DUALITY IN TAILS OF MULTIPLE ZETA VALUES

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Abstract. Duality relations are deduced for tails of multiple zeta values using elementary methods. These formulas extend the classical duality formulas for multiple zeta values.

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1. Introduction

There are countless formulae relating infinite sums of *harmonic numbers*

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \tag{1.1}$$

to values of the Riemann zeta-function, from the well-known expression for $\zeta(2)$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{H_n}{n(n+1)}, \tag{1.2}$$

to more involved results involving powers of H_n such as

$$\zeta(4) = \frac{4}{11} \sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = \frac{4}{17} \sum_{n=1}^{\infty} \frac{H_n^2}{n^2}, \tag{1.3}$$

which can arise from a number of different ways (see, for example, De Doelder [6] or Borwein and Borwein [2], as well as [3, p. 173, Problem 9]), to the infinite classes of identities discovered by W. Chu [5] using evaluations of hypergeometric series.

These formulae are, as one might expect due to the appearance of the harmonic numbers, also related to *Euler-Zagier sums*, or *multiple zeta-values* (MZVs), given by

$$\zeta(a_1, \ldots, a_k) := \sum_{n_1 > \cdots > n_k > 0} \prod_{i=1}^{k} \frac{1}{n_i^{a_i}}, \tag{1.4}$$

which converge when $a_1 > 1$. In what follows, we use the standard convention of abbreviating a repeating sequence S in the argument of an MZV that repeats k times by replacing it with $\{S\}^k$ in the argument. For example, $\zeta(2,1,1,2,1) = \zeta(2,\{1\}^2,2,1)$ and $\zeta(2,1,2,1,3) = \zeta(\{2,1\}^2,3)$.

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MZVs are known to satisfy what is known as *MZV duality*, the simplest case of which is the famous identity of Euler:

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \zeta(2,1) = \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}. \tag{1.5}$$

The first author and D. Bradley [4] have collected 32 different proofs of (1.5), covering many diverse ideas which can be used to attack sums of this type.

Our present study stems from the following intriguing generalization of (1.2), which we learned from Problem 854 in the May 2007 issue of the College Mathematics Journal [8, p. 228]. A slightly rephrased version of it is:

Theorem 1. *For each positive integer m we have*

$$\zeta(2) - \sum_{n=1}^{m-1} \frac{1}{n^2} = m! \sum_{n=1}^{\infty} \frac{H_n}{n(n+1)_m}, \tag{1.6}$$

where

$$(n+1)_m := (n+1)\cdots(n+m)$$

is the Pochhammer symbol*.*

Equation (1.2) follows from the case $m = 1$. Theorem 1 may be interpreted as an expression for the "tail" (that is, the error) in the $m-1$st partial sum of $\zeta(2)$. Note that the right-hand side of (1.6) is much more computationally efficient for the evaluation of $\zeta(2)$ even for small m, as it converges like $O(n^{-(m+1)})$ while the classical definition of $\zeta(2)$ converges like $O(n^{-2})$.

In the next section, we give an elementary proof of Theorem 1. Then in Section 3 we consider extensions to tails of MZVs, and we are led to a duality relation that generalizes the classical duality formula for MZVs. In Section 4 we apply our ideas to alternating zeta functions and more general types of multiple zeta-functions.

2. The $\zeta(2)$ case

In this section we give an inductive proof of Theorem 1. We require the following lemma.

Lemma 2. *For any positive integer m, we have*

$$\sum_{n\geq 0} \frac{1}{\binom{n+m+1}{m+1}} = \frac{m+1}{m}. \tag{2.1}$$

Proof. In this proof, as well as the rest of the paper, we will make heavy use of the following partial fraction identity, often without mention.

$$\frac{1}{n(n+m)} = \frac{1}{m}\left(\frac{1}{n} - \frac{1}{n+m}\right). \tag{2.2}$$

Applying (2.2) we find that

$$\sum_{n=0}^{N} \frac{1}{\binom{n+m+1}{m+1}} = \sum_{n=0}^{N} \frac{(m+1)!n!}{(n+m+1)!}$$

$$= \sum_{n=0}^{N} \frac{(m+1)!(n+1)!}{(n+m)!} \frac{1}{(n+1)(n+m+1)}$$

$$= \frac{m+1}{m} \sum_{n=0}^{N} \frac{m!(n+1)!}{(n+m)!} \left(\frac{1}{n+1} - \frac{1}{n+m+1} \right)$$

$$= \frac{m+1}{m} \sum_{n=0}^{N} \left(\frac{1}{\binom{n+m}{m}} - \frac{1}{\binom{n+1+m}{m}} \right),$$

which is a telescoping sum. Letting N tend to infinity gives the desired result. □

Remark 3. We remark that (2.1) goes back at least to Euler, and can be found in [7, p. 44]. In comparison, modern computer algebra systems can easily provide a machine-proof. For example, the following line of code works in *Maple* 10 or 11 and many other versions.

```
S:=Sum(1/(binomial(n+m+1,m+1)),n=0..infinity):S=value(S);
```

We now give our proof of Theorem 1.

Proof of Theorem 1. Let $f(m)$ denote the right-hand side of (1.6), then Theorem 1 follows if we show that $f(1) = \zeta(2)$ and $f(m) - f(m+1) = 1/m^2$ for $m \geq 1$. As we mentioned, the fact that $f(1) = \zeta(2)$ is exactly (1.2), but for completeness we supply a proof:

$$\sum_{n\geq 1} \frac{H_n}{n(n+1)} = \sum_{n\geq 1} \left(\frac{H_n}{n} - \frac{H_n}{n+1} \right)$$

$$= \sum_{n\geq 1} \left(\frac{1}{n^2} + \frac{H_{n-1}}{n} - \frac{H_n}{n+1} \right) = \zeta(2),$$

where we use the convention $H_0 = 0$.

Now it remains to show that for all $m \geq 1$,

$$m! \sum_{n=1}^{\infty} \frac{H_n}{n(n+1)\cdots(n+m)} - (m+1)! \sum_{n=1}^{\infty} \frac{H_n}{n(n+1)\cdots(n+m+1)} = \frac{1}{m^2}. \tag{2.3}$$

Using (2.2), we find that

$$f(m) = (m-1)! \sum_{n\geq 1} \frac{H_n}{(n+1)_{m-1}} \left(\frac{1}{n} - \frac{1}{n+m} \right)$$

$$= f(m-1) - (m-1)! \sum_{n\geq 1} \frac{H_{n+1} - 1/(n+1)}{(n+1)(n+2)_{m-1}}$$

$$= f(m-1) - \left(f(m-1) - (m-1)! \sum_{n\geq 1} \frac{1}{n^2(n+1)_{m-1}} \right) \tag{2.4}$$

since $H_1 = 1$. Therefore,

$$\begin{aligned}
f(m) - f(m+1) &= (m-1)! \sum_{n\geq 1} \frac{1}{n^2(n+1)_{m-1}} - m! \sum_{n\geq 1} \frac{1}{n^2(n+1)_m} \\
&= (m-1)! \sum_{n\geq 1} \frac{1}{n^2(n+1)_{m-1}} \left(1 - \frac{m}{n+m}\right) \\
&= (m-1)! \sum_{n\geq 1} \frac{1}{n(n+1)\cdots(n+m)} = \frac{1}{m(m+1)} \sum_{n\geq 1} \frac{1}{\binom{n+m}{m+1}} \\
&= \frac{1}{m(m+1)} \cdot \frac{m+1}{m} = \frac{1}{m^2} \qquad (2.5)
\end{aligned}$$

as required. □

Let us at this point define the *hypergeometric series* ${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; x\right)$ by

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; x\right) := \sum_{n\geq 0} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!} x^n, \qquad (2.6)$$

as well as the k-th *polylogarithm*, $\mathrm{Li}_k(x)$, given by

$$\mathrm{Li}_k(x) := \sum_{n\geq 1} \frac{x^n}{n^k}, \qquad (2.7)$$

see [1]. We may combine (2.4) with Theorem 1 and rewrite it in hypergeometric form.

Corollary 4. *For any positive integer m we have*

$${}_3F_2\left(\begin{matrix} m, m, 1 \\ m+1, m+1 \end{matrix} ; 1\right) = m\, {}_3F_2\left(\begin{matrix} 1, 1, 1 \\ m+1, 2 \end{matrix} ; 1\right). \qquad (2.8)$$

Proof. Begin by noting that we have the power series identity

$$\begin{aligned}
\mathrm{Li}_k(x) - \sum_{n=1}^{m-1} \frac{x^n}{n^k} &= x^m \sum_{n\geq 0} \frac{x^n}{(n+m)^k} = x^m \sum_{n\geq 0} \frac{(m)_n^k}{m^k (m+1)_n^k} x^n \\
&= \frac{x^m}{m^k}\, {}_{k+1}F_k\left(\begin{matrix} m, \ldots, m, 1 \\ m+1, \ldots, m+1 \end{matrix} ; x\right). \qquad (2.9)
\end{aligned}$$

By Theorem 1,

$$f(m) = \mathrm{Li}_2(1) - \sum_{n=1}^{m-1} \frac{1}{n^2} = \frac{1}{m^2}\, {}_3F_2\left(\begin{matrix} m, m, 1 \\ m+1, m+1 \end{matrix} ; 1\right).$$

But by (2.4),

$$f(m) = \sum_{n\geq 0} \frac{(m-1)!}{(n+1)(n+1)_m} = \frac{1}{m}\, {}_3F_2\left(\begin{matrix} 1, 1, 1 \\ m+1, 2 \end{matrix} ; 1\right).$$

□

Remark 5. Corollary 4 is a special case of a result of Kummer [1, p. 142, Cor. 3.3.5], which states

$$ {}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};1\right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)}\,{}_3F_2\left(\begin{matrix} a,d-b,d-c \\ d,d+e-b-c \end{matrix};1\right), $$

and is itself a limiting case of a result of Whipple [1, p. 140, Thm. 3.3.3]. Corollary 4 can also be thought of as a ${}_3F_2$ analogue of Euler's Transformation:

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};x\right) = (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,c-b \\ c \end{matrix};x\right). \tag{2.10} $$

3. Extensions to Multiple Zeta Values

In this section we prove some MZV analogues to Theorem 1. For convenience we introduce some notation. For any finite sequence $S = \{a_1, \ldots, a_k\}$ of positive integers denote by $\zeta_m(S)$ the mth *outer-partial sum* of $\zeta(S)$ and by $\iota_m(S)$ the mth *inner-partial sum.* That is,

$$ \zeta_m(S) := \sum_{m\geq n_1>\cdots>n_k>0} \prod_{i=1}^{k} \frac{1}{n_i^{a_i}}, \qquad \iota_m(S) := \sum_{n_k=1}^{m} \sum_{n_1>\cdots>n_k} \prod_{i=1}^{k} \frac{1}{n_i^{a_i}}. \tag{3.1} $$

Note that $\zeta_m(S)$ is finite and therefore converges for any choice of S whereas $\iota_m(S)$ is infinite and converges if and only if $a_1 > 1$. Also, for any function $g : \mathbb{N} \to \mathbb{C}$, with values g_n, we define the function $\mathcal{F}$ by

$$ \mathcal{F}(k,m;g_n) := m! \sum_{n\geq 1} \frac{g_n}{n^k (n+1)_m}. \tag{3.2} $$

All of our results in this section depend on the following properties of $\mathcal{F}(k,m;g_n)$.

Proposition 6. *For any positive integers k and m, we have*

$$ \mathcal{F}(k,m;g_n) = \mathcal{F}(k,0;g_n) - \sum_{j=1}^{m} \frac{1}{j}\mathcal{F}(k-1,j;g_n), \tag{3.3} $$

provided both sides converge. In particular, if there exists an $\varepsilon > 0$ such that $g_n = O(n^{1-\varepsilon})$ as $n \to \infty$, then for every positive integer k we have

$$ \mathcal{F}(k,0;g_n) = \sum_{j=1}^{\infty} \frac{1}{j}\,\mathcal{F}(k-1,j;g_n), \tag{3.4} $$

provided that both sides converge.

Proof. First note that for each k and $m \geq 1$ we have

$$ \begin{aligned} &\mathcal{F}(k,m-1;g_n) - \mathcal{F}(k,m;g_n) \\ &\quad= (m-1)! \sum_{n\geq 1} \frac{g_n}{n^k (n+1)_{m-1}} - (m-1)! \sum_{n\geq 1} \frac{g_n}{n^{k-1}(n+1)_{m-1}} \left(\frac{1}{n} - \frac{1}{n+m}\right) \\ &\quad= (m-1)! \sum_{n\geq 1} \frac{g_n}{n^{k-1}(n+1)_m} \end{aligned} $$

$$= \frac{1}{m}\mathcal{F}(k-1,m;g_n), \tag{3.5}$$

provided the far right-hand side converges. (3.3) follows from summing on m. To prove (3.4), it suffices to show that if $g_n = O(n^{1-\varepsilon})$ then $\lim_{m\to\infty} \mathcal{F}(k,m;g_n) = 0$ for each $k \geq 1$. Since $\mathcal{F}(k,m;|g_n|) < \mathcal{F}(1,m;|g_n|)$, we need only to prove the case $k=1$. Now, since $g_n = O(n^{1-\varepsilon})$, we know that

$$\sum_{n=1}^{\infty} \frac{|g_n|}{n^2} < \infty,$$

and so

$$\begin{aligned}
\mathcal{F}(1,m;|g_n|) &= \sum_{n\geq 1} \frac{m!|g_n|}{n(n+1)_m} \leq \sum_{n\geq 1} \frac{|g_n|}{n(n+m)} \\
&= \sum_{1\leq n\leq\sqrt{m}} \frac{|g_n|}{n(n+m)} + \sum_{n>\sqrt{m}} \frac{|g_n|}{n(n+m)} \\
&\leq \frac{1}{\sqrt{m}} \sum_{1\leq n\leq\sqrt{m}} \frac{|g_n|}{n^2} + \sum_{n>\sqrt{m}} \frac{|g_n|}{n^2} \\
&= O(m^{-1/2}) + o(1)
\end{aligned}$$

as $m \to \infty$, as required. □

Remark 7. The growth condition $g_n = O(n^{1-\varepsilon})$ in Proposition 6 is nowhere near best possible, since we used very loose bounds on the factorial and Pochhammer symbols. However, it more than suffices for the sequences that we consider in the remainder of this paper.

In this notation, we may rewrite (2.4) as

$$\mathcal{F}(1,m;H_n) = \mathcal{F}(2,m-1;1),$$

where we write 1 for 1_n.

Theorem 8. *Let k and m be non-negative integers with $k \geq 2$. Then we have*

$$\begin{aligned}
\mathcal{F}(k,m;1) &= \sum_{n_1>\cdots>n_{k-1}>m} \frac{1}{n_1^2 n_2 \cdots n_{k-1}} \\
&= \zeta(2,\{1\}^{k-2}) - \iota_m(2,\{1\}^{k-2}).
\end{aligned} \tag{3.6}$$

Proof. We induct on k. The case $k=2$ is Theorem 1. Suppose (3.6) holds for some $k \geq 2$. Then by the recurrence relation (3.3) we find that

$$\mathcal{F}(k+1,m;1) = \zeta(k+1) - \sum_{j=1}^{m} \frac{1}{j} \sum_{n_1>n_2>\cdots>n_{k-1}>j} \frac{1}{n_1^2 n_2 \cdots n_{k-1}}.$$

Let m tend to ∞. Since $\lim\limits_{m\to\infty}\mathcal{F}(k+1,m;1)=0$, we find that $\zeta(k+1)=\zeta(2,\{1\}^{k-1})$, and so we have

$$\begin{aligned}\mathcal{F}(k+1,m;1) &= \sum_{n_k>m}\frac{1}{n_k}\sum_{n_1>n_2>\cdots>n_k}\frac{1}{n_1^2 n_2\cdots n_{k-1}}\\ &= \sum_{n_1>n_2>\cdots>n_k>m}\frac{1}{n_1^2 n_2\cdots n_k}\end{aligned}$$

as required. □

Theorem 9. *For each integer $k\ge 2$ and $m\ge 0$ we have*

$$\mathcal{F}(k,m;H_{n-1})=\zeta(3,\{1\}^{k-2})-\iota_m(3,\{1\}^{k-2})=\sum_{n_1>\cdots>n_{k-1}>m}\frac{1}{n_1^3 n_2\cdots n_{k-1}}. \qquad (3.7)$$

Proof. We begin by considering $\mathcal{F}(k,m;H_n)$ and showing that

$$\mathcal{F}(k,m;H_n)=\zeta(2,\{1\}^{k-1})-\iota_m(2,\{1\}^{k-1})+\zeta(3,\{1\}^{k-2})-\iota_m(3,\{1\}^{k-2})$$

for each $k\ge 2$. By (3.3) and Theorem 1 we have

$$\begin{aligned}\mathcal{F}(2,m;H_n) &= \mathcal{F}(2,0;H_n)-\sum_{j=1}^{m}\frac{1}{j}\mathcal{F}(1,j;H_n)\\ &= \zeta(2,1)+\zeta(3)-\sum_{j=1}^{m}\frac{1}{j}\sum_{n=j}^{\infty}\frac{1}{n^2}\\ &= \zeta(2,1)+\zeta(3)-\iota_m(2,1)-\iota_m(3).\end{aligned}$$

Thus, for $k>2$ we find inductively that

$$\begin{aligned}\mathcal{F}(k,m;H_n) &= \mathcal{F}(k,0;H_n)-\sum_{j=1}^{m}\frac{1}{j}\mathcal{F}(k-1,j;H_n)\\ &= \zeta(2,\{1\}^{k-1})+\zeta(3,\{1\}^{k-2})-\sum_{j=1}^{m}\frac{1}{j}\sum_{n_1>\cdots>n_{k-1}>j}\frac{1}{n_1^2 n_2\cdots n_{k-1}}\\ &\quad -\sum_{j=1}^{m}\frac{1}{j}\sum_{n_1>\cdots>n_{k-2}>j}\frac{1}{n_1^3 n_2\cdots n_{k-2}}\\ &= \zeta(2,\{1\}^{k-1})+\zeta(3,\{1\}^{k-2})-\iota_m(2,\{1\}^{k-1})-\iota_m(3,\{1\}^{k-2}).\end{aligned}$$

Finally, w2e note that $\mathcal{F}(k,m;H_{n-1})=\mathcal{F}(k,m;H_n)-\mathcal{F}(k+1,m;1)$ and then apply Theorem 8. □

Before we continue, we record two appealing special cases of Theorem 9.

Corollary 10.

$$\sum_{n\ge 1}\frac{m!\,H_{n-1}}{n^2(n+1)_m}=\sum_{n>m}\frac{1}{n^3}, \qquad (3.8)$$

$$\sum_{n\geq 1}\frac{m!\,H_{n-1}}{n^3(n+1)_m}=\sum_{n>m}\frac{H_{n-1}}{n^3}. \tag{3.9}$$

Equation (3.8) is an expression for the tail of $\zeta(3)$, but can also be interpreted as an extension of Euler's identity $\zeta(2.1)=\zeta(3)$, which can be recovered by setting $m=0$. Equation (3.9) is the first case where we do not get a connection formula between MZVs by setting $m=0$ (indeed, one obtains $\zeta(3,1)=\zeta(3,1)$ in this case). Such MZVs are *self-dual* (the reason for this is evident from the duality formula for MZVs below). However, a reduction formula (an MZV connection formula that relates $\zeta(S)$ to an MZV whose argument is a string of shorter length than S) for $\zeta(3,1)$ exists in the form of *Zagier's identity* $\zeta(\{3,1\}^n)=4^{-n}\zeta(\{4\}^n)$, see [3, p. 160, Cor. 3.13].

The proofs of Theorems 8 and 9 suggest that the recursive nature of $\mathcal{F}$ should allow us to inductively build up more general tail formulas. Indeed, Theorem 9 is the base case of the following generalization.

Theorem 11. *For any positive integers k,m,N with $k\geq 2$ we have*

$$\begin{aligned}\mathcal{F}(k,m;\zeta_n(\{1\}^N)) &= \zeta(N+1,\{1\}^{k-1})-\iota_m(N+1,\{1\}^{k-1})\\ &\qquad +\zeta(N+2,\{1\}^{k-2})-\iota_m(N+2,\{1\}^{k-2})\\ &= \sum_{n_1>\cdots>n_k>m}\frac{1}{n_1^{N+1}n_2\cdots n_k}+\sum_{n_1>\cdots n_{k-1}>m}\frac{1}{n_1^{N+2}n_2\cdots n_{k-1}},\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\mathcal{F}(k,m;\zeta_{n-1}(\{1\}^N) &= \zeta(N+2,\{1\}^{k-2})-\iota_m(N+2,\{1\}^{k-2})\\ &= \sum_{n_1>\cdots>n_{k-1}>m}\frac{1}{n_1^{N+2}n_2\cdots n_{k-1}}.\end{aligned} \tag{3.11}$$

Proof. Our proof is by double induction on N and k. The base case is Theorem 9, which corresponds to $N=1$ and is true for all $k\geq 2$. We begin by deriving a generalization of (2.4). For each positive integer n and N we have the relation

$$\begin{aligned}\mathcal{F}(1,m;\zeta_n(\{1\}^N) &= m!\sum_{n\geq 1}\frac{\zeta_n(\{1\}^N)}{n(n+1)_m}\\ &= (m-1)!\left(\sum_{n\geq 1}\frac{\zeta_n(\{1\}^N)}{n(n+1)_{m-1}}-\sum_{n\geq 1}\frac{\zeta_{n+1}(\{1\}^N)-\zeta_n(\{1\}^{N-1})/(n+1)}{(n+1)_m}\right)\\ &= \mathcal{F}(2,m-1;\zeta_{n-1}(\{1\}^{N-1})),\end{aligned}$$

since $\zeta_1(\{1\}^N)=0$ for $N\geq 2$. Now, suppose both (3.10) and (3.11) are true for all $k\geq 2$ if $N<N'$. Then, by (3.3) we have

$$\mathcal{F}(2,m;\zeta_n(\{1\}^{N'}))=\mathcal{F}(2,0;\zeta_n(\{1\}^{N'}))-\sum_{j=1}^{m}\frac{1}{j}\,\mathcal{F}(1,j;\zeta_n(\{1\}^{N'}))$$

$$
\begin{aligned}
&= \mathcal{F}(2,0;\zeta_n(\{1\}^{N'})) - \sum_{j=1}^{m} \frac{1}{j}\,\mathcal{F}(2,j-1;\zeta_{n-1}(\{1\}^{N'-1}))\\
&= \zeta(2,\{1\}^{N'}) + \zeta(3,\{1\}^{N'-1}) - \sum_{j=1}^{m}\frac{1}{j}\sum_{n_1\geq j}\frac{1}{n_1^{N'+1}}\\
&= \zeta(2,\{1\}^{N'}) - \sum_{n_1=1}^{m}\frac{1}{n_1^{N'+2}} + \zeta(3,\{1\}^{N'-1}) - \sum_{j=1}^{m}\frac{1}{j}\sum_{n_1>j}\frac{1}{n_1^{N'+1}}\\
&= \zeta(N'+2) - \iota_m(N'+2) + \zeta(N'+1,1) - \iota_m(N'+1,1)
\end{aligned}
$$

by Theorems 8 and 9 at $m=0$. Suppose further that (3.10) is true for $N=N'$ and for $2\leq k<k'$. Then we have for each $m\geq 1$,

$$
\begin{aligned}
&\sum_{j=1}^{m}\frac{1}{j}\,\mathcal{F}(k'-1,j,\zeta_n(\{1\}^{N'}))\\
&\quad= \sum_{j=1}^{m}\frac{1}{j}\sum_{n_1>\cdots>n_{k'-1}>j}\frac{1}{n_1^{N'+1}n_2\cdots n_{k'-1}} + \sum_{j=1}^{m}\frac{1}{j}\sum_{n_1>\cdots>n_{k'-2}>j}\frac{1}{n_1^{N'+2}n_2\cdots n_{k'-2}}\\
&\quad= \iota_m(N'+1,\{1\}^{k'-1}) + \iota_m(N'+2,\{1\}^{k'-2}).
\end{aligned}
$$

Since $\zeta_n(\{1\}^N)=O(\log(n)^N)$ for every $N\geq 0$, we may apply (3.4) to find that

$$
\begin{aligned}
\mathcal{F}(k',0;\zeta_n(\{1\}^{N'})) &= \zeta(k'+1,\{1\}^{N'-1}) + \zeta(k',\{1\}^{N'})\\
&= \zeta(N'+1,\{1\}^{k'-1}) + \zeta(N'+2,\{1\}^{k'-2}),
\end{aligned}
$$

and combining this with (3.3) we obtain

$$
\begin{aligned}
\mathcal{F}(k',m;\zeta_n(\{1\}^{N'})) &= \mathcal{F}(k',0;\zeta_n(\{1\}^{N'})) - \sum_{j=1}^{m}\frac{1}{j}\,\mathcal{F}(k'-1,j,\zeta_n(\{1\}^{N'}))\\
&= \zeta(N'+1,\{1\}^{k'-1}) + \zeta(N'+2,\{1\}^{k'-2})\\
&\qquad - \iota_m(N'+1,\{1\}^{k'-1}) - \iota_m(N'+2,\{1\}^{k'-2})
\end{aligned}
$$

as required. To prove (3.11), we simply apply (3.10), the inductive hypothesis, and the identity

$$
\begin{aligned}
\mathcal{F}(k,m;\zeta_{n-1}(\{1\}^N)) &= m!\sum_{n\geq 1}\frac{\zeta_{n-1}(\{1\}^N)}{n^k(n+1)_m} = m!\sum_{n\geq 1}\frac{\zeta_n(\{1\}^N)-\zeta_{n-1}(\{1\}^{N-1})/n}{n^k(n+1)_m}\\
&= \mathcal{F}(k,m;\zeta_n(\{1\}^N)) - \mathcal{F}(k+1,m;\zeta_{n-1}(\{1\}^{N-1}))
\end{aligned}
$$

which holds for every positive integer k and N. □

Remark 12. We remark that (3.11) can be proved without first proving (3.10), using the fact that

$$
\begin{aligned}
\mathcal{F}(1,m;\zeta_{n-1}(\{1\}^N) &= \mathcal{F}(1,m;\zeta_n(\{1\}^N) - \mathcal{F}(2,m;\zeta_{n-1}(\{1\}^{N-1})\\
&= \mathcal{F}(2,m-1;\zeta_{n-1}(\{1\}^N)) - \mathcal{F}(2,m;\zeta_{n-1}(\{1\}^{N-1}))
\end{aligned}
$$

and then proceeding inductively as in the proof of (3.10). We choose, however, to include both formulas because (3.10) is a more direct generalization of Theorem 1 and also because both the formulas and their proofs are very closely intertwined.

As a consequence of (3.11) at $m = 0$ we obtain a special case of the famous duality result for MZVs.

Corollary 13 (Special MZV duality). *For any non-negative integers k and N we have*

$$\zeta(k+2, \{1\}^N) = \zeta(N+2, \{1\}^k). \tag{3.12}$$

Theorem 11 can be thought of as a connection formula between the MZV $\zeta(k, \{1\}^N)$ and the inner-partial sums of its dual $\iota_m(N+2, \{1\}^{k-2})$. Specifically, that the tail of the dual is obtained by dividing the outermost summands by a Pochhammer symbol. This naturally leads us to the following elementary development of the formula for general MZV duality [3, Thm. 3.6.1].

Theorem 14 (General MZV duality). *Let $a_1, \ldots, a_N$, $b_1, \ldots, b_N$, and m be non-negative integers. If we let S be the sequence*

$$\{1\}^{b_1}, a_2+2, \{1\}^{b_2}, a_3+2, \ldots, a_N+2, \{1\}^{b_N}$$

and S' be its dual sequence

$$b_N+2, \{1\}^{a_N}, b_{N-1}+2, \{1\}^{a_{N-1}}, \ldots, b_1+2,$$

then we have

$$\mathcal{F}(a_1+2, m; \zeta_{n-1}(S)) = \zeta(S', \{1\}^{a_1}) - \iota_m(S', \{1\}^{a_1}). \tag{3.13}$$

In particular, upon setting $m = 0$ we recover the general duality formula

$$\zeta(a_1+2, S) = \zeta(S', \{1\}^{a_1}). \tag{3.14}$$

Proof. We induct on the length of S, counting each 1 in $\{1\}^{b_i}$ as a single term in the sequence. So the *length* of S is $b_1+b_2+\cdots+b_N+N-1$. We also adopt the convention that $\zeta_n(\emptyset) \equiv 1$ for all n. The base case (when $S = \emptyset$, so $N = 1$ and $b_1 = 0$) is the identity

$$\mathcal{F}(a_1+2, m; 1) = \zeta(2, \{1\}^{a_1}) - \iota_m(2, \{1\}^{a_1}),$$

which is a restatement of Theorem 8.

For the inductive step, let the sequence T be given by

$$T := \begin{cases} \{1\}^{b_1-1}, a_2+2, \ldots, \{1\}^{b_N}, & \text{if } b_1 \geq 1, \\ \{1\}^{b_2}, a_3+2, \ldots, \{1\}^{b_N}, & \text{if } b_1 = 0. \end{cases}$$

Then since

$$\mathcal{F}(1, m; \zeta_{n-1}(S)) = \begin{cases} \mathcal{F}(1, m; \zeta_n(S)) - \mathcal{F}(2, m; \zeta_{n-1}(T)), & \text{if } b_1 \geq 1, \\ \mathcal{F}(1, m; \zeta_n(S)) - \mathcal{F}(a_2+3, m; \zeta_{n-1}(T)), & \text{if } b_1 = 0, \end{cases}$$

we have that

$$\mathcal{F}(1, m; \zeta_{n-1}(S)) = \begin{cases} \mathcal{F}(2, m-1; \zeta_{n-1}(T)) - \mathcal{F}(2, m; \zeta_{n-1}(T)), & \text{if } b_1 \geq 1, \\ \mathcal{F}(a_2+3, m-1; \zeta_{n-1}(T)) - \mathcal{F}(a_2+3, m; \zeta_{n-1}(T)), & \text{if } b_1 = 0. \end{cases}$$

Applying (3.3) and inducting on a_1 for fixed S (separating the two cases above) as in the proof of Theorem 11 yields the result. □

4. Alternating Zetas and κ-Duality

In the previous section we considered evaluations of $\mathcal{F}(k,m;g_n)$ where $g_n=\zeta_n(S)$ is an outer-partial sum of a multiple zeta value. It is natural to ask whether similar evaluations exist for alternating zeta values, such as $-\ln 2=\sum_{n=1}^{\infty}(-1)^n/n$. Considering $\mathcal{F}(1,m;(-1)^n)$ leads us to the pair of equations

$$\frac{1}{2^m}\mathcal{F}(1,m;(-1)^n)=\sum_{n>m}\frac{1}{n2^n}=-\ln 2+\sum_{n=1}^{m}\frac{1}{n2^n}, \tag{4.1}$$

and

$$(-1)^m\mathcal{F}(1,m;(1/2)^n)=-\sum_{n>m}\frac{(-1)^n}{n}=-\ln(1/2)+\sum_{n=1}^{m}\frac{(-1)^n}{n}. \tag{4.2}$$

The pair $(-1,1/2)$ is reminiscent of the so-called *kappa-to-unit-Euler* duality formula [3, p. 154, Thm. 3.8]. To work with these more general functions, it is convenient to define the multivariate ζ-function

$$\zeta(\mathbf{x};a_1,\dots,a_k):=\sum_{n_1>\cdots>n_k>0}\prod_{i=1}^{k}\frac{x_j^{n_i}}{n_i^{a_j}}, \tag{4.3}$$

where $\mathbf{x}=(x_1,\cdots,x_k)$, as well as the multivariate analogues of (3.1):

$$\zeta_m(\mathbf{x};S):=\sum_{m\geq n_1>\cdots>n_k>0}\prod_{i=1}^{k}\frac{x_i^{n_i}}{n_i^{a_i}},\qquad \iota_m(\mathbf{x};S):=\sum_{n_k=1}^{m}\sum_{n_1>\cdots>n_k}\prod_{i=1}^{k}\frac{x_i^{n_i}}{n_i^{a_i}}. \tag{4.4}$$

Continuing to follow convention, we may drop the vector $\mathbf{x}$ if all the x_i's are ±1, and indicate the positions of the -1s by placing a bar over the corresponding a_j. For example, $\zeta((-1,1,1);\{3\}^2,1)=\zeta(\bar{3},3,1)$. We also denote by $\kappa(S)$ the MZV $\zeta((\frac{1}{2},1,1,\dots,1);S)$.

In this notation, the kappa-to-unit-Euler duality formula states

$$\kappa(a_1,\dots,a_k)=(-1)^k\zeta((\tau_1,\tau_2/\tau_1,\dots,\tau_s/\tau_{s-1});\{1\}^s), \tag{4.5}$$

where $s=a_1+\cdots+a_k$ and $(\tau_1,\dots,\tau_s)=(-1,\{1\}^{a_k-1},-1,\{1\}^{a_{k-1}-1},\dots,-1,\{1\}^{a_1-1})$. In particular, we have the following analogue of (3.14).

$$\begin{aligned}&\kappa(a_1+2,\{1\}^{b_1},\dots,a_k+2,\{1\}^{b_k})=\\&\qquad(-1)^{b_1+\cdots+b_k+k}\zeta(\bar{1},\{1\}^{b_k},\bar{1},\{1\}^{a_k},\dots,\bar{1},\{1\}^{b_1},\bar{1},\{1\}^{a_1}).\end{aligned} \tag{4.6}$$

It is not surprising, then, that we can derive a κ-analogue of Theorem 14.

Theorem 15 (κ duality)**.** *Let a_1, …, a_N, and m be non-negative integers, $s=a_1+\cdots+a_N$, $(\tau_1,\dots,\tau_s)=(-1,\{1\}^{a_N-1},-1,\{1\}^{a_{N-1}-1},\dots,-1,\{1\}^{a_1-1})$, and $\mathbf{x}=(\tau_1,\tau_2/\tau_1,\dots,\tau_s/\tau_{s-1})$. Then we have*

$$(-1)^N\mathcal{F}(a_1,m;2^{-n}\zeta_{n-1}(a_2,\dots,a_N))$$

$$= \begin{cases} (-1)^m \left(\zeta(\mathbf{x}; \{1\}^s) - \iota_m(\mathbf{x}; \{1\}^s)\right), & \textit{if } a_1 = 1, \\ \zeta(\mathbf{x}; \{1\}^s) - \iota_m(\mathbf{x}; \{1\}^s), & \textit{if } a_1 > 1. \end{cases} \tag{4.7}$$

Proof. We induct on N. The base cases

$$(-1)^{m+1}\mathcal{F}(1, m; 2^{-n}) = \zeta(\bar{1}) - \iota_m(\bar{1}),$$

and

$$-\mathcal{F}(a_1, m; 2^{-n}) = \zeta(\bar{1}, \bar{1}, \{1\}^{a_1-2}) - \iota_m(\bar{1}, \bar{1}, \{1\}^{a_1-2})$$

follow from (4.2) and (3.3). Suppose now that (4.7) holds for all $1 \le N < N'$. We first consider the case $a_1 = 1$. In this case, $(\tau_1, \dots, \tau_s) = (-1, \{1\}^{a_{N'}-1}, \dots, -1, \{1\}^{a_2-1}, -1)$. Set $\mathbf{x} = (\tau_1, \tau_2/\tau_1, \dots, \tau_s/\tau_{s-1})$ and $\mathbf{x}' = (\tau_1, \tau_2/\tau_1, \dots, \tau_{s-1}/\tau_{s-2})$. Note that we have the identity

$$\begin{aligned} &\mathcal{F}(1, m; 2^{-n}\zeta_{n-1}(a_2, \dots, a_{N'})) = \\ &\quad - \mathcal{F}(1, m-1; 2^{-n}\zeta_{n-1}(a_2, \dots, a_{N'})) + \frac{1}{m}\mathcal{F}(a_2, m; 2^{-n}\zeta_{n-1}(a_3, \dots, a_{N'})). \end{aligned}$$

If $a_2 = 1$, then $\tau_s = \tau_{s-1} = -1$ so $\mathbf{x} = (\mathbf{x}', 1)$. Thus,

$$\begin{aligned} &(-1)^m\mathcal{F}(1, m; 2^{-n}\zeta_{n-1}(a_2, \dots, a_{N'})) \\ &\quad = -\mathcal{F}(1, 0; 2^{-n}\zeta_{n-1}(1, a_3, \dots, a_{N'})) + \sum_{j=1}^{m} \frac{(-1)^j}{j}\mathcal{F}(1, j; 2^{-n}\zeta_{n-1}(a_3, \dots, a_{N'})) \\ &\quad = -\mathcal{F}(1, 0; 2^{-n}\zeta_{n-1}(1, a_3, \dots, a_{N'})) \\ &\qquad + \sum_{j=1}^{m} \frac{(-1)^{N'-1}}{j} \left(\zeta(\mathbf{x}', \{1\}^{s-1}) - Z_j(\mathbf{x}', \{1\}^{s-1})\right) \\ &\quad = -\mathcal{F}(1, 0; 2^{-n}\zeta_{n-1}(1, a_3, \dots, a_{N'})) - (-1)^{N'}\iota_m(\mathbf{x}, \{1\}^s), \end{aligned}$$

where we used the inductive hypothesis in the penultimate step. Letting m tend to infinity and rearranging yields

$$(-1)^{N'+m}\mathcal{F}(1, m; 2^{-n}\zeta_{n-1}(a_2, \dots, a_{N'})) = \zeta(\mathbf{x}, \{1\}^s) - \iota_m(\mathbf{x}, \{1\}^s)$$

as desired. If $a_2 > 1$, then $\tau_{s-1} = 1$ so $\mathbf{x} = (\mathbf{x}', -1)$. In this case the $(-1)^j$ does not cancel, so that

$$\begin{aligned} &(-1)^m\mathcal{F}(1, m; 2^{-n}\zeta_{n-1}(a_2, \dots, a_{N'})) \\ &\quad = -\mathcal{F}(1, 0; 2^{-n}\zeta_{n-1}(1, a_3, \dots, a_{N'})) \\ &\qquad + \sum_{j=1}^{m} \frac{(-1)^{N'-1}(-1)^j}{j} \left(\zeta(\mathbf{x}', \{1\}^{s-1}) - Z_j(\mathbf{x}', \{1\}^{s-1})\right) \\ &\quad = -\mathcal{F}(1, 0; 2^{-n}\zeta_{n-1}(1, a_3, \dots, a_{N'})) - (-1)^{N'}\iota_m(\mathbf{x}, \{1\}^s), \end{aligned}$$

which again is what we want. To deduce the general case when $a_1 > 1$ one simply applies the recurrence (3.3) as in the proof of the base case. □

One may deduce formulas for the general multivariate ζ-function $\zeta(\mathbf{x}; a_1, \dots, a_k)$ using the inductive procedure as we have done above and the identity

$$\mathcal{F}(1, m; x^n \zeta_{n-1}(\mathbf{y}; a_2, \dots, a_N)) = \left(\frac{x-1}{x}\right) \mathcal{F}(1, m-1; x^n \zeta_{n-1}(\mathbf{y}; a_2, \dots, a_N)) - \frac{1}{m} \mathcal{F}(a_2, m; (xy)^n \zeta_{n-1}(\mathbf{y}'; a_3, \dots, a_N)), \tag{4.8}$$

with $\mathbf{y} = (y, y_2, \dots, y_{N-1})$ and $\mathbf{y}' = (y_2, \dots, y_{N-1})$ for values of the parameters such that both sides converge. As the general formulas obtained from this procedure are not as elegant, we do not work them out here. We remark, however, that the simplest case (where $N = 1$) is a special case of Pfaff's Transformation

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) = (1-x)^{-a} {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix} ; \frac{x}{x-1} \right), \tag{4.9}$$

with $a = b = 1$ and $c = m + 2$. This is not surprising, since both Pfaff's Transformation and MZV Duality can be deduced from the same type of change-of-variable transformation applied to an appropriate integral representation. See [1, p. 68] and [3, p. 153].

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