\alpha^*}{\|a_\alpha^*\|} \rangle \geq \delta - \frac{(\|a_\alpha\| + \delta)M}{\|a_\alpha^*\|}, \quad \forall \alpha \in \Gamma. \tag{15}$$

By Fact 3.1, there exists a weak* convergent *subnet* $(a_\beta^*)_{\beta\in I}$ of $(a_\alpha^*)_{\alpha\in\Gamma}$, say

$$\frac{a_\beta^*}{\|a_\beta^*\|} \rightharpoonup_{\mathrm{w}^*} a_\infty^* \in X^*. \tag{16}$$

Then taking the limit along the subnet in (15), by (12) and (13), we have

$$\langle x, a_\infty^* \rangle \geq \delta. \tag{17}$$

On the other hand, by (12), we have

$$\langle x, a_\alpha^* \rangle \longrightarrow \langle x, x^* \rangle. \tag{18}$$

Dividing by $\|a_\alpha^*\|$ in both sides of (18), then by (13) and (16) we take the limit along the subnet again to get

$$\langle x, a_\infty^* \rangle = 0. \tag{19}$$

The above inequality contradicts (17). Hence we have $(a_\alpha, a_\alpha^*)_{\alpha\in\Gamma}$ is eventually bounded. $\square$

Corollary 4.1 [Norm-weak* closed graph] Let $A : X \rightrightarrows X^*$ be maximally monotone such that $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then $\operatorname{gra} A$ is norm $\times$ weak* closed.

Proof. Apply Fact 3.5 and Theorem 4.1. $\square$

Example 4.1 [Failure of graph to be norm-weak* closed] In [23], the authors showed that the statement of Corollary 4.1 cannot hold without the assumption of the nonempty interior domain even for the subdifferential operators — actually it fails in the bw* topology. More precisely (see [23] or [4, Example 21.5]): Let $f : \ell^2(\mathbb{N}) \to \left]-\infty, +\infty\right]$ be defined by

$$x \mapsto \max\left\{1 + \langle x, e_1\rangle, \sup_{2\leq n\in\mathbb{N}} \langle x, \sqrt{n}e_n\rangle\right\}, \tag{20}$$

where $e_n := (0, \ldots, 0, 1, 0, \cdots, 0)$: the nth entry is 1 and the others are 0. Then f is proper lower semicontinuous and convex, but ∂f is not norm $\times$ weak* closed. A more general construction in an infinite-dimensional Banach space E is also given in [23, Section 3]. It is as follows:

Let Y be an infinite dimensional separable subspace of E, and $(v_n)_{n\in\mathbb{N}}$ be a *normalized Markushevich basis* of Y with the dual coefficients $(v_n^*)_{n\in\mathbb{N}}$. We defined $v_{p,m}$ and $v_{p,m}^*$ by

$$v_{p,m} := \frac{1}{p}(v_p + v_{p^m}) \quad \text{and} \quad v_{p,m}^* := v_p^* + (p-1)v_{p^m}^*, \quad m \in \mathbb{N}, p \,\text{is prime}.$$

Let $f : E \to \left]-\infty, +\infty\right]$ be defined by

$$x \mapsto \iota_Y(x) + \max\left\{1 + \langle x, v_1^*\rangle, \sup_{2\leq m\in\mathbb{N},\, p\,\text{is prime}} \langle x, v_{p,m}^*\rangle\right\}. \tag{21}$$

Then f is proper lower semicontinuous and convex. We have that ∂f is not norm $\times$ bw* closed and hence ∂f is not norm $\times$ weak* closed. $\Diamond$

Corollary 4.2 Let $A : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Assume that $S \subseteq \operatorname{dom} A$. Then $\operatorname{gra} A_S \subseteq \operatorname{gra} A$ and in consequence $\overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}^*} \subseteq Ax, \forall x \in \operatorname{dom} A$. Moreover, $Ax = A_S(x), \forall x \in S$ and hence $Ax = A_{\mathrm{int}}(x), \forall x \in \operatorname{int}\operatorname{dom} A$.

Proof. By (3) and Corollary 4.1, $\operatorname{gra} A_S \subseteq \operatorname{gra} A$. Since A is maximally monotone, (for every $x \in \operatorname{dom} A$), Ax is convex and weak* closed. Thus $\overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}^*} \subseteq Ax, \forall x \in \operatorname{dom} A$. Let $x \in S$. Then by (3) again, $Ax \subseteq A_S(x)$ and hence $Ax = A_S(x)$. Thus we have $A = A_{\mathrm{int}}$ on $\operatorname{int}\operatorname{dom} A$. $\square$

We now turn to consequences of these boundedness results.

5 Structure of Maximally Monotone Operators

A useful consequence of the Hahn-Banach separation principle [5] is:

Proposition 5.1 Let D, F be nonempty subsets of X^*, and C be a convex set of X with $\operatorname{int} C \neq \varnothing$. Assume that $x \in C$ and that for every $v \in \operatorname{int} T_C(x)$,

$$\sup\langle D, v\rangle \leq \sup\langle F, v\rangle < +\infty.$$

Then

$$D \subseteq \overline{\operatorname{conv} F + N_C(x)}^{\mathrm{w}^*}. \tag{22}$$

Proof. The separation principle ensures that suffices to show

$$\sup\big\langle D, h\big\rangle \leq \sup\big\langle N_C(x) + F, h\big\rangle, \quad \forall h \in X. \tag{23}$$

We consider two cases.

Case 1: $h \notin T_C(x)$. We have $\sup \big\langle N_C(x) + F, h\big\rangle = +\infty$ since $\sup \big\langle N_C(x), h\big\rangle = +\infty$. Hence (23) holds.

Case 2: $h \in T_C(x)$. Let $v \in \operatorname{int} T_C(x)$. Then (for every $t > 0$) $h + tv \in \operatorname{int} T_C(x)$ by [27, Fact 2.2(ii)]. Now $z \mapsto \sup \big\langle D, z\big\rangle$ is lower semicontinuous, and so by the assumption, we have

$$\begin{aligned}
\sup \big\langle D, h\big\rangle &\leq \liminf_{t\to 0^+} \sup \big\langle D, h + tv\big\rangle \leq \liminf_{t\to 0^+} \sup \big\langle F, h + tv\big\rangle \\
&\leq \sup \big\langle F, h\big\rangle + \liminf_{t\to 0^+} t \sup \big\langle F, v\big\rangle \\
&= \sup \big\langle F, h\big\rangle \quad (\text{ since } \sup \big\langle F, v\big\rangle \text{ is finite}) \\
&\leq \sup \big\langle N_C(x) + F, h\big\rangle.
\end{aligned}$$

Hence (23) holds and we have (22) holds. $\square$

The proof of Proposition 5.1 was inspired partially by that of [27, Theorem 4.5].

Remark 5.1 Dr. Robert Csetnek kindly communicated to us the following alternative proof of Proposition 5.1:

Let σ_D be the *support function* of the set D, i.e., $\sigma_D(z) := \sup_{d^*\in D}\langle z, d^*\rangle$, $\forall z \in X$. The hypotheses imply $\sigma_D \leq \sigma_F + \iota_{\operatorname{int} T_C(x)}$, hence taking the conjugates we have $\iota_{\overline{\operatorname{conv} D}^{\mathrm{w}^*}} \geq \big(\sigma_F + \iota_{\operatorname{int} T_C(x)}\big)^*$; since $\operatorname{int} T_C(x) \subseteq \operatorname{dom} \sigma_F$, we can apply [12, Theorem 2.8.7 (iii)] and obtain

$$\iota_{\overline{\operatorname{conv} D}^{\mathrm{w}^*}} \geq \sigma_F^* \Box \iota_{\operatorname{int} T_C(x)}^* = \iota_{\overline{\operatorname{conv} F}^{\mathrm{w}^*}} \Box \sigma_{T_C(x)} = \iota_{\overline{\operatorname{conv} F}^{\mathrm{w}^*}} \Box \iota_{N_C(x)} = \iota_{\overline{\operatorname{conv} F}^{\mathrm{w}^*} + N_C(x)}.$$

Thus,

$$\overline{\operatorname{conv} D}^{\mathrm{w}^*} \subseteq \overline{\operatorname{conv} F}^{\mathrm{w}^*} + N_C(x) \subseteq \overline{\operatorname{conv} F + N_C(x)}^{\mathrm{w}^*}. \tag{24}$$

We can now provide our final technical proposition.

Proposition 5.2 Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \operatorname{int} \operatorname{dom} A \neq \varnothing$ such that S is dense in $\operatorname{int} \operatorname{dom} A$. Assume that $x \in \operatorname{dom} A$ and $v \in H_{\overline{\operatorname{dom} A}}(x) = \operatorname{int} T_{\overline{\operatorname{dom} A}}(x)$. Then there exists $x_0^* \in A_S(x)$ such that

$$\sup \big\langle A_S(x), v \big\rangle = \big\langle x_0^*, v \big\rangle = \sup \big\langle Ax, v \big\rangle. \tag{25}$$

In particular, $\operatorname{dom} A_S = \operatorname{dom} A$.

Proof. By Corollary 4.2, $\operatorname{gra} A_S \subseteq \operatorname{gra} A$ and hence

$$\sup \big\langle A_S(x), v \big\rangle \leq \sup \big\langle Ax, v \big\rangle. \tag{26}$$

Now we show that

$$\sup \big\langle A_S(x), v \big\rangle \geq \sup \big\langle Ax, v \big\rangle. \tag{27}$$

Appealing now to Fact 3.6, we can and do suppose that $v = x_0 - x$, where $x_0 \in \operatorname{int} \overline{\operatorname{dom} A} = \operatorname{int} \operatorname{dom} A$ by Fact 3.4. Using Lemma 4.1 select $M, \delta > 0$ such that $x_0 + 2\delta B_X \subseteq \operatorname{dom} A$ and

$$\sup_{a \in [x_0 + \delta B_X,\, x[,\ a^* \in Aa} \|a^*\| \leq M < +\infty. \tag{28}$$

Let $t \in \,]0, 1[$. Then by Fact 3.4 again,

$$x + tB_\delta(v) = (1-t)x + tx_0 + t\delta B_X \subseteq \operatorname{int} \overline{\operatorname{dom} A} = \operatorname{int} \operatorname{dom} A. \tag{29}$$

Then by the monotonicity of A (for every $a^* \in A(x+tw)$, $x^* \in Ax, w \in B_\delta(v)$)

$$t\langle a^* - x^*, w\rangle = \langle a^* - x^*, x + tw - x\rangle \geq 0. \tag{30}$$

There exists a sequence $(x_n^*)_{n\in\mathbb{N}}$ in Ax such that

$$\langle x_n^*, v\rangle \longrightarrow \sup\langle Ax, v\rangle. \tag{31}$$

Combining (31) and (30), we have

$$\langle a^* - x_n^*, v + w - v\rangle \geq 0, \quad \forall a^* \in A(x+tw),\, w \in B_\delta(v),\, n \in \mathbb{N}. \tag{32}$$

Fix $1 < n \in \mathbb{N}$. Thus, appealing to (28) and (32) yields,

$$\begin{aligned}\langle a^*, v\rangle &\geq \langle x_n^*, v\rangle - \langle a^* - x_n^*, w - v\rangle \\ &\geq \langle x_n^*, v\rangle - (M + \|x_n^*\|)\cdot\|w - v\| \quad \forall a^* \in A(x+tw),\, w \in B_\delta(v),\, n \in \mathbb{N}. \end{aligned} \tag{33}$$

Take $\varepsilon_n := \min\{\frac{1}{n(M+\|x_n^*\|)}, \delta\}$ and $t_n := \frac{1}{n}$.

Since S is dense in $\operatorname{int}\operatorname{dom} A$ and $x + t_n B_{\varepsilon_n}(v) \subseteq \operatorname{int}\operatorname{dom} A$ by (29), $S \cap [x + t_n B_{\varepsilon_n}(v)] \neq \varnothing$. Then there exists $w_n \in X$ such that

$$w_n \in B_{\varepsilon_n}(v), \qquad x + t_n w_n \in S \quad \text{and then} \quad x + t_n w_n \longrightarrow x. \tag{34}$$

Hence, by (33),

$$\langle a^*, v\rangle \geq \langle x_n^*, v\rangle - \frac{1}{n}, \quad \forall a^* \in A(x + t_n w_n). \tag{35}$$

Let $a_n^* \in A(x + t_n w_n)$. Then by (35),

$$\langle a_n^*, v\rangle \geq \langle x_n^*, v\rangle - \frac{1}{n}. \tag{36}$$

By (28) and (29), $(a_n^*)_{n\in\mathbb{N}}$ is bounded. Then by Fact 3.1, there exists a weak* convergent subnet of $(a_\alpha^*)_{\alpha\in I}$ of $(a_n^*)_{n\in\mathbb{N}}$ such that

$$a_\alpha^* \rightharpoondown_{\mathrm{w}^*} x_0^* \in X^*. \tag{37}$$

Then by (34), $x_0^* \in A_S(x)$ and thus by (36), (37) and (31)

$$\sup \langle A_S(x), v\rangle \geq \langle x_0^*, v\rangle \geq \sup \langle Ax, v\rangle.$$

Hence (27) holds and so does (25) by (26). The last conclusion then follows from Corollary 4.2 directly. □

An easy consequence is the reconstruction of A on the interior of its domain. In the language of [5,7,28–30] this is asserting the minimality of A as a w*-*cusco.*

Corollary 5.1 Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \operatorname{int}\operatorname{dom} A \neq \varnothing$. For any S dense in $\operatorname{int}\operatorname{dom} A$, we have $\overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}^*} = Ax = A_{\mathrm{int}}(x), \forall x \in \operatorname{int}\operatorname{dom} A$.

Proof. Corollary 4.2 shows $\operatorname{gra} A_S \subseteq \operatorname{gra} A$. Thus A_S is monotone. By Proposition 5.2, $A_S(x) \neq \varnothing$ on $\operatorname{dom} A$. Then apply [7, Theorem 7.13 and Corollary 7.8] and Corollary 4.2 to obtain

$$\overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}^*} = Ax = A_{\mathrm{int}}(x), \quad \forall x \in \operatorname{int}\operatorname{dom} A,$$

as required. □

There are many possible extensions of this sort of result along the lines studied in [28]. Applying Proposition 5.2 and Lemma 4.1, we can also quickly recapture [31, Theorem 2.1].

Theorem 5.1 [Directional boundedness in Euclidean space] Suppose that X is finite-dimensional. Let $A : X \rightrightarrows X^*$ be maximally monotone and $x \in \operatorname{dom} A$.

Assume that there exist $d \in X$ and $\varepsilon_0 > 0$ such that $x + \varepsilon_0 d \in \operatorname{int} \operatorname{dom} A$. Then

$$[Ax]_d := \big\{x^* \in Ax \mid \langle x^*, d\rangle = \sup\langle Ax, d\rangle\big\}$$

is nonempty and compact. Moreover, if a sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} A$ is such that $x_n \longrightarrow x$ and

$$\lim \frac{x_n - x}{\|x_n - x\|} = d, \tag{38}$$

then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$A(x_n) \subseteq [Ax]_d + \varepsilon B_{X^*}, \quad \forall n \geq N. \tag{39}$$

Proof. By Fact 3.6, we have

$d = \frac{1}{\varepsilon_0}(x + \varepsilon_0 d - x) \in \frac{1}{\varepsilon_0}[\operatorname{int} \operatorname{dom} A - x] \subseteq \operatorname{int} T_{\overline{\operatorname{dom} A}}(x)$. Then by Proposition 5.2 and Corollary 4.2, there exists $v^* \in Ax$ such that

$$\sup\langle Ax, d\rangle = \langle v^*, d\rangle. \tag{40}$$

Hence $v^* \in [Ax]_d$ and thus $[Ax]_d \neq \varnothing$.

We next show that $[Ax]_d$ is compact. Let $x^* \in [Ax]_d$. By Fact 4.1, there exist $\delta > 0$ and $M > 0$ such that

$-\varepsilon_0\langle d, x^*\rangle = \langle x - (x + \varepsilon_0 d), x^*\rangle \geq \delta\|x^*\| - (\|\varepsilon_0 d\| + \delta)M$. Then by (40), $\delta\|x^*\| \leq (\|\varepsilon_0 d\| + \delta)M - \varepsilon_0\langle d, x^*\rangle = (\|\varepsilon_0 d\| + \delta)M - \varepsilon_0\langle d, v^*\rangle < +\infty$. Hence $[Ax]_d$ is bounded. Clearly, $[Ax]_d$ is closed and so $[Ax]_d$ is compact.

Finally, we show that (39) holds. By Lemma 4.1 and $x + \varepsilon_0 d \in \operatorname{int} \operatorname{dom} A$, there exists $\delta_1 > 0$ such that

$$\sup_{a \in [x + \varepsilon_0 d + \delta_1 B_X,\, x[,\ a^* \in Aa} \|a^*\| < +\infty. \tag{41}$$

By (38), we have $\|d\| = 1$ and there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$0 < \|x_n - x\| < \varepsilon_0$ and $x_n \in x + \|x_n - x\| d + \|x_n - x\| \frac{\delta_1}{\varepsilon_0} B_X \subseteq [x + \varepsilon_0 d + \delta_1 B_X, x[$.

Then by (41),

$$\sup_{a^* \in A(x_n),\, n \geq N} \|a^*\| < +\infty. \tag{42}$$

Suppose to the contrary that (39) does not holds. Then there exists $\varepsilon_1 > 0$ and a subsequence $(x_{n,k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and $x^*_{n,k} \in A(x_{n,k})$ such that

$$x^*_{n,k} \notin [Ax]_d + \varepsilon_1 B_{X^*}, \quad \forall k \in \mathbb{N}. \tag{43}$$

By (42), there exists a convergent subsequence of $(x^*_{n,k})_{k \in \mathbb{N}}$, for convenience, still denoted by $(x^*_{n,k})_{k \in \mathbb{N}}$ such that

$$x^*_{n,k} \longrightarrow x^*_\infty. \tag{44}$$

Since $x_{n,k} \longrightarrow x$, by (44),

$$(x, x^*_\infty) \in \operatorname{gra} A. \tag{45}$$

We claim that

$$x^*_\infty \in [Ax]_d\,. \tag{46}$$

By the monotonicity of A, recalling (40), we have $\langle x^*_{n,k} - v^*, x_{n,k} - x \rangle \geq 0$, $\forall k \in \mathbb{N}$. Hence

$$\langle x^*_{n,k} - v^*, \frac{x_{n,k} - x}{\|x_{n,k} - x\|} \rangle \geq 0, \quad \forall k \in \mathbb{N}. \tag{47}$$

Combining (44), (38) and (47),

$$\langle x^*_\infty - v^*, d \rangle \geq 0. \tag{48}$$

By (40), (48) and (45), $x^*_\infty \in [Ax]_d$ and hence (46) holds. Then $x^*_\infty + \varepsilon_1 B_X \subseteq [Ax]_d + \varepsilon_1 B_X$ and $x^*_\infty + \varepsilon_1 B_X$ contains infinitely many terms of $(x^*_{n,k})_{k\in\mathbb{N}}$, which contradicts (43). Hence, (39) holds as asserted. $\square$

Remark 5.2 In the statement of [31, Theorem 2.1], the "$x - x_n$" in Eq (2.0) should be read as "$x_n - x$". In his proof, the author considered it as "$x_n - x$". $\Diamond$

We next recall an alternate and well-known *recession cone* description of $N_{\operatorname{dom} A}$. (We give the proof for completeness and because it is often done in restricted settings.) Consider

$$\begin{aligned} &\operatorname{rec} A(x) \\ &:= \left\{x^* \in X^* \mid \exists t_n \to 0^+, (a_n, a_n^*) \in \operatorname{gra} A \,\text{such that}\, a_n \longrightarrow x,\, t_n a_n^* \rightharpoonup_{\mathrm{w}^*} x^*\right\}. \end{aligned} \tag{49}$$

Proposition 5.3 [Recession cone] Let $A : X \rightrightarrows X^*$ be maximally monotone. Then for every $x \in \operatorname{dom} A$ one has

$$N_{\overline{\operatorname{dom} A}}(x) = \operatorname{rec} A(x).$$

Proof. Let $x \in \operatorname{dom} A$. We first show that

$$\operatorname{rec} A(x) \subseteq N_{\overline{\operatorname{dom} A}}(x). \tag{50}$$

Let $x^* \in \operatorname{rec} A(x)$. There are $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(a_n, a_n^*)_{n\in\mathbb{N}}$ in $\operatorname{gra} A$ such that

$$t_n \longrightarrow 0^+,\, a_n \longrightarrow x \quad \text{and} \quad t_n a_n^* \rightharpoonup_{\mathrm{w}^*} x^*. \tag{51}$$

By [19, Corollary 2.6.10], $(t_n a_n^*)_{n\in\mathbb{N}}$ is bounded. By the monotonicity of A,

$$\langle a_n - a, a_n^*\rangle \geq \langle a_n - a, a^*\rangle, \quad \forall (a, a^*) \in \operatorname{gra} A.$$

Therefore,

$$\langle a_n - a, t_n a_n^* \rangle \geq t_n \langle a_n - a, a^* \rangle, \quad \forall (a, a^*) \in \operatorname{gra} A. \tag{52}$$

Taking the limit in (52), by (51), we have $\langle x - a, x^* \rangle \geq 0, \quad \forall a \in \operatorname{dom} A$. Thus, $x^* \in N_{\overline{\operatorname{dom} A}}(x)$. Hence (50) holds.

It remains to show that

$$N_{\overline{\operatorname{dom} A}}(x) \subseteq \operatorname{rec} A(x). \tag{53}$$

Let $y^* \in N_{\overline{\operatorname{dom} A}}(x)$ and $n \in \mathbb{N}$. Take $v^* \in Ax$. Since $A = N_{\overline{\operatorname{dom} A}} + A$, we have $ny^* + v^* \in Ax$. Set $a_n := x, a_n^* := ny^* + v^*$ and $t_n := \frac{1}{n}$. Then we have

$$a_n \longrightarrow x, t_n \longrightarrow 0^+ \quad \text{and} \quad t_n a_n^* = y^* + \frac{1}{n} v^* \longrightarrow y^*.$$

Hence $y^* \in \operatorname{rec} A(x)$ and then (53) holds. Combining (50) and (53), we have $N_{\overline{\operatorname{dom} A}}(x) = \operatorname{rec} A(x)$. □

We are now ready for our main result, Theorem 5.2, the proof of which was inspired partially by that of [32, Theorem 3.1].

Theorem 5.2 [Reconstruction of A, I] Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \operatorname{int} \operatorname{dom} A \neq \varnothing$ and with S dense in $\operatorname{int} \operatorname{dom} A$. Then

$$Ax = N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv} [A_S(x)]}^{\mathrm{w}^*} = \operatorname{rec} A(x) + \overline{\operatorname{conv} [A_S(x)]}^{\mathrm{w}^*}, \quad \forall x \in X, \tag{54}$$

where $\operatorname{rec} A(x)$ is as in (49).

Proof. We first show that

$$Ax = \overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv} [A_S(x)]}^{\mathrm{w}^*}, \quad \forall x \in X. \tag{55}$$

By Corollary 4.2, we have $\operatorname{conv}[A_S(x)] \subseteq Ax,\ \forall x \in X$. Since likewise $A = A + N_{\overline{\operatorname{dom} A}}$,

$$\overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv}[A_S(x)]}^{\mathrm{w}^*} \subseteq Ax, \quad \forall x \in X. \tag{56}$$

It remains show that

$$Ax \subseteq \overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv}[A_S(x)]}^{\mathrm{w}^*}, \quad \forall x \in \operatorname{dom} A. \tag{57}$$

Let $x \in \operatorname{dom} A$. By the maximal monotonicity of A and Proposition 5.2, both Ax and $A_S(x)$ are nonempty sets. Then applying Proposition 5.1 and Proposition 5.2 directly, we have (57) holds and hence (55) holds.

We must still show

$$Ax = N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}^*}, \quad \forall x \in X. \tag{58}$$

Now, for every two sets $C, D \subseteq X^*$, we have $C + \overline{D}^{\mathrm{w}^*} \subseteq \overline{C + D}^{\mathrm{w}^*}$. Then by (55), it suffices to show that

$$\overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv}[A_S(x)]}^{\mathrm{w}^*} \subseteq N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}^*}, \quad \forall x \in \operatorname{dom} A. \tag{59}$$

We again can and do suppose that $0 \in \operatorname{int} \operatorname{dom} A$ and $(0,0) \in \operatorname{gra} A$. Let $x \in \operatorname{dom} A$ and $x^* \in \overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv}[A_S(x)]}^{\mathrm{w}^*}$. Then there exists nets $(x^*_\alpha)_{\alpha \in I}$ in $N_{\overline{\operatorname{dom} A}}(x)$ and $(y^*_\alpha)_{\alpha \in I}$ in $\operatorname{conv}[A_S(x)]$ such that

$$x^*_\alpha + y^*_\alpha \rightharpoonup_{\mathrm{w}^*} x^*. \tag{60}$$

Now we claim that

$$(x^*_\alpha)_{\alpha \in I} \text{ is eventually bounded.} \tag{61}$$

Suppose to the contrary that $(x^*_\alpha)_{\alpha\in I}$ is not eventually bounded. Then there exists a subnet of $(x^*_\alpha)_{\alpha\in I}$, for convenience, still denoted by $(x^*_\alpha)_{\alpha\in I}$, such that

$$\lim_\alpha \|x^*_\alpha\| = +\infty. \tag{62}$$

We can and do suppose that $x^*_\alpha \neq 0, \forall \alpha \in I$. By $0 \in \operatorname{int}\operatorname{dom} A$ and $x^*_\alpha \in N_{\overline{\operatorname{dom} A}}(x)$ (for every $\alpha \in I$), there exists $\delta > 0$ such that $\delta B_X \subseteq \overline{\operatorname{dom} A}$ and hence we have

$$\langle x, x^*_\alpha\rangle \geq \sup_{b\in B_X} \langle x^*_\alpha, \delta b\rangle = \delta\|x^*_\alpha\|. \tag{63}$$

Thence, we have

$$\langle x, \frac{x^*_\alpha}{\|x^*_\alpha\|}\rangle \geq \delta. \tag{64}$$

By Fact 3.1, there exists a weak* convergent subnet $(x^*_\beta)_{\beta\in\Gamma}$ of $(x^*_\alpha)_{\alpha\in I}$, say

$$\frac{x^*_\beta}{\|x^*_\beta\|} \rightharpoonup_{\mathrm{w}^*} x^*_\infty \in X^*. \tag{65}$$

Taking the limit along the subnet in (64), by (65), we have

$$\langle x, x^*_\infty\rangle \geq \delta. \tag{66}$$

By (60) and (62), we have

$$\frac{x^*_\alpha}{\|x^*_\alpha\|} + \frac{y^*_\alpha}{\|x^*_\alpha\|} \rightharpoonup_{\mathrm{w}^*} 0. \tag{67}$$

And so by (65),

$$\frac{y^*_\beta}{\|x^*_\beta\|} \rightharpoonup_{\mathrm{w}^*} -x^*_\infty. \tag{68}$$

By Corollary 4.2, $\operatorname{conv}[A_S(x)] \subseteq Ax$, and hence $(y^*_\alpha)_{\alpha\in I}$ is in Ax. Since $(0,0) \in \operatorname{gra} A$, we have $\langle y^*_\alpha, x\rangle \geq 0$ and so

$$\Big\langle \frac{y^*_\beta}{\|x^*_\beta\|}, x\Big\rangle \geq 0. \tag{69}$$

Using (68) and taking the limit along the subnet in (69) we get

$$\big\langle -x^*_\infty, x\big\rangle \geq 0, \tag{70}$$

which contradicts (66). Hence, $(x^*_\alpha)_{\alpha\in I}$ is eventually bounded and thus (61) holds.

Then by Fact 3.1 again, there exists a weak* convergent subset of $(x^*_\alpha)_{\alpha\in I}$, for convenience, still denoted by $(x^*_\alpha)_{\alpha\in I}$ which lies in the normal cone, such that $x^*_\alpha \rightharpoondown_{\mathrm{w}^*} w^* \in X^*$. Hence $w^* \in N_{\overline{\operatorname{dom} A}}(x)$ and $y^*_\alpha \rightharpoondown_{\mathrm{w}^*} x^* - w^* \in \overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}^*}$ by (60). Hence $x^* \in N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}^*}$ so that (59) holds. Then we apply Proposition 5.3 to get (54) directly. $\square$

Remark 5.3 Using (24), Dr. Robert Csetnek kindly showed us an elegant proof of Theorem 5.2:

Indeed, (by Proposition 5.2) we have now

$$Ax \subseteq N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}^*} \subseteq \overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv}[A_S(x)]}^{\mathrm{w}^*} \subseteq Ax,$$

where the last inclusion follows from (56); hence (58) holds.

Remark 5.4 If X is a *weak Asplund space* (as holds if X has a Gâteaux smooth equivalent norm, see [7,28,30]), the nets defined in A_S in Proposition 5.2 and Theorem 5.2 can be replaced by sequences. By [33, Chap. XIII, Notes and Remarks, page 239], B_{X^*} is weak* sequentially compact. In fact, see [5, Chpt. 9], this holds somewhat more generally.

Hence, throughout the proof of Proposition 5.2, we can obtain weak* convergent subsequences instead of subnets. The rest of each subsequent argument is unchanged. $\diamondsuit$

In various classes of Banach space we can choose useful structure for $S \in S_A$, where

$$S_A := \{S \subseteq \operatorname{int} \operatorname{dom} A \mid S \text{ is dense in } \operatorname{int} \operatorname{dom} A\}.$$

Corollary 5.2 [Specification of S_A] Let $A : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{int} \operatorname{dom} A \neq \varnothing$. We may choose the dense set $S \in S_A$ to be as follows:

1. In a Gâteaux smooth space, entirely within the residual set of non-σ porous points of $\operatorname{dom} A$,
2. In an Asplund space, to include only a subset of the generic set points of single-valuedness and norm to norm continuity of A,
3. In a separable Asplund space, to hold only countably many angle-bounded points of A,
4. In a weak Asplund space, to include only a subset of the generic set of points of single-valuedness (and norm to weak* continuity) of A,
5. In a separable space, to include only points of single-valuedness (and norm to weak* continuity) of A whose complement is covered by a countable union of Lipschitz surfaces.
6. In finite dimensions, to include only points of differentiability of A which are of full measure.

Proof. It suffices to determine in each case that the points of the given kind are dense. 1: See [34, Theorem 5.1]. 2: See [7, Lemma 2.18 and Theorem 2.30]. 3: See [7, Theorem 2.19 and Theorem 2.11]. 4: See [30, Proposition 1.1(iii) and Theorem 1.6] or [7, Theorem 4.31 and Example 7.2]. 5: See [35, 36]. 6: See [9, Corollary 12.66(a)] or [5, Exercise 9.1.1(2), page 412]. □

These classes are sufficient but not necessary: for example, there are Asplund spaces with no equivalent Gâteaux smooth renorm [5]. Note also that in 5 and 6 we also know that $A \setminus S$ is a null set in the senses discussed [37].

We now restrict attention to convex functions.

Corollary 5.3 [Convex subgradients] Let $f : X \to \,]-\infty, +\infty]$ be proper lower semicontinuous and convex with $\operatorname{int}\operatorname{dom} f \neq \varnothing$. Let $S \subseteq \operatorname{int}\operatorname{dom} f$ be given with S dense in $\operatorname{dom} f$. Then (for every $x \in X$)

$$\partial f(x) = N_{\overline{\operatorname{dom} f}}(x) + \overline{\operatorname{conv}\left[(\partial f)_S(x)\right]}^{\mathrm{w}^*} = N_{\operatorname{dom} f}(x) + \overline{\operatorname{conv}\left[(\partial f)_S(x)\right]}^{\mathrm{w}^*}.$$

Proof. By [7, Proposition 3.3 and Proposition 1.11], $\operatorname{int}\operatorname{dom}\partial f \neq \varnothing$. By the Brøndsted-Rockafellar Theorem (see [7, Theorem 3.17] or [12, Theorem 3.1.2]), $\overline{\operatorname{dom}\partial f} = \overline{\operatorname{dom} f}$. Then we may apply Fact 3.2 and Theorem 5.2 to get (for every $x \in X$) $\partial f(x) = N_{\overline{\operatorname{dom} f}}(x) + \overline{\operatorname{conv}\left[(\partial f)_S(x)\right]}^{\mathrm{w}^*}$. We have
$N_{\overline{\operatorname{dom} f}}(x) = N_{\operatorname{dom} f}(x), \forall x \in \operatorname{dom}\partial f$. Hence we have
$\partial f(x) = N_{\operatorname{dom} f}(x) + \overline{\operatorname{conv}\left[(\partial f)_S(x)\right]}^{\mathrm{w}^*}, \forall x \in X.$ □

In this case Corollary 5.2 specifies settings in which only points of differentiability need be used (in 6 we recover Alexandroff's theorem on twice differentiability of convex functions), see [5] for more details.

Remark 5.5 Results closely related to Corollary 5.3 have been obtained in [8,27,38,39] and elsewhere. Interestingly, in the convex case we have obtained as much information more easily than by the direct convex analysis approach of [27]. $\Diamond$

We finish this section by refining Corollary 5.1 and Theorem 5.2.

Let $A : X \rightrightarrows X^*$. We define $\widehat{A} : X \rightrightarrows X^*$ by

$$\operatorname{gra} \widehat{A} := \Big\{(x, x^*) \in X \times X^* \mid x^* \in \bigcap_{\varepsilon>0} \overline{\operatorname{conv}\left[A(x+\varepsilon B_X)\right]}^{\mathrm{w}^*}\Big\}. \tag{71}$$

Clearly, we have $\overline{\operatorname{gra} A}^{\|\cdot\|\times \mathrm{w}^*} \subseteq \operatorname{gra} \widehat{A}$.

Theorem 5.3 [Reconstruction of A, II] Let $A : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$.

1. Then $\widehat{A} = A$.

   In particular, A has property (Q); and so has a norm $\times$ weak* closed graph.

2. Moreover, if $S \subseteq \operatorname{int}\operatorname{dom} A$ is dense in $\operatorname{int}\operatorname{dom} A$ then (for every $x \in X$)

$$\widehat{A_S}(x) := \bigcap_{\varepsilon>0} \overline{\operatorname{conv}\left[A(S \cap (x+\varepsilon B_X))\right]}^{\mathrm{w}^*} \supseteq \overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}^*}. \tag{72}$$

   Thence

$$Ax = \widehat{A_S}(x) + \operatorname{rec} A(x), \quad \forall x \in X. \tag{73}$$

Proof. *Part 1.* We first show that $\operatorname{gra} \widehat{A} \subseteq \operatorname{gra} A$. Let $(x, x^*) \in \operatorname{gra} \widehat{A}$. Now we show that $x \in \operatorname{dom} A$. We suppose that $0 \in \operatorname{int}\operatorname{dom} A$. Since

$x^* \in \overline{\operatorname{conv}\left[A(x+\frac{1}{n}B_X)\right]}^{\mathrm{w}^*}$ (for all $n \in \mathbb{N}$),

$$\inf \left\langle A(x+\tfrac{1}{n}B_X), x\right\rangle = \inf \left\langle \operatorname{conv}\left[A(x+\tfrac{1}{n}B_X)\right], x\right\rangle$$
$$= \inf \left\langle \overline{\operatorname{conv}\left[A(x+\tfrac{1}{n}B_X)\right]}^{\mathrm{w}^*}, x\right\rangle < \left\langle x, x^*\right\rangle + 1.$$

Then there exists $z_n^* \in A(z_n)$ such that

$$\langle z_n^*, x\rangle \leq \langle x^*, x\rangle + 1, \tag{74}$$

where $z_n \in x + \frac{1}{n}B_X$. By Fact 4.1, there exist $\delta_0 > 0$ and $M_0 > 0$ such that

$$\delta_0\|z_n^*\| \leq \langle z_n, z_n^*\rangle + (\|z_n\| + \delta)M_0 = \langle z_n - x, z_n^*\rangle + \langle x, z_n^*\rangle + (\|z_n\| + \delta)M_0$$
$$\leq \frac{1}{n}\|z_n^*\| + \langle x^*, x\rangle + 1 + (\|x\| + 1 + \delta)M_0, \quad \forall n \in \mathbb{N} \quad \text{(by (74))}.$$

Hence $(z_n^*)_{n\in\mathbb{N}}$ is bounded. By Fact 3.1, there exists a weak* convergent limit z_∞^* of a subnet of $(z_n^*)_{n\in\mathbb{N}}$. Then $z_n \longrightarrow x$ and the maximal monotonicity of A, imply that $(x, z_\infty^*) \in \operatorname{gra} A$ and so $x \in \operatorname{dom} A$.

Now let $v \in \operatorname{int} T_{\overline{\operatorname{dom} A}}(x)$. We claim that

$$\sup \left\langle \widehat{A}(x), v\right\rangle \leq \sup \left\langle Ax, v\right\rangle. \tag{75}$$

By Fact 3.6, we can and do suppose that $v = x_0 - x$,

where $x_0 \in \operatorname{int}\overline{\operatorname{dom} A} = \operatorname{int}\operatorname{dom} A$ by Fact 3.4. There exists a sequence $(y_n^*)_{n\in\mathbb{N}}$ in $\widehat{A}x$ such that

$$\langle y_n^*, v\rangle \longrightarrow \sup\langle \widehat{A}x, v\rangle. \tag{76}$$

Using Lemma 4.1 select $M, \delta > 0$ such that $x_0 + 2\delta B_X \subseteq \operatorname{dom} A$ and

$$\sup_{a\in[x_0+\delta B_X,\, x[,\, a^*\in Aa} \|a^*\| \leq M < +\infty. \tag{77}$$

Then by Fact 3.4 again,

$$[x_0 + \delta B_X,\, x[\subseteq \operatorname{int} \overline{\operatorname{dom} A} = \operatorname{int} \operatorname{dom} A. \tag{78}$$

Fix $\frac{1}{\delta} < n \in \mathbb{N}$. Since $y_n^* \in \overline{\operatorname{conv}\left[A(x + \frac{1}{n}B_X)\right]}^{\mathrm{w}^*}$, then $\left\langle y_n^*, v\right\rangle \leq \sup\left\langle A(x + \frac{1}{n}B_X), v\right\rangle$. Then there exist $x_n \in (x + \frac{1}{n}B_X)$ and $x_n^* \in A(x_n)$ such that

$$\langle x_n^*, v\rangle \geq \langle y_n^*, v\rangle - \frac{1}{n}. \tag{79}$$

Set $t_n := \frac{1}{\delta\, n}$. Then,

$$\begin{aligned} a_n &: = x_n + t_n v = x_n - x + x + t_n(x_0 - x) = t_n\left(x_0 + \frac{x_n - x}{t_n}\right) + (1 - t_n)x \\ &\in t_n(x_0 + \delta B_X) + (1 - t_n)x. \end{aligned} \tag{80}$$

Select $a_n^* \in A(a_n)$ by (78). Then by the monotonicity of A, $t_n\langle a_n^* - x_n^*, v\rangle = \langle a_n^* - x_n^*, a_n - x_n\rangle \geq 0$. Hence $\langle a_n^*, v\rangle \geq \langle x_n^*, v\rangle$. Using (79), we have

$$\langle a_n^*, v\rangle \geq \langle y_n^*, v\rangle - \frac{1}{n}, \quad \forall \tfrac{1}{\delta} < n \in \mathbb{N}. \tag{81}$$

Thus, appealing to (77) and (80) shows that $(a_n^*)_{n\in\mathbb{N}}$ is bounded. Fact 3.1, now yields a weak* convergent subnet of $(a_\alpha^*)_{\alpha\in I}$ of $(a_n^*)_{n\in\mathbb{N}}$ such that

$$a_\alpha^* \rightharpoonup_{\mathrm{w}^*} x_0^* \in X^*. \tag{82}$$

By Corollary 4.1 and $a_n \longrightarrow x$, we have $x_0^* \in Ax$. Combining (81), (82) and (76), we obtain

$$\sup\left\langle Ax, v\right\rangle \geq \left\langle x_0^*, v\right\rangle \geq \sup\left\langle \widehat{A}x, v\right\rangle.$$

Hence (75) holds. Now applying Proposition 5.1 and Proposition 5.2, we have $\widehat{A}x \subseteq \overline{Ax + N_{\overline{\operatorname{dom} A}}(x)}^{\mathrm{w}^*} = Ax$. Hence $\operatorname{gra} \widehat{A} \subseteq \operatorname{gra} A$.

Since $\operatorname{gra} A \subseteq \operatorname{gra} \widehat{A}$, we have $\widehat{A} = A$. It is immediate A has property (Q) so has a norm $\times$ weak* closed graph.

Part 2. It only remains to prove (72). We first show that

$$A_S(x) \subseteq \widehat{A_S}(x), \quad \forall x \in X. \tag{83}$$

By Proposition 5.2, $\operatorname{dom} A_S = \operatorname{dom} A$. Let $w \in X$. If $w \notin \operatorname{dom} A$, then clearly, $A_S(w) \subseteq \widehat{A_S}(w)$. Assume that $w \in \operatorname{dom} A$ and $w^* \in A_S(w)$. Then by (3), there exist a net $(w_\alpha, w^*_\alpha)_{\alpha \in I}$ in $\operatorname{gra} A \cap (S \times X^*)$ such that $w_\alpha \longrightarrow w$ and $w^*_\alpha \rightharpoonup_{\mathrm{w}^*} w^*$. The for every $\varepsilon > 0$, there exists $\alpha_0 \in I$ such that $w_\alpha \in w + \varepsilon B_X, \quad \forall \alpha \succeq_I \alpha_0$. Thus

$$w_\alpha \in S \cap (w + \varepsilon B_X) \quad \text{and then} \quad w^*_\alpha \in A\big(S \cap (w + \varepsilon B_X)\big), \quad \forall \alpha \succeq_I \alpha_0.$$

Hence $w^* \in \overline{A\big(S \cap (w + \varepsilon B_X)\big)}^{\mathrm{w}^*} \subseteq \overline{\operatorname{conv}\big[A\big(S \cap (w + \varepsilon B_X)\big)\big]}^{\mathrm{w}^*}$ and thus (83) holds.

By (83), we have

$$\overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}^*} \subseteq \widehat{A_S}(x), \quad \forall x \in X. \tag{84}$$

Then by Proposition 5.3,

$$\overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}^*} + \operatorname{rec} A(x) \subseteq \widehat{A_S}(x) + \operatorname{rec} A(x) \subseteq Ax + \operatorname{rec} A(x) = Ax, \quad \forall x \in X.$$

Thus, on appealing to Theorem 5.2, we obtain (73). $\square$

Remark 5.6 Property (Q) first introduced by Cesari in Euclidean space, was recently established for maximally monotone operators with nonempty domain interior in a barreled normed space by Voisei in [15, Theorem 42] (See also [15, Theorem 43] for the result under more general hypotheses.). Several interesting characterizations of maximally monotone operators in finite dimensional spaces, including the property (Q) were studied by Löne [40]. $\Diamond$

6 Further Examples and Applications

In general, we do not have $Ax = \overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}^*}, \forall x \in \operatorname{dom} A$, for a maximally monotone operator $A : X \rightrightarrows X^*$ with $S \subseteq \operatorname{int}\operatorname{dom} A \neq \varnothing$ such that S is dense in $\operatorname{dom} A$.

We give a simple example to demonstrate this.

Example 6.1 Let C be a closed convex subset of X with $S \subseteq \operatorname{int} C \neq \varnothing$ such that S is dense in C. Then N_C is maximally monotone and $\operatorname{gra}(N_C)_S = C \times \{0\}$, but $N_C(x) \neq \overline{\operatorname{conv}[(N_C)_S(x)]}^{\mathrm{w}^*}, \forall x \in \operatorname{bdry} C$. We have

$\bigcap_{\varepsilon>0} \overline{\operatorname{conv}[N_C(x+\varepsilon B_X)]}^{\mathrm{w}^*} = N_C(x), \forall x \in X.$

Proof. The maximal monotonicity of N_C is directly from Fact 3.2. Since, for every $x \in \operatorname{int} C$, $N_C(x) = \{0\}$, $\operatorname{gra}(N_C)_S = C \times \{0\}$ by (3) and Proposition 5.2. Hence $\overline{\operatorname{conv}[(N_C)_S(x)]}^{\mathrm{w}^*} = \{0\}, \forall x \in C$. However, $N_C(x)$ is unbounded, $\forall x \in \operatorname{bdry} C$. Hence $N_C(x) \neq \overline{\operatorname{conv}[(N_C)_S(x)]}^{\mathrm{w}^*}, \forall x \in \operatorname{bdry} C$.

By contrast, on applying Theorem 5.3, we have

$\bigcap_{\varepsilon>0} \overline{\operatorname{conv}[N_C(x+\varepsilon B_X)]}^{\mathrm{w}^*} = N_C(x), \forall x \in X.$ $\Box$

While the subdifferential operators in Example 4.1 necessarily fail to have property (Q), it is possible for operators with no points of continuity to possess the property. Considering any closed linear mapping A from a reflexive space X to its dual, we have $\widehat{A} = A$ and hence A has property (Q). More generally:

Example 6.2 Suppose that X is reflexive. Let $A : X \rightrightarrows X^*$ be such that $\operatorname{gra} A$ is nonempty closed and convex. Then $\widehat{A} = A$ and hence A has property (Q).

Proof. It suffices to show that $\operatorname{gra} \widehat{A} \subseteq \operatorname{gra} A$. Let $(x, x^*) \in \operatorname{gra} \widehat{A}$. Then we have

$$x^* \in \bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}\left[A(x + \frac{1}{n}B_X)\right]}^{\mathrm{w}^*} = \bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}\left[A(x + \frac{1}{n}B_X)\right]} = \bigcap_{n\in\mathbb{N}} \overline{A(x + \frac{1}{n}B_X)}.$$

Then there exists a sequence $(a_n, a_n^*)_{n\in\mathbb{N}}$ in $\operatorname{gra} A$ such that $a_n \longrightarrow x, a_n^* \longrightarrow x^*$. The closedness of $\operatorname{gra} A$ implies that $(x, x^*) \in \operatorname{gra} A$. Then $\operatorname{gra} \widehat{A} \subseteq \operatorname{gra} A$. $\square$

It would be interesting to know whether $\widehat{A}$ and A can differ for a maximal operator with norm $\times$ weak* closed graph.

Finally, we illustrate what Corollary 5.3 says in the case of $x \mapsto \iota_{B_X}(x) + \frac{1}{p}\|x\|^p$.

Example 6.3 Let $p > 1$ and $f : X \to \left]-\infty, +\infty\right]$ be defined by

$$x \mapsto \iota_{B_X}(x) + \frac{1}{p}\|x\|^p.$$

Then for every $x \in \operatorname{dom} f$, we have

$$N_{\operatorname{dom} f}(x) = \begin{cases} \mathbb{R}_+ \cdot Jx, & \text{if } \|x\| = 1; \\ \{0\}, & \text{if } \|x\| < 1 \end{cases} \tag{85}$$

$$(\partial f)_{\mathrm{int}}(x) = \begin{cases} \|x\|^{p-2} \cdot Jx, & \text{if } \|x\| \neq 0; \\ \{0\}, & \text{otherwise} \end{cases} \tag{86}$$

where $J := \partial\frac{1}{2}\|\cdot\|^2$ and $\mathbb{R}_+ := [0, +\infty[$. Moreover,

$\partial f = N_{\operatorname{dom} f} + (\partial f)_{\text{int}} = N_{\operatorname{dom} f} + \partial\frac{1}{p}\|\cdot\|^p$, and then

$\partial f(x) \neq (\partial f)_{\text{int}}(x) = \overline{\operatorname{conv}\left[(\partial f)_{\text{int}}(x)\right]}^{\mathrm{w}^*}, \forall x \in \operatorname{bdry}\operatorname{dom} f$. We also have

$\bigcap_{\varepsilon>0} \overline{\operatorname{conv}\left[\partial f(x+\varepsilon B_X)\right]}^{\mathrm{w}^*} = \partial f(x)$, $\forall x \in X$.

Proof. By Fact 3.2, ∂f is maximally monotone. We have

$$\partial f = \partial\frac{1}{p}\|\cdot\|^p, \quad \forall x \in \operatorname{int}\operatorname{dom}\partial f. \tag{87}$$

By [27, Lemma 6.2],

$$\partial\frac{1}{p}\|\cdot\|^p(x) = \begin{cases} \|x\|^{p-2}\cdot Jx, & \text{if } \|x\| \neq 0; \\ \{0\}, & \text{otherwise.} \end{cases} \tag{88}$$

Now we show that

$$(\partial f)_{\text{int}}(x) = \partial\frac{1}{p}\|\cdot\|^p(x), \quad \forall x \in \operatorname{dom} f. \tag{89}$$

Let $x \in \operatorname{dom} f$. By Corollary 4.1 and (87), we have

$$(\partial f)_{\text{int}}(x) \subseteq \partial\frac{1}{p}\|\cdot\|^p(x). \tag{90}$$

Let $x^* \in \partial\frac{1}{p}\|\cdot\|^p(x)$. We first show that $(x, x^*) \in \operatorname{gra}(\partial f)_{\text{int}}$. If $\|x\| < 1$, then $x \in \operatorname{int}\operatorname{dom} f$ and hence by (87) and Corollary 4.2, $x^* \in \partial f(x) = (\partial f)_{\text{int}}(x)$. Now we suppose that $\|x\| = 1$. By (88), $x^* \in Jx$. Then $\frac{n-1}{n}x^* \in J(\frac{n-1}{n}x)$ and hence $(\frac{n-1}{n})^{p-1}x^* \in \partial\frac{1}{p}\|\cdot\|^p(\frac{n-1}{n}x)$ by (88), $\forall n \in \mathbb{N}$. By (87),

$$(\tfrac{n-1}{n})^{p-1}x^* \in \partial f(\tfrac{n-1}{n}x), \quad \forall n \in \mathbb{N}. \tag{91}$$

Since $0 \in \operatorname{int}\operatorname{dom} f$, $\frac{n-1}{n}x \in \operatorname{int}\operatorname{dom} f = \operatorname{int}\operatorname{dom}\partial f, \forall n \in \mathbb{N}$. Since $\frac{n-1}{n}x \longrightarrow x, (\frac{n-1}{n})^{p-1}x^* \longrightarrow x^*$, by (91), $x^* \in (\partial f)_{\text{int}}(x)$. Hence

$\partial\frac{1}{p}\|\cdot\|^p(x) \subseteq (\partial f)_{\text{int}}(x)$. Thus by (90), we have (89) holds and then we obtain (86) by (88).

By (89),

$$(\partial f)_{\text{int}}(x) = \overline{\text{conv}\,[(\partial f)_{\text{int}}(x)]}^{\text{w}^*}, \quad \forall x \in \text{dom}\, f. \tag{92}$$

On the other hand, since $N_{\text{dom}\, f} = N_{B_X}$, we can immediately get (85).

Then by Corollary 5.3, (92) and (89), we have

$$\partial f(x) = N_{\text{dom}\, f}(x) + (\partial f)_{\text{int}}(x) = N_{\text{dom}\, f}(x) + \partial \frac{1}{p}\|\cdot\|^p(x), \quad \forall x \in X. \tag{93}$$

Let $x \in \text{bdry}\,\text{dom}\, f$. Then $\|x\| = 1$. On combining (93), (85) and (86),

$$\partial f(x) = [1, +\infty[\,\cdot Jx \neq Jx = (\partial f)_{\text{int}}(x) = \overline{\text{conv}\,[(\partial f)_{\text{int}}(x)]}^{\text{w}^*}.$$

Theorem 5.3 again implies that

$\bigcap_{\varepsilon>0} \overline{\text{conv}\,[\partial f(x + \varepsilon B_X)]}^{\text{w}^*} = \partial f(x), \forall x \in X.$ □

7 Concluding Remarks

We have provided explicit structure formulas for maximally monotone operators in Banach space whose domains have nonempty interior (see Theorem 5.2 and Theorem 5.3). In the process, we also gave new proofs of some results of Voisei and one due to Auslender.

The results herein reinforces the need for answers to the three questions listed below.

- How does one give characterizations of the structure of maximally monotone operators with no interior point. The article [28]) treats various cases—for both subgradients and monotone operators—where the domain while having empty interior is large in category. It might be possible to extend and make more uniform the results therein.

- How does one refine the recession cone component in our main results so as to better generalize the use of *horizon* subgradients used in nonsmooth analysis (see, for example, [9])? That is, to represent any member of the recession cone as a limit of scaled multiples of nearby elements of the range of the operator.

- In [41], Veselý shows among other results that: The domain of the subdifferential operator for a closed convex function is arcwise and locally arcwise connected. When the space has a Fréchet renorm, and the function is not affine, then the range of the subdifferential is locally pathwise connected.

  This naturally raises this question: Can such results be extended to the domain of some or all maximally monotone operators? The difficulty here would appear to be in determining how to exploit some variant of the Fitzpatrick function—to replace the use of the sum of the function and its conjugate. More generally, what can be said topologically about the domain of a maximally monotone operator?

As discussed in [1–3, 5], the two most central open questions in monotone operator theory in a general real Banach space are almost certainly the following:

(i) Assume that two maximally monotone operators S, T satisfy *Rockafellar's* *constraint qualification*, i.e., the domain of one operator meets the interior domain of another operator [42]. *Is the sum operator $S + T$ necessarily maximally monotone*?

(ii) *Is the closure of every maximally monotone operator necessarily convex?* Rockafellar showed that the answer is 'yes' for every operator that has nonempty interior domain [22] and it is now known to hold for most classes of maximally monotone operators.

A positive answer to various restricted versions of (i) implies a positive answer to (ii) [5, 11]. See Simons' monograph [11] and [1–3, 5, 25, 32, 43] for recent developments of (i). Recent progress regarding (ii) can be found in [44].

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