Structure theory for maximally monotone operators with points of continuity

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Outline

Bauschke, Borwein and Combettes provided a new explicit construction for the subdifferential operator ∂f as follows:

For every $x \in X$,

$$\partial f(x) = \overline{N_{\operatorname{dom} f}(x) + \overline{\operatorname{conv}\left[(\partial f)_{\operatorname{int}}(x)\right]}^{\mathrm{w*}}}^{\mathrm{w*}},$$

where

- $\operatorname{dom} f$ is the *domain of f*;
- $N_{\operatorname{dom} f}$ is the *normal cone operator of* $\operatorname{dom} f$;
- $(\partial f)_{\operatorname{int}}$ is the operator whose graph is the norm-weak* closure of $\operatorname{gra} \partial f \cap (\operatorname{int} \operatorname{dom} f \times X^*)$.

We now extend it into every maximally monotone operator.

Notation and definitions

Throughout this talk,

- X is a general real Banach space, with continuous dual X^*, with the pairing $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.
- Let $A : X \rightrightarrows X^*$. The *graph* of A, $\operatorname{gra} A := \{(x, x^*) \mid x^* \in Ax\}$.
- $\operatorname{dom} A := \{x \in X \mid Ax \neq \varnothing\}$ and $\operatorname{ran} A := A(X)$.
- We say a net $(a_\alpha)_{\alpha \in \Gamma}$ in X is *eventually bounded* if there exist $\alpha_0 \in \Gamma$ and $M \geq 0$ such that

$$\boxed{\|a_\alpha\| \leq M, \quad \forall \alpha \succeq_\Gamma \alpha_0.}$$

- The *closed unit ball* in X is $B_X := \big\{x \in X \mid \|x\| \leq 1\big\}$, and $B_\delta(x) := x + \delta B_X$.

- $A : X \rightrightarrows X^*$ is *monotone* $\Leftrightarrow \langle x^* - y^*, x - y\rangle \geq 0$, whenever $(x, x^*), (y, y^*) \in \operatorname{gra} A$.
- We say $(x, x^*) \in X \times X^*$ is *monotonically related to* $\operatorname{gra} A$ if

$$\boxed{\langle x - y, x^* - y^*\rangle \geq 0, \quad \forall (y, y^*) \in \operatorname{gra} A.}$$

- A monotone mapping $A : X \rightrightarrows X^*$ is *maximally monotone* if no proper enlargement of A is monotone.

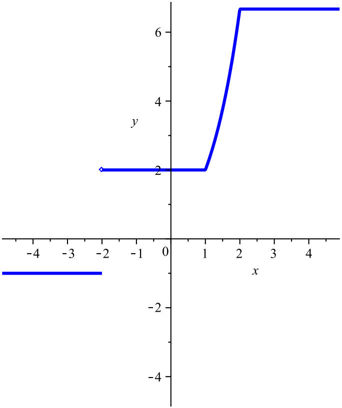

Figure: The graph of a monotone operator

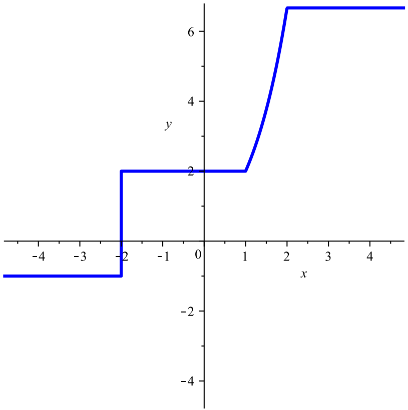

Figure: The graph of a maximally monotone operator

Notation and definitions

- f is *convex* $\Leftrightarrow f\big((1-\lambda)x+\lambda y\big) \le (1-\lambda)f(x)+\lambda f(y), \quad \lambda \in]0,1[.$
- Let $C \subseteq X$. The *interior* of C is $\operatorname{int} C$ and $\overline{C}$ is the *norm closure* of C.
- The *convex hull* of C is $\operatorname{conv} C$.
- For the set $D \subseteq X^*$, $\overline{D}^{\mathrm{w}*}$ is the *weak* closure* of D, and the *norm $\times$ weak* closure* of $C \times D$ is $\overline{C \times D}^{\|\cdot\| \times \mathrm{w}*}$.
- The *indicator function* ι_C is defined by

$$\iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

- *Subdifferential operator* $\partial f : X \rightrightarrows X^*$ via

$$\boxed{x^* \in \partial f(x) \quad \Leftrightarrow \quad (\forall y \in X) \;\; f(x) + \langle y - x, x^* \rangle \le f(y).}$$

- The *normal cone operator* of C , $N_C := \partial \iota_C$, The *tangent cone operator* of C is T_C.
- The *duality map on X*, $J := \partial \frac{1}{2}\|\cdot\|^2$.
- Let A be such that $\operatorname{dom} A \neq \varnothing$ and consider a set $S \subseteq \operatorname{dom} A$. We define $A_S : X \rightrightarrows X^*$ by

$$\begin{aligned}
\operatorname{gra} A_S &:= \overline{\operatorname{gra} A \cap (S \times X^*)}^{\|\cdot\| \times \mathrm{w}^*} \\
&= \Big\{ (x, x^*) \mid \exists \text{ a net } (x_\alpha, x_\alpha^*)_{\alpha \in \Gamma} \text{ in } \operatorname{gra} A \cap (S \times X^*) \\
&\qquad \text{such that } x_\alpha \longrightarrow x, x_\alpha^* \rightharpoonup_{\mathrm{w}^*} x^* \Big\}.
\end{aligned}$$

Set $A_{\mathrm{int}} := A_{\operatorname{int} \operatorname{dom} A}$. We note that

$$\operatorname{gra} A_{\operatorname{dom} A} = \overline{\operatorname{gra} A}^{\|\cdot\| \times \mathrm{w}^*} \supseteq \operatorname{gra} A.$$

Fact 1. (Banach–Alaoglu, 1932)
The closed unit ball B_{X^*} in X^* is weak* compact.

Fact 2. (Rockafellar, 1970)
Let $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then ∂f is maximally monotone.

Fact 3. (Rockafellar, 1969)
Let $A : X \rightrightarrows X^*$ be monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then A is locally bounded at $x \in \operatorname{int}\operatorname{dom} A$, i.e., there exist $\delta > 0$ and $K > 0$ such that

$$\sup_{y^* \in Ay} \|y^*\| \leq K, \quad \forall y \in (x + \delta B_X) \cap \operatorname{dom} A.$$

Fact 4. (Rockafellar, 1969)
Let $A : X \rightrightarrows X^*$ be maximal monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then $\operatorname{int}\operatorname{dom} A = \operatorname{int}\overline{\operatorname{dom} A}$ and $\overline{\operatorname{dom} A}$ is convex.

Fact 5.
Let $A : X \rightrightarrows X^*$ be monotone and $x \in \operatorname{int}\operatorname{dom} A$. Then there exist $\delta > 0$ and $M > 0$ such that $x + \delta B_X \subseteq \operatorname{dom} A$ and $\sup_{a \in x+\delta B_X} \|Aa\| \leq M$. Assume that (z, z^*) is monotonically related to $\operatorname{gra} A$. Then

$$\langle z - x, z^* \rangle \geq \delta\|z^*\| - (\|z - x\| + \delta)M.$$

Lemma 1. [Strong directional boundedness]
Let $A : X \rightrightarrows X^*$ be monotone and $x \in \operatorname{int}\operatorname{dom} A$. Then there exist $\delta > 0$ and $M > 0$ such that $x + 2\delta B_X \subseteq \operatorname{dom} A$ and $\sup_{a \in x+2\delta B_X} \|Aa\| \leq M$. Assume also that (x_0, x_0^*) is monotonically related to $\operatorname{gra} A$. Then

$$\sup_{a \in [x+\delta B_X, x_0[,\, a^* \in Aa} \|a^*\| \leq \frac{1}{\delta}\left(\|x_0 - x\| + 1\right)\left(\|x_0^*\| + 2M\right),$$

where $[x + \delta B_X, x_0[:= \big\{(1-t)y + tx_0 \mid 0 \leq t < 1, y \in x + \delta B_X\big\}$.

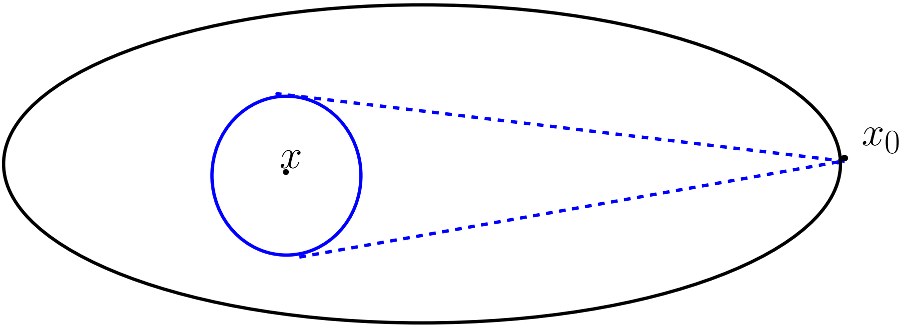

Figure: Strong directional boundedness

Theorem 1. [Voisei] Let $A : X \rightrightarrows X^*$ be monotone such that $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then every norm $\times$ weak* convergent net in $\operatorname{gra} A$ is eventually bounded.

Proof. We can and do suppose that $0 \in \operatorname{int}\operatorname{dom} A$. Let $(a_\alpha, a_\alpha^*)_{\alpha\in\Gamma}$ in $\operatorname{gra} A$ be such that

$$(a_\alpha, a_\alpha^*) \text{ norm } \times \text{ weak}^* \text{ converges to } (x, x^*).$$

Clearly, it suffices to show that

$$\boxed{(a_\alpha^*)_{\alpha\in\Gamma} \text{ is eventually bounded.}}$$

Suppose to the contrary that $(a_\alpha^*)_{\alpha\in\Gamma}$ is not eventually bounded. We can and do suppose that

$$\lim_\alpha \|a_\alpha^*\| = +\infty.$$

By Fact 5, there exist $\delta > 0$ and $M > 0$ such that

$$\langle a_\alpha, a_\alpha^* \rangle \geq \delta\|a_\alpha^*\| - (\|a_\alpha\| + \delta)M, \quad \forall \alpha \in \Gamma.$$

Then we have

$$\langle a_\alpha, \frac{a_\alpha^*}{\|a_\alpha^*\|}\rangle \geq \delta - \frac{(\|a_\alpha\| + \delta)M}{\|a_\alpha^*\|}, \quad \forall \alpha \in \Gamma. \qquad (*)$$

By Fact 1 (Banach-Alaoglu theorem), there exists a weak* convergent *subnet* $(a_\beta^*)_{\beta \in I}$ of $(a_\alpha^*)_{\alpha \in \Gamma}$, say

$$\frac{a_\beta^*}{\|a_\beta^*\|} \rightharpoonup_{\mathrm{w}*} a_\infty^* \in X^*. \qquad (**)$$

Then taking the limit along the subnet in $(*)$, we have

$$\boxed{\langle x, a_\infty^*\rangle \geq \delta.} \qquad (\triangle)$$

On the other hand, since $a_\alpha^* \rightharpoonup_{\mathrm{w}*} x^*$, we have

$$\langle x, a_\alpha^*\rangle \longrightarrow \langle x, x^*\rangle.$$

Dividing by $\|a_\alpha^*\|$ in both sides of above equation, then by $(**)$ we take the limit along the subnet again to get

$$\boxed{\langle x, a_\infty^*\rangle = 0, \quad \text{which contradicts } (\triangle).}$$

Corollary 1.
Let $A : X \rightrightarrows X^*$ be maximally monotone such that $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then $\operatorname{gra} A$ is norm $\times$ weak* closed, i.e., $\operatorname{gra} A = \overline{\operatorname{gra} A}^{\|\cdot\|\times \mathrm{w}^*}$.

Example 1. [Failure of graph to be norm-weak* closed]
Borwein, Fitzpatrick, and Girgensohn showed statement of Corollary 1 cannot hold without the assumption of the nonempty interior domain: The following example is as simplified by Bauschke and Combettes.
Let $f : \ell^2(\mathbb{N}) \to \left]-\infty, +\infty\right]$ be defined by

$$\boxed{x \mapsto \max\left\{1 + \langle x, e_1\rangle, \sup_{2\leq n\in\mathbb{N}} \langle x, \sqrt{n} e_n\rangle\right\},}$$

where $e_n := (0, \ldots, 0, 1, 0, \cdots, 0)$: the nth entry is 1 and the others are 0. Then f is proper lower semicontinuous and convex, but

$$\boxed{\partial f \text{ is not norm} \times \text{weak}^* \text{ closed.}}$$

Corollary 2

Let $A : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Assume that $S \subseteq \operatorname{dom} A$. Then

1. $\operatorname{gra} A_S \subseteq \operatorname{gra} A$.
2. $\overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}*} \subseteq Ax, \forall x \in \operatorname{dom} A$.
3. $Ax = A_S(x), \forall x \in S$ and hence $Ax = A_{\mathrm{int}}(x), \forall x \in \operatorname{int}\operatorname{dom} A$.

Proposition 1

Let D, F be nonempty subsets of X^*, and C be a convex set of X with $\operatorname{int} C \neq \varnothing$. Assume that $x \in C$ and that for every $v \in \operatorname{int} T_C(x)$,

$$\sup\langle D, v\rangle \leq \sup\langle F, v\rangle < +\infty.$$

Then

$$\boxed{D \subseteq \overline{\operatorname{conv} F + N_C(x)}^{\mathrm{w}*}.}$$

Next is our key technical part.

Proposition 2

Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \operatorname{int}\operatorname{dom} A \neq \varnothing$ such that S is dense in $\operatorname{int}\operatorname{dom} A$. Assume that $x \in \operatorname{dom} A$ and $v \in \operatorname{int} T_{\overline{\operatorname{dom} A}}(x)$. Then there exists $x_0^* \in A_S(x)$ such that

$$\boxed{\sup\left\langle A_S(x), v\right\rangle = \left\langle x_0^*, v\right\rangle = \sup\left\langle Ax, v\right\rangle.}$$

In particular, $\operatorname{dom} A_S = \operatorname{dom} A$.

Outline proof of Proposition 2

Proof. By Corollary 2, $\operatorname{gra} A_S \subseteq \operatorname{gra} A$ and hence

$$\boxed{\sup \langle A_S(x), v\rangle \leq \sup \langle Ax, v\rangle.} \qquad (*)$$

Appealing now to $v \in \operatorname{int} T_{\overline{\operatorname{dom} A}}(x)$, we can and do suppose that $v = x_0 - x$, where $x_0 \in \operatorname{int} \overline{\operatorname{dom} A} = \operatorname{int} \operatorname{dom} A$ by Fact 4.

Using Lemma 1 select $M, \delta > 0$ such that $x_0 + 2\delta B_X \subseteq \operatorname{dom} A$ and

$$\boxed{\sup_{a \in [x_0 + \delta B_X, x[,\, a^* \in Aa} \|a^*\| \leq M < +\infty.} \qquad (**)$$

Proof continued:

Let $t \in]0,1[$. Then,

$$\boxed{x + tB_\delta(v) = (1-t)x + tx_0 + t\delta B_X \subseteq \operatorname{int}\overline{\operatorname{dom} A} = \operatorname{int}\operatorname{dom} A.} \qquad (***)$$

Then by the monotonicity of A,

$$t\langle a^* - x^*, w\rangle$$
$$= \langle a^* - x^*, x + tw - x\rangle \geq 0, \quad \forall a^* \in A(x+tw),\ x^* \in Ax, w \in B_\delta(v).$$

There exists a sequence $(x_n^*)_{n\in\mathbb{N}}$ in Ax such that

$$\boxed{\langle x_n^*, v\rangle \longrightarrow \sup\langle Ax, v\rangle.} \qquad (\triangle)$$

Combining above two equations, we have

$$\langle a^* - x_n^*, v + w - v\rangle \geq 0, \quad \forall a^* \in A(x+tw),\ w \in B_\delta(v),\ n \in \mathbb{N}.$$

Proof continued:

Fix $1 < n \in \mathbb{N}$. Thus, appealing to $(**)$ and the above equation yields,

$$\begin{aligned}\langle a^*, v\rangle &\geq \langle x_n^*, v\rangle - \langle a^* - x_n^*, w - v\rangle \\ &\geq \langle x_n^*, v\rangle - (M + \|x_n^*\|)\cdot\|w - v\| \quad \forall a^* \in A(x + tw),\ w \in B_\delta(v).\end{aligned}$$

Take $\varepsilon_n := \min\{\frac{1}{n(M+\|x_n^*\|)}, \delta\}$ and $t_n := \frac{1}{n}$.

Since S is dense in $\operatorname{int}\operatorname{dom} A$ and $x + t_n B_{\varepsilon_n}(v) \subseteq \operatorname{int}\operatorname{dom} A$ by $(***)$, $S \cap [x + t_n B_{\varepsilon_n}(v)] \neq \varnothing$. Then there exists $w_n \in X$ such that

$$\boxed{w_n \in B_{\varepsilon_n}(v), \qquad x + t_n w_n \in S \quad \text{and then} \quad x + t_n w_n \longrightarrow x.} \qquad (\triangle\triangle)$$

Thus,

$$\langle a^*, v\rangle \geq \langle x_n^*, v\rangle - \frac{1}{n}, \quad \forall a^* \in A(x + t_n w_n).$$

Proof concluded

Let $a_n^* \in A(x + t_n w_n)$. Then by the previous equation,

$$\boxed{\langle a_n^*, v\rangle \geq \langle x_n^*, v\rangle - \frac{1}{n}.} \qquad (\triangle\triangle\triangle)$$

By $(**)$ and $(***)$, $(a_n^*)_{n\in\mathbb{N}}$ is bounded. Then by the Banach-Alaoglu theorem, there exists a weak* convergent subnet of $(a_\alpha^*)_{\alpha\in I}$ of $(a_n^*)_{n\in\mathbb{N}}$ such that

$$\boxed{a_\alpha^* \rightharpoonup_{\mathrm{w}*} x_0^* \in X^*.}$$

Then by $(\triangle\triangle)$, $x_0^* \in A_S(x)$ and thus by $(\triangle\triangle\triangle)$& $(\triangle)$

$$\sup \left\langle A_S(x), v\right\rangle \geq \left\langle x_0^*, v\right\rangle \geq \sup \left\langle Ax, v\right\rangle.$$

Hence by $(*)$, we obtain $\sup \left\langle A_S(x), v\right\rangle = \left\langle x_0^*, v\right\rangle = \sup \left\langle Ax, v\right\rangle$.

We next recall an alternate *recession cone* description of $N_{\operatorname{dom} A}$. Consider

$$\operatorname{rec} A(x) := \{x^* \in X^* \mid \exists t_n \to 0^+, (a_n, a_n^*) \in \operatorname{gra} A \text{ such that } a_n \longrightarrow x,\ t_n a_n^* \rightharpoonup_{\mathrm{w}*} x^*\}.$$

Remark

When A is maximally monotone,

$$\boxed{(N_{\overline{\operatorname{dom} A}} =)\, N_{\operatorname{dom} A} = \operatorname{rec} A \quad \text{on} \quad \operatorname{dom} A.}$$

Theorem 2. [Reconstruction of A, 1]

Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \operatorname{int}\operatorname{dom} A \neq \varnothing$ and with S dense in $\operatorname{int}\operatorname{dom} A$. Then for every $x \in X$,

$$Ax = N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}\,[A_S(x)]}^{\mathrm{w}*} = \operatorname{rec} A(x) + \overline{\operatorname{conv}\,[A_S(x)]}^{\mathrm{w}*}.$$

Proof. By Remark ($N_{\overline{\operatorname{dom} A}} = \operatorname{rec} A$ on $\operatorname{dom} A$), we only need show

$$Ax = N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}*}.$$

Applying Propositions 1&2,

$$\boxed{Ax = \overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}*}, \quad \forall x \in X.}$$

We must still show

$$\boxed{Ax = N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}*}, \quad \forall x \in X}$$

Now, for every two sets $C, D \subseteq X^*$, we have $C + \overline{D}^{\mathrm{w}*} \subseteq \overline{C + D}^{\mathrm{w}*}$. Thus, it suffices to show that for every $x \in \operatorname{dom} A$,

$$\boxed{\overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}*} \subseteq N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}*}.}$$

We again can and do suppose that $0 \in \operatorname{int}\operatorname{dom} A$ and $(0,0) \in \operatorname{gra} A$. Let $x \in \operatorname{dom} A$ and $x^* \in \overline{N_{\overline{\operatorname{dom} A}}(x) + \operatorname{conv}[A_S(x)]}^{\mathrm{w}*}$. Now we show that

$$x^* \in N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}[A_S(x)]}^{\mathrm{w}*}.$$

Then there exists nets $(x^*_\alpha)_{\alpha\in I}$ in $N_{\overline{\operatorname{dom} A}}(x)$ and $(y^*_\alpha)_{\alpha\in I}$ in $\operatorname{conv}[A_S(x)]$ such that

$$\boxed{x^*_\alpha + y^*_\alpha \rightharpoonup_{\mathrm{w}*} x^*.}$$

Now we claim that

$$(x^*_\alpha)_{\alpha\in I} \text{ is eventually bounded.}$$

Suppose to the contrary that $(x^*_\alpha)_{\alpha\in I}$ is not eventually bounded. We can and do suppose that

$$\lim_\alpha \|x^*_\alpha\| = +\infty.$$

Proof continued:

By $0 \in \operatorname{int} \operatorname{dom} A$ and $x_\alpha^* \in N_{\overline{\operatorname{dom} A}}(x)$ (for every $\alpha \in I$), there exists $\delta > 0$ such that $\delta B_X \subseteq \overline{\operatorname{dom} A}$ and hence we have

$$\langle x, x_\alpha^* \rangle \geq \sup_{b \in B_X} \langle x_\alpha^*, \delta b \rangle = \delta \|x_\alpha^*\|.$$

Thence, we have

$$\boxed{\left\langle x, \frac{x_\alpha^*}{\|x_\alpha^*\|} \right\rangle \geq \delta.} \qquad (**)$$

By Fact 1, there exists a weak* convergent subnet $(x_\beta^*)_{\beta \in \Gamma}$ of $(x_\alpha^*)_{\alpha \in I}$, say

$$\frac{x_\beta^*}{\|x_\beta^*\|} \rightharpoonup_{\mathrm{w}^*} x_\infty^* \in X^*.$$

Proof continued:

Taking the limit along the subnet in $(**)$, we have

$$\boxed{\langle x, x^*_\infty\rangle \geq \delta.} \qquad (\triangle)$$

Since $x^*_\alpha + y^*_\alpha \rightharpoonup_{w*} x^*$, we have

$$\frac{x^*_\alpha}{\|x^*_\alpha\|} + \frac{y^*_\alpha}{\|x^*_\alpha\|} \rightharpoonup_{w*} 0.$$

And so by $\frac{x^*_\beta}{\|x^*_\beta\|} \rightharpoonup_{w*} x^*_\infty$,

$$\frac{y^*_\beta}{\|x^*_\beta\|} \rightharpoonup_{w*} -x^*_\infty.$$

By Corollary 2, $\operatorname{conv}[A_S(x)] \subseteq Ax$, and hence $(y^*_\alpha)_{\alpha\in I}$ is in Ax. Since $(0,0) \in \operatorname{gra} A$, we have $\langle y^*_\alpha, x\rangle \geq 0$ and so

$$\boxed{\langle \frac{y^*_\beta}{\|x^*_\beta\|}, x\rangle \geq 0.}$$

Using the equation $\frac{y^*_\beta}{\|x^*_\beta\|} \rightharpoonup_{\mathrm{w}*} -x^*_\infty$ and taking the limit along the subnet in above equation we get

$$\boxed{\big\langle -x^*_\infty, x\big\rangle \geq 0, \ \text{which contradicts that } \langle x^*_\infty, x\rangle \geq \delta.}$$

Hence, $(x^*_\alpha)_{\alpha\in I}$ is eventually bounded.

Then by Fact 1 (Banach- Alaoglu theorem) again, there exists a weak* convergent subset of $(x^*_\alpha)_{\alpha\in I}$, for convenience, still denoted by $(x^*_\alpha)_{\alpha\in I}$ which lies in the normal cone, such that $x^*_\alpha \rightharpoonup_{\mathrm{w}*} w^* \in X^*$. Hence $w^* \in N_{\overline{\operatorname{dom} A}}(x)$ and $y^*_\alpha \rightharpoonup_{\mathrm{w}*} x^* - w^* \in \overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}*}$. Hence

$$\boxed{x^* \in N_{\overline{\operatorname{dom} A}}(x) + \overline{\operatorname{conv}\left[A_S(x)\right]}^{\mathrm{w}*}.}$$

Corollary 2. [Convex subgradients]
Let $f : X \to\ ]-\infty, +\infty]$ be proper lower semicontinuous and convex with $\operatorname{int}\operatorname{dom} f \neq \varnothing$. Let $S \subseteq \operatorname{int}\operatorname{dom} f$ be given with S dense in $\operatorname{dom} f$. Then

$$\partial f(x) = N_{\operatorname{dom} f}(x) + \overline{\operatorname{conv}\left[(\partial f)_S(x)\right]}^{\mathrm{w}^*}, \quad \forall x \in X.$$

In various classes of Banach space we can choose useful structure for $S \in S_A$, where

$$S_A := \{S \subseteq \operatorname{int}\operatorname{dom} A \mid S \text{ is dense in } \operatorname{int}\operatorname{dom} A\}.$$

Corollary 3. [Specification of S_A]

Let $A : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. We may choose the dense set $S \in S_A$ to be as follows:

1. In a Gâteaux smooth space, entirely within the residual set of non-σ porous points of $\operatorname{dom} A$,
2. In an Asplund space, to include only a subset of the generic set points of single-valuedness and norm to norm continuity of A,
3. In a separable Asplund space, to hold only countably many angle-bounded points of A,

4. In a weak Asplund space, to include only a subset of the generic set of points of single-valuedness (and norm to weak* continuity) of A,
5. In a separable space, to include only points of single-valuedness (and norm to weak* continuity) of A whose complement is covered by a countable union of Lipschitz surfaces.
6. In finite dimensions, to include only points of differentiability of A which are of full measure.

Let $A : X \rightrightarrows X^*$. We define $\widehat{A} : X \rightrightarrows X^*$ by

$$\operatorname{gra}\widehat{A} := \left\{(x, x^*) \in X \times X^* \mid x^* \in \bigcap_{\varepsilon>0} \overline{\operatorname{conv}\left[A(x + \varepsilon B_X)\right]}^{\mathrm{w}*}\right\}.$$

Clearly, we have $\overline{\operatorname{gra} A}^{\|\cdot\|\times \mathrm{w}*} \subseteq \operatorname{gra}\widehat{A}$.

We say A has the upper-semicontinuity property *property (Q)* if for every net $(x_\alpha)_{\alpha\in J}$ in X such that $x_\alpha \longrightarrow x$, we have

$$\bigcap_{\alpha\in J} \overline{\operatorname{conv}\left[\bigcup_{\beta\succeq_J \alpha} A(x_\beta)\right]}^{\mathrm{w}*} \subseteq Ax.$$

The following directly follows from above:

$$\widehat{A} = A \Rightarrow \Big(A \text{ has property (Q)}\Big) \Rightarrow \Big(\operatorname{gra} A = \overline{\operatorname{gra} A}^{\|\cdot\|\times \mathrm{w}*}\Big).$$

Theorem 3. [Reconstruction of A, 2]
Let $A : X \rightrightarrows X^*$ be maximally monotone with $\operatorname{int}\operatorname{dom} A \neq \varnothing$. Then $\boxed{\widehat{A} = A}$. In particular, A has property (Q); and so has a norm $\times$ weak* closed graph.

Recall that

$$\boxed{\operatorname{gra}\widehat{A} := \left\{(x, x^*) \in X \times X^* \mid x^* \in \bigcap_{\varepsilon>0} \overline{\operatorname{conv}\left[A(x + \varepsilon B_X)\right]}^{\mathrm{w}^*}\right\}.}$$

In general, we do not have

$$Ax = \overline{\operatorname{conv}\left[A_S(x)\right]}^{w*}, \quad \forall x \in \operatorname{dom} A.$$

Example 2

Let C be a closed convex subset of X with $S \subseteq \operatorname{int} C \neq \varnothing$ such that S is dense in C. Then

1. N_C is maximally monotone and $\operatorname{gra}(N_C)_S = C \times \{0\}$.
2. $N_C(x) \neq \overline{\operatorname{conv}\left[(N_C)_S(x)\right]}^{w*}, \forall x \in \operatorname{bdry} C$.
3. $\bigcap_{\varepsilon>0} \overline{\operatorname{conv}\left[N_C(x+\varepsilon B_X)\right]}^{w*} = N_C(x), \forall x \in X$.

There always exists an operator A even with no interior point such that $\widehat{A} = A$ and hence A has property (Q). More generally:

Example 3

Suppose that X is reflexive. Let $A : X \rightrightarrows X^*$ be such that $\operatorname{gra} A$ is nonempty closed and convex. Then

$$\boxed{\widehat{A} = A \text{ and hence } A \text{ has property (Q).}}$$

Example 4

Let $p > 1$ and $f : X \to \left]-\infty, +\infty\right]$ be defined by

$$x \mapsto \iota_{B_X}(x) + \frac{1}{p}\|x\|^p.$$

Then for every $x \in \operatorname{dom} f$, we have

$$N_{\operatorname{dom} f}(x) = \begin{cases} \mathbb{R}_+ \cdot Jx, & \text{if } \|x\| = 1; \\ \{0\}, & \text{if } \|x\| < 1 \end{cases}$$

$$(\partial f)_{\text{int}}(x) = \begin{cases} \|x\|^{p-2} \cdot Jx, & \text{if } \|x\| \neq 0; \\ \{0\}, & \text{otherwise.} \end{cases}$$

Moreover,

1. $\partial f = N_{\operatorname{dom} f} + (\partial f)_{\text{int}} = N_{\operatorname{dom} f} + \partial \frac{1}{p}\|\cdot\|^p$.
2. $\partial f(x) \neq (\partial f)_{\text{int}}(x) = \overline{\operatorname{conv}\left[(\partial f)_{\text{int}}(x)\right]}^{\text{w*}}, \forall x \in \operatorname{bdry} \operatorname{dom} f$.
3. $\bigcap_{\varepsilon > 0} \overline{\operatorname{conv}\left[\partial f(x + \varepsilon B_X)\right]}^{\text{w*}} = \partial f(x), \ \forall x \in X$.

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