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A CLOSED FORM FOR THE DENSITY FUNCTIONS OF RANDOM WALKS IN ODD DIMENSIONS

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Abstract

We derive an explicit piecewise-polynomial closed form for the probability density function of the distance traveled by a uniform random walk in an odd-dimensional space, based on recent work of Borwein, Straub, and Vignat [1] and by R. García-Pelayo [3].

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1. Preliminaries

In [1], the authors explore the distance traveled by a uniform n-step random walk in $\mathbb{R}^d$ with unit step length. Following their lead, we denote the probability density function of this distance by $p_n(m-1/2; x)$, where $m = \frac{d-1}{2}$.
We recall that the density can be expressed in terms of an integral engaging the *normalized Bessel function of the first kind* of order ν, defined by

$$j_\nu(x) = \nu! \left(\frac{2}{x}\right)^\nu J_\nu(x) = \nu! \sum_{k\geq 0} \frac{(-x^2/4)^k}{k!(k+\nu)!}. \tag{1.1}$$

With this normalization, we have $j_\nu(0) = 1$ and obtain:

Theorem 1 (Bessel representation [1, 4]). *The probability density function of the distance to the origin in $d \geq 2$ dimensions after $n \geq 2$ steps is, for $x > 0$,*

$$p_n(m-1/2; x) = \frac{2^{-m+1/2}}{\Gamma(m+1/2)} \int_0^\infty (tx)^{m+1/2} J_{m-1/2}(tx) j_{m-1/2}^n(t)\, \mathrm{d}t, \tag{1.2}$$

wherein $m = \frac{d-1}{2}$.

The study of the density $p_n(\nu; x)$ is quite classical, originating in the early 20th century [2, 4–7]. The most fundamental cases are that of two dimensions [2] and three dimensions [7]. The Bessel representation of the density is valuable for its generality and its analytically-pleasing structure, which form the basis for many related results [1, 4]. Additionally, when Theorem 1 is used for half-integer m, one can symbolically integrate any given small-order case, although the structure of the closed form is obscured in the process.

While some probabilistic results such as Theorem 1 hold in all dimensions, many arithmetic and analytic results are distinct between odd and even dimensions. Indeed, even dimensional results often involve elliptic integrals [1, 2], while odd dimensional results are typically resolvable in terms of elementary functions. For instance, noting that $j_{1/2}(x) = \operatorname{sinc}(x) = \sin(x)/x$ partly explains why analysis in three-dimensional space is relatively simple. More generally, $j_\nu(x)$ is elementary when ν is a proper half-integer [1, 4, 7]. In light of this discussion, it is striking that the next result is very recent.

Theorem 2 (Convolution formula for density in odd dimensions [3]). *Assume that the dimension $d = 2m+1$ is an odd number. Then for $x \geq 0$,*

$$p_n(m-1/2;x) = \frac{(2x)^{2m}\Gamma(m)}{\Gamma(2m)}\left(-\frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_{m,n}(x) \tag{1.3}$$

where $P_{m,n}$ is the piecewise polynomial obtained from convolving

$$f_m(x) := \frac{\Gamma(m+1/2)}{\Gamma(1/2)\Gamma(m)}\begin{cases}\left(1-x^2\right)^{m-1} & \text{if } x \in [-1,1] \\ 0 & \text{otherwise}\end{cases}$$

$n-1$ times with itself.

The expression in Theorem 2 above is both elegant and compact. It shows easily that in odd dimensions the density is a piecewise polynomial, but it can be difficult to manipulate or compute with or without a computer algebra system such as *Maple* or *Mathematica*. Note also that $p_n(m-1/2;x) = p_n(m-1/2;-x)$ in all cases.

2. Main result

We now use Theorem 2 to obtain an entirely explicit and tractable, convolution and differentiation free formula for $p_n(m-1/2;x)$, valid for all lengths and in all odd dimensions. We begin with a preliminary result which simplifies $P_{m,n}(x)$. We shall employ the *Heaviside* step function $H(x)$ which has $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$, and $H(0) = 1/2$. We also use the notation $[x^j]Q(x)$ to denote the coefficient of x^j in a polynomial Q.

Proposition 3. *Let $n \geq 1$ and $m \geq 1$. Then for $|x| \leq n$ we have $P_{m,n}(x) =$*

$$\left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}H(n-2r+x)\sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}[x^j]C_m(x)^r C_m(-x)^{n-r} \tag{2.1}$$

where

$$C_m(x) := \sum_{k=0}^{m-1}\frac{(m-1+k)!}{2^k k!(m-1-k)!}x^k. \tag{2.2}$$

Note that $C_m(x)$ satisfies the useful recurrence

$$C_m(x) = (2m-3)\, x\, C_{m-1}(x) + C_{m-2}(x).$$

Moreover, in terms of hypergeometric functions $C_m(x) = {}_2\mathrm{F}_0(m, 1-m;\ ;\ -x/2)$.

PROOF. By the convolution theorem for the Fourier transform,

$$\mathcal{F}\left(P_{m,n}(x)\right) = \mathcal{F}\left(f_m(x)\right)^n = \left(\frac{\Gamma(m+1/2)}{\Gamma(1/2)\Gamma(m)} \int_{-1}^{1} \left(1-x^2\right)^{m-1} e^{-iwx} dx\right)^n.$$

Observe that, for $m \geq 3$, $\mathcal{F}\left(f_m(x)\right)$ satisfies the recurrence

$$T_m = \frac{(2m-1)(2m-3)}{w^2}\left(T_{m-1} - T_{m-2}\right)$$

which is also satisfied by

$$G_m(w) := \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right) \sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!} (-1)^m \frac{2\cos(w + \frac{\pi}{2}(m+k))}{w^{m+k}}.$$

This can be checked by hand. It can also easily be shown with the following *Maple 18* code.

```
with(inttrans,fourier):
f:=m -> piecewise(-1<=x and x<=1,
   GAMMA(m+1/2)/(GAMMA(m)*GAMMA(1/2)) * (1-x^2)^(m-1),0):
F:=m -> fourier(f(m),x,w):
simplify(F(m)-(2*m-1)*(2*m-3)/w^2*(F(m-1)-F(m-2)));
```

The above code returns 0 to indicate that $\mathcal{F}\left(f_m(x)\right)$ satisfies the recurrence.
Correspondingly, we may execute the following *Maple 18* code.

```
G := m -> (GAMMA(2*m)/(2^m*GAMMA(m)))
     * sum( (m-1+k)!/(2^k*k!*(m-1-k)!)
     * (-1)^m * (2*cos(w+Pi/2*(m+k))/w^(m+k)), k=0..m-1 ):
simplify(G(m)-(2*m-1)*(2*m-3)/w^2*(G(m-1)-G(m-2)));
```

This returns 0 to show that $G_m(x)$ satisfies the same recurrence.
We can easily check that $\mathcal{F}\left(f_m(x)\right)$ and $G_m(x)$ agree for $m = 1$ and $m = 2$, and so we may conclude that $\mathcal{F}\left(f_m(x)\right) = G_m(x)$ for all $m \geq 1$.

Therefore,

$$\begin{aligned}
\mathcal{F}(P_{m,n}(x)) &= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \left(\sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot(-1)^m \frac{2\cos(w+\frac{\pi}{2}(m+k))}{w^{m+k}}\right)^n \\
&= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \left(\sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot(-1)^m \frac{e^{iw+i\frac{\pi}{2}(m+k)}+e^{-iw-i\frac{\pi}{2}(m+k)}}{w^{m+k}}\right)^n \\
&= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \left(\sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot(-1)^m \frac{(-1)^{m+k}e^{iw}+e^{-iw}}{(iw)^{m+k}}\right)^n \\
&= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \left(\sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot \frac{(-1)^m e^{iw}}{(-iw)^{m+k}} + \sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!(m-1-k)!}\cdot \frac{(-1)^m e^{-iw}}{(iw)^{m+k}}\right)^n \\
&= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \left(e^{iw}\left(\frac{1}{iw}\right)^m C_m\left(\frac{-1}{iw}\right) + e^{-iw}\left(\frac{-1}{iw}\right)^m C_m\left(\frac{1}{iw}\right)\right)^n
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n}\binom{n}{r}\frac{e^{iw(n-2r)}}{(iw)^{mn}}\left((-1)^m C_m\left(\frac{1}{iw}\right)\right)^r C_m\left(\frac{-1}{iw}\right)^{n-r} \\
&= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{(m-1)n}\frac{e^{iw(n-2r)}}{(iw)^{mn+j}}[x^j]C_m(x)^r C_m(-x)^{n-r}.
\end{aligned}$$

We can now reconstruct $P_{m,n}(x)$ from its Fourier transform, since

$$\mathcal{F}^{-1}\left(\frac{e^{iw(n-2r)}}{(iw)^{mn+j}}\right) = \frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x) - \frac{1}{2}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}.$$

Thus, taking the inverse Fourier transform of $\mathcal{F}(P_{m,n}(x))$,

$$\begin{aligned}
P_{m,n}(x) &= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x)[x^j]C_m(x)^r C_m(-x)^{n-r} \\
&+ \frac{1}{2}\left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}[x^j]C_m(x)^r C_m(-x)^{n-r}. \quad (2.3)
\end{aligned}$$

It remains only to show that the second term above is zero. Observe that when $x < -n$, $P_{m,n}(x)$ simplifies to

$$\frac{1}{2}\left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}[x^j]C_m(x)^r C_m(-x)^{n-r}. \quad (2.4)$$

From the definition of convolution, we can easily deduce that $P_{m,n}(x)$ vanishes for $|x| > n$. It follows that (2.4) is zero for $x < -n$, but since it is a polynomial it must be zero everywhere. Thus, the latter term in (2.3) is zero, yielding (2.1). □

Next, we deal with the differential operator in Theorem 2.

LEMMA 4. *For all $F(x)$ and $m \geq 1$,*

$$\left(-\frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^m F(x) = \sum_{k=1}^{m} \frac{(-1)^k(2m-1-k)!}{2^{2m-k}(m-k)!(k-1)!}\frac{1}{x^{2m-k}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k F(x). \tag{2.5}$$

PROOF. We proceed by induction. It is trivial to see that (2.5) is true for $m = 1$. Suppose it holds for some $m \geq 1$. Then

$$\begin{aligned}
\left(-\frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+1} F(x) &= \left(-\frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x}\right)\sum_{k=1}^{m} \frac{(-1)^k(2m-1-k)!}{2^{2m-k}(m-k)!(k-1)!}\frac{1}{x^{2m-k}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k F(x) \\
&= \sum_{k=1}^{m} \frac{(-1)^{k+1}(2m-1-k)!}{2^{2m-k+1}(m-k)!(k-1)!}\left(\frac{1}{x^{2m-k+1}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k+1} F(x) - \frac{2m-k}{x^{2m-k+2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k F(x)\right) \\
&= \sum_{k=2}^{m+1} \frac{(-1)^k(2m-k)!}{2^{2m-k+2}(m+1-k)!(k-2)!}\frac{1}{x^{2m-k+2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k F(x) \\
&\qquad + \sum_{k=1}^{m} \frac{(-1)^k(2m-k)!}{2^{2m-k+1}(m-k)!(k-1)!}\frac{1}{x^{2m-k+2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k F(x) \\
&= \sum_{k=1}^{m+1} \frac{(-1)^k(2m+1-k)!}{2^{2m+2-k}(m+1-k)!(k-1)!}\frac{1}{x^{2m+2-k}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k F(x).
\end{aligned}$$

Thus, (2.5) holds for all $m \geq 1$, proving the lemma. □

We are now ready to approach the probability density. Combining our previous results will allow us to fully expand $p_n(m-1/2;x)$.

THEOREM 5 (Densities in odd dimensions). *Let $n \geq 2$ and $m \geq 1$. Then for $x \geq 0$,*

$$\begin{aligned}
p_n(m-1/2;x) &= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}H(n-2r+x) \\
\times \sum_{k=1}^{m}(-2)^k\binom{m-1}{k-1}&\frac{(2m-1-k)!}{(2m-1)!}x^k \sum_{j=0}^{(m-1)n}\frac{(n-2r+x)^{mn-1+j-k}}{(mn-1+j-k)!}[x^j]C_m(x)^r C_m(-x)^{n-r}
\end{aligned} \tag{2.6}$$

where $H(x)$ is the Heaviside step function and

$$C_m(x) = \sum_{k=0}^{m-1}\frac{(m-1+k)!}{2^k k!(m-1-k)!}x^k. \tag{2.7}$$

Proof. By Theorem 2, Lemma 4, and Proposition 3, we arrive at

$$
\begin{aligned}
p_n(m-1/2;x) &= \frac{(2x)^{2m}\Gamma(m)}{\Gamma(2m)}\left(-\frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_{m,n}(x)\\
&= \frac{(2x)^{2m}\Gamma(m)}{\Gamma(2m)}\sum_{k=1}^{m}\frac{(-1)^k(2m-1-k)!}{2^{2m-k}(m-k)!(k-1)!}\frac{1}{x^{2m-k}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k P_{m,n}(x)\\
&= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n\sum_{k=1}^{m}(-2)^k\binom{m-1}{k-1}\frac{(2m-1-k)!}{(2m-1)!}x^k\\
&\times\sum_{r=0}^{n}\binom{n}{r}(-1)^{mr}\sum_{j=0}^{mn-n}[x^j]C_m(x)^rC_m(-x)^{n-r}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x).
\end{aligned}
$$

We can evaluate the derivative above directly, but we must be careful since there are jump discontinuities at $n-2r$ for $0\le r\le n$. We shall see that these points are not an issue. Applying the general Leibniz rule, we obtain

$$
\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x)\\
&=\sum_{a=0}^{k}\binom{k}{a}\left(\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^a\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}\right)\left(\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-a}H(n-2r+x)\right)\\
&=\sum_{a=0}^{k}\binom{k}{a}\frac{(n-2r+x)^{mn-1+j-a}}{(mn-1+j-a)!}\left(\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-a}H(n-2r+x)\right)
\end{aligned}
$$

We shall see that the terms of this sum vanish except when $a=k$. Suppose $a<k$ and consider one such term. Clearly, $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-a}H(n-2r+x)=0$ for $x\neq -n+2r$. Additionally, since $a<k\le m$ and $n\ge 2$ the exponent $mn-1+j-a$ is strictly positive, so $(n-2r+x)^{mn-1+j-a}=0$ at $x=-n+2r$. Thus, the summand above vanishes for $a<k$, yielding

$$
\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\frac{(n-2r+x)^{mn-1+j}}{(mn-1+j)!}H(n-2r+x)=\frac{(n-2r+x)^{mn-1+j-k}}{(mn-1+j-k)!}H(n-2r+x)
$$

We now apply this relation above and the result follows from a simple rearrangement. □

The formula we have presented is derived from the convolution form in Theorem 2 and produces an even function. However, $p_n(m-1/2;x)$ is the probability density function of a non-negative random variable, so it must be 0 for negative values of x. We may use this fact to significantly reduce the number of terms in our formula, halving the time needed to compute $p_n(m-1/2;x)$ for given values of n and m.

Corollary 6. *Let $n \geq 2$ and $m \geq 1$. Then for $x \geq 0$,*

$$p_n(m-1/2;x) = \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{r}(-1)^{mr} H(n-2r-x)$$
$$\times \sum_{k=1}^{m} 2^k \binom{m-1}{k-1} \frac{(2m-1-k)!}{(2m-1)!} x^k \sum_{j=0}^{(m-1)n} \frac{(n-2r-x)^{mn-1+j-k}}{(mn-1+j-k)!} [x^j] C_m(x)^r C_m(-x)^{n-r} \tag{2.8}$$

Proof. Since our formula 2.6 is even (easily seen in Theorem 2), for $x \geq 0$ we have

$$\begin{aligned} p_n(m-1/2;x) &= p_n(m-1/2;-x) \\ &= \left(\frac{\Gamma(2m)}{2^m\Gamma(m)}\right)^n \sum_{r=0}^{n} \binom{n}{r}(-1)^{mr} H(n-2r-x) \\ &\quad \times \sum_{k=1}^{m} 2^k \binom{m-1}{k-1} \frac{(2m-1-k)!}{(2m-1)!} x^k \sum_{j=0}^{(m-1)n} \frac{(n-2r-x)^{mn-1+j-k}}{(mn-1+j-k)!} [x^j] C_m(x)^r C_m(-x)^{n-r} \end{aligned}$$

by Theorem 5. Observe that when $r > \lfloor (n-1)/2 \rfloor$, $H(n-2r-x)$ is zero on $(0,\infty)$. At $x = 0$, every term is 0 for all values of r. Thus, when $x \geq 0$, we may simply omit the terms where $r > \lfloor (n-1)/2 \rfloor$. So we let r range from 0 to $\lfloor (n-1)/2 \rfloor$ in the sum, which yields our result directly. □

We finish with two examples echoing the direct analyses in [7]:

Example 7 (Density in three dimensions). *In $\mathbb{R}^3$, we have $C_1(x) = 1$ so for $n \geq 2$ and $x \geq 0$, the density reduces to*

$$p_n(1/2;x) = \frac{-x}{2^{n-1}} \sum_{r=0}^{n} \binom{n}{r}(-1)^r H(n-2r+x) \frac{(n-2r+x)^{n-2}}{(n-2)!}.$$

In particular, we have

$$p_2(1/2;x) = \begin{cases} 0 & \text{if } x < 0 \\ x/2 & \text{if } x \in [0,2) \\ 0 & \text{if } x > 2 \end{cases}$$

$$p_3(1/2;x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } x \in [0,1) \\ -\frac{1}{4}x^2 + \frac{3}{4}x & \text{if } x \in [1,3) \\ 0 & \text{if } x > 3 \end{cases}$$

$$p_4(1/2;x) = \begin{cases} 0 & \text{if } x < 0 \\ -\frac{3}{16}x^3 + \frac{1}{2}x^2 & \text{if } x \in [0,2) \\ \frac{1}{16}x^3 - \frac{1}{2}x^2 + x & \text{if } x \in [2,4) \\ 0 & \text{if } x > 4 \end{cases}$$

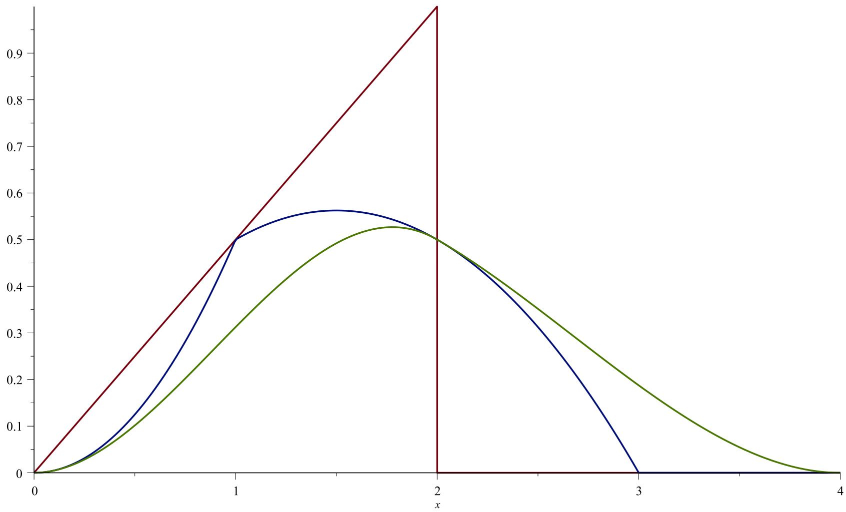

FIGURE 1. $p_n(1/2; x)$ for $n = 2, 3, 4$.

EXAMPLE 8 (Density in five dimensions). *In $\mathbb{R}^5$, we have $C_2(x) = 1 + x$ so for $n \geq 2$ and $x \geq 0$, the density reduces to*

$$p_n(3/2; x) = \left(\frac{3}{2}\right)^{n-1} \sum_{r=0}^{n} \binom{n}{r} H(n - 2r + x)$$
$$\times \sum_{j=0}^{n} \frac{(n - 2r + x)^{2n-3+j}}{(2n - 3 + j)!} \left(x^2 - x\frac{(n - 2r + x)}{(2n - 2 + j)}\right) \sum_{l=0}^{j} (-1)^{j-l} \binom{r}{l} \binom{n - r}{j - l}.$$

In particular, we have

$$p_2(3/2; x) = \begin{cases} 0 & \text{if } x < 0 \\ -\frac{3}{16}x^5 + \frac{3}{4}x^3 & \text{if } x \in [0, 2) \\ 0 & \text{if } x > 2 \end{cases}$$

$$p_3(3/2; x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{3}{560}x^8 - \frac{9}{40}x^6 + \frac{9}{16}x^4 & \text{if } x \in [0, 1) \\ -\frac{3}{1120}x^8 + \frac{9}{80}x^6 - \frac{9}{32}x^5 - \frac{9}{32}x^4 + \frac{81}{80}x^3 - \frac{243}{1120}x & \text{if } x \in [1, 3) \\ 0 & \text{if } x > 3 \end{cases}$$

As these examples demonstrate, Theorem 5 always provides an explicit, workable expression for $p_n(m - 1/2; x)$ with clearly indicated structure. We finish by observing that since the *moment function* is defined by $W_n(m-1/2, s) := \int_{x=0}^{n} x^s \, p_n(m-1/2; x) \mathrm{d}x$, we may also obtain an explicit formula for $W_n(m - 1/2, s)$.

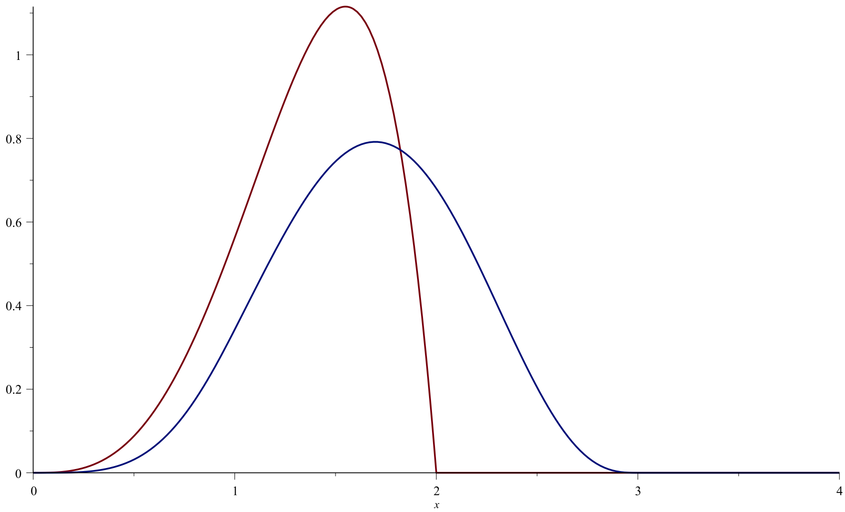

FIGURE 2. $p_n(3/2; x)$ for $n = 2, 3$.

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