The Life of π: History and Computation
A Talk for Pi Day or Other Days

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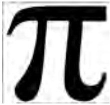
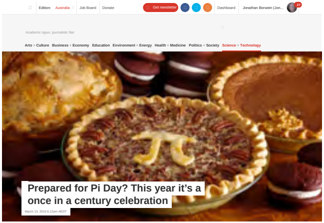
University of Newcastle
http://carma.newcastle.edu.au/jon/piday-16.pdf

3.14 pm, March 14, 2016 Revised 28.01.16 for *Western* 08.04.16



The Life of Pi: From this extended on line presentation we shall sample


Prepared for Pi Day? This year it's a once in a century celebration

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Something very special about this year's Pi Day. Flickr/Dennis Wilkinson, CC BY-NC-SA

Authors

Prepared for Pi Day_ This year it's a once in a century celebration.html[28/01/2016 2:23:44 PM]

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- Why Pi? From utility to ... normality.
- Recent computations and digit extraction methods.

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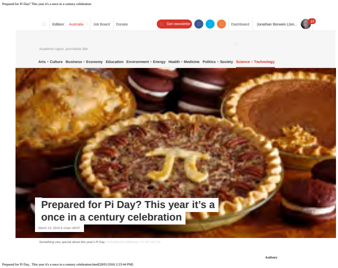

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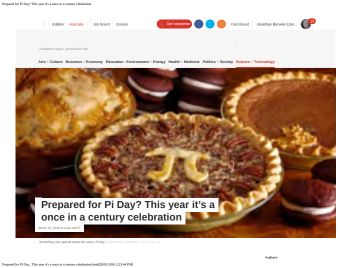

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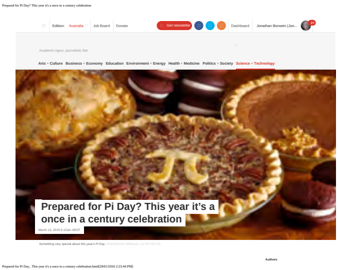

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- Why Pi? From utility to ... normality.
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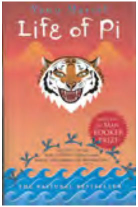
Outline. We will cover Some of:

IBM

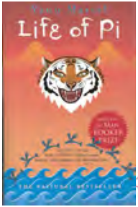
Introduction: Pi is ubiquitous

- The desire to understand π, the challenge, and originally the need, to calculate ever more accurate values of π, the ratio of the circumference of a circle to its diameter, has captured mathematicians — great and less great — for eons.
- And, especially recently, π has provided compelling examples of computational mathematics.



Pi, uniquely in mathematics,
is pervasive in popular culture
and the popular imagination.

In this talk I shall intersperse
a largely chronological account of π's
mathematical and numerical status
with examples of its ubiquity.



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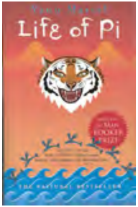

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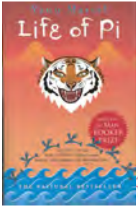

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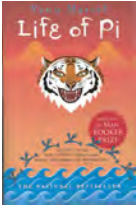

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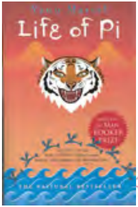

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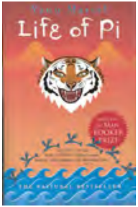

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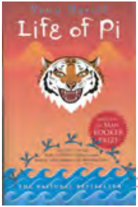
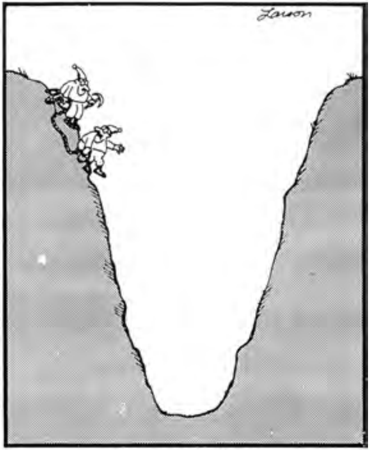


The Life of Pi Teaches a Great Deal:

We shall learn that scientists are humans and see a lot:

- of important mathematics;
- of its history and philosophy;
- about the evolution of computers and computation;
- of general history, philosophy and science;
- proof and truth (certainty and likelihood);
- of just plain interesting — sometimes *weird* — stuff.



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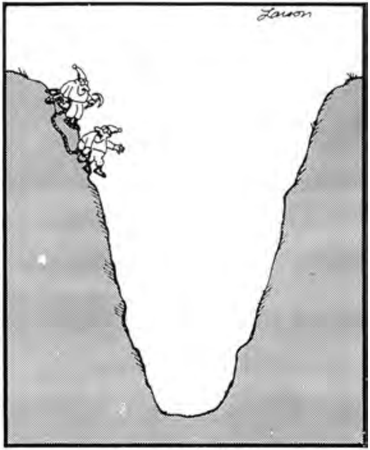

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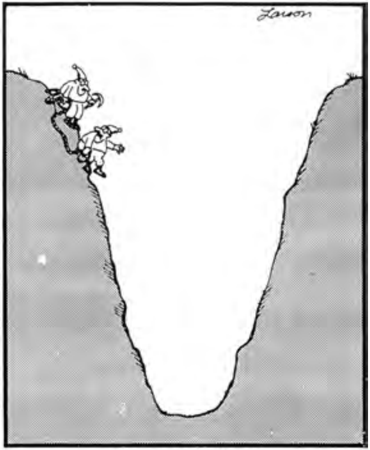

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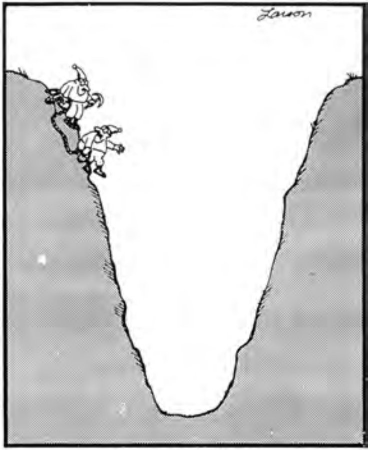

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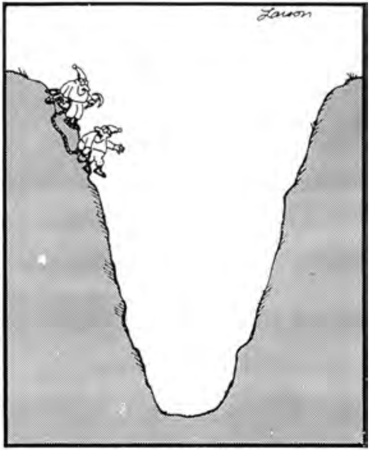

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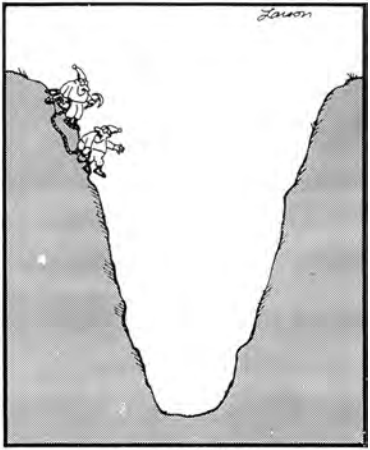

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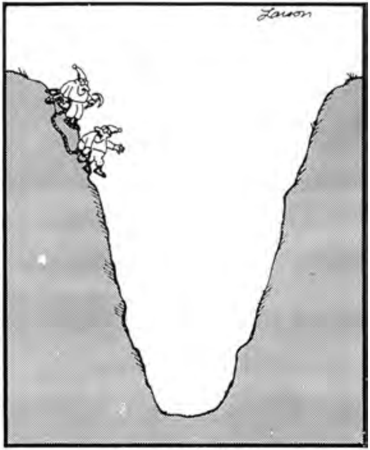

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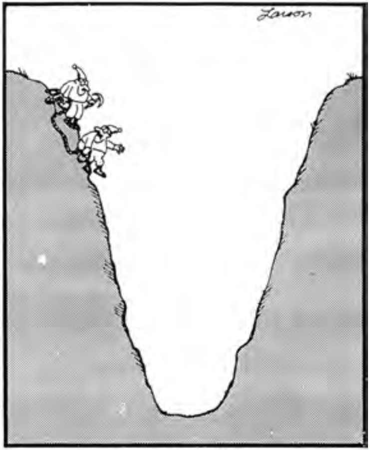

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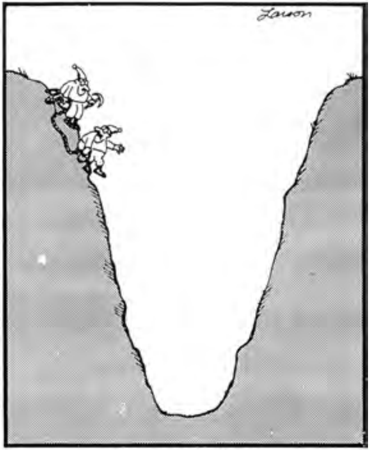

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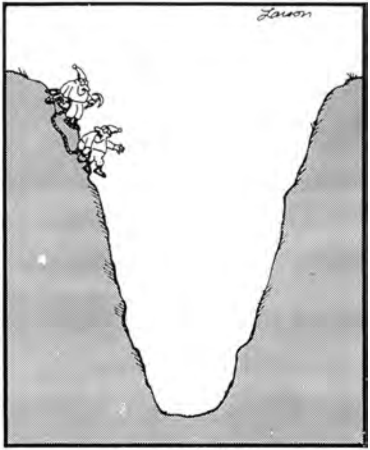

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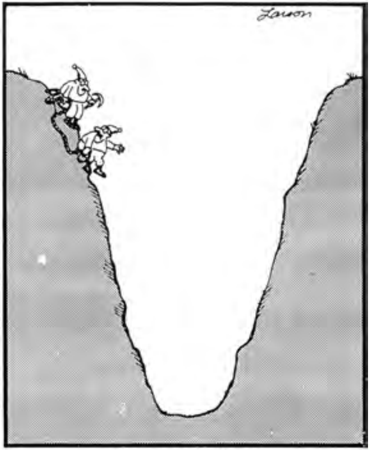

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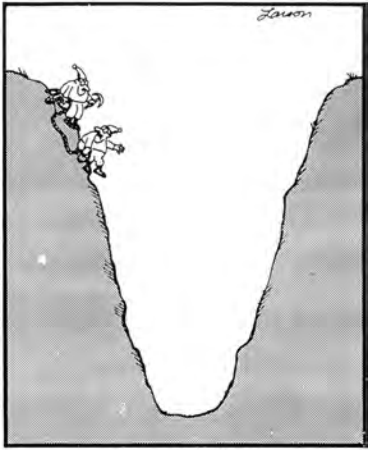

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Mnemonics for Pi Abound: Piems — Word lengths give digits

toc

"When you're young, it comes naturally, but when you get a little older, you have to rely on mnemonics."

Now I, even I, would celebrate
(**3 1 4 1 5 9**)
In rhymes inapt, the great
(**2 6 5 3 5**)
Immortal Syracusan, rivaled nevermore,
Who in his wondrous lore,
Passed on before
Left men for guidance
How to circles mensurate.

– punctuation is always ignored

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Life of Pi (2001): Yann Martel's 2002 Booker Prize novel starts

''My name is
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For good measure I added
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and I then drew a large circle
which I sliced in two with a
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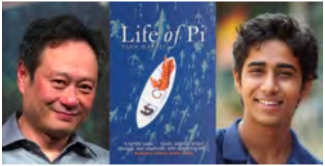
lesson of geometry.''



2013 Ang Lee's movie version (4 Oscars)

- **1706**. Notation of π introduced by William Jones.
- **1737**. Leonhard Euler (**1707-83**) popularized π.
  - One of the three or four greatest mathematicians of all times:
  - He introduced much of our modern notation: $\int, \Sigma, \phi, e, \Gamma, \ldots$.

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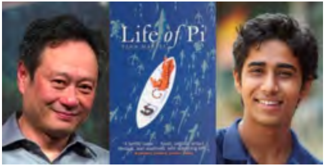

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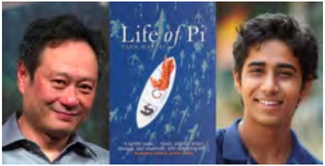

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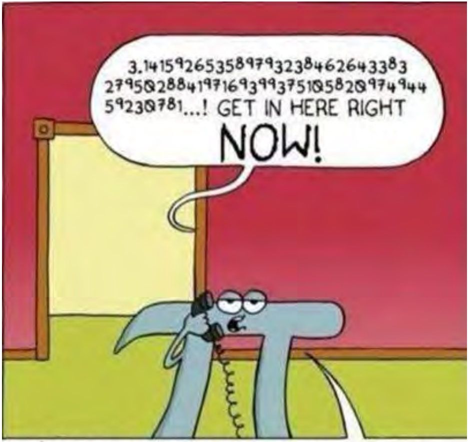
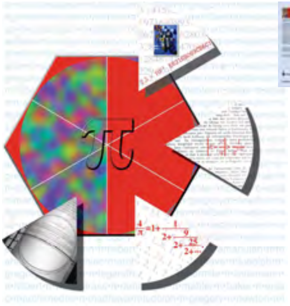
Wife of Pi (2013)

Life of Pi (2014)



I'VE GOT TO GO. MY MOM ONLY USES MY FULL NAME WHEN I'M IN BIG TOUBLE.

Pi: the Source Book (1997)



- Berggren, Borwein and Borwein, 3rd Ed, Springer, **2004**. (3,650 years of copyright releases. *E-rights* for Ed. 4 are in process.)
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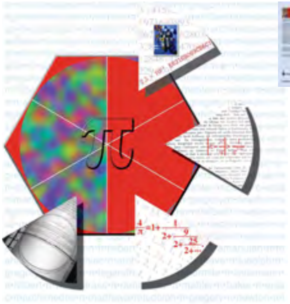

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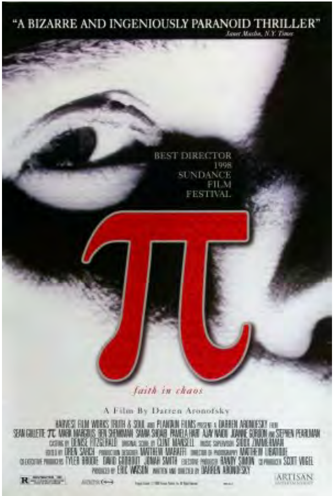
Pi: in The Matrix (1999)

Keanu Reeves, Neo, only has **314** seconds to enter "The Source."
(Do we need Parts 4 and 5?)

- From http://www.freakingnews.com/Pi-Day-Pictures--1860.asp

Pi the Movie (1998): a Sundance screenplay winner



Roger Ebert gave the film 3.5 stars out of 4: "Pi is a thriller. I am not very thrilled these days by whether the bad guys will get shot or the chase scene will end one way instead of another. You have to make a movie like that pretty skillfully before I care.

"But I am thrilled when a man risks his mind in the pursuit of a dangerous obsession."

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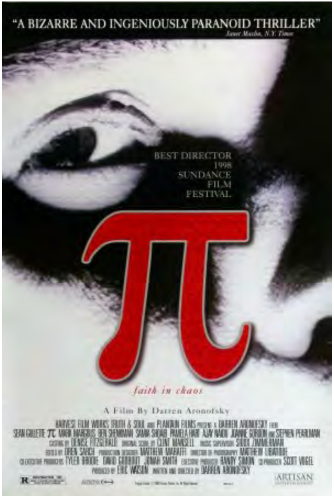

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Pi the **URL**

Pi to 1,000,000 places

Pi to one MILLION decimal places

3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679
8214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196
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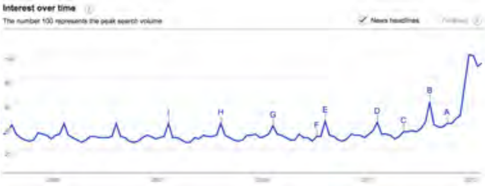
file:///C|/Documents%20and%20Settings/Jon%20Borwein/My%20Documents/Le...%20Opt%20Talks/Life%20of%20Pi%20Talk/Pi%20to%201,000,000%20places.htm (1 of 257)26/10/2005 10:23:03 PM

- From 3.14159265358979323846264338327950288419716939937510582097494459 2.com/
  This **2005 URL** seems to have *disappeared*.

2014: πDay turns **26**: Our book Pi and the AGM is **27**



- From `www.google.com/trends?q=Pi+`
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CARMA

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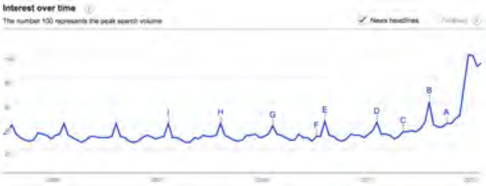

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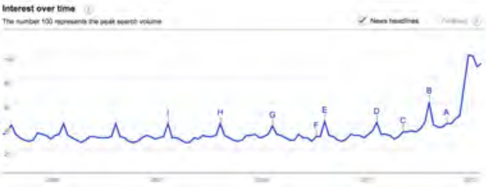

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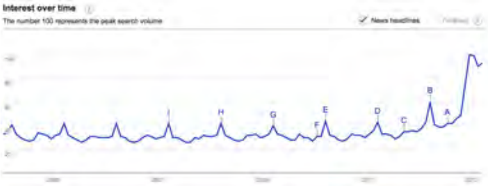

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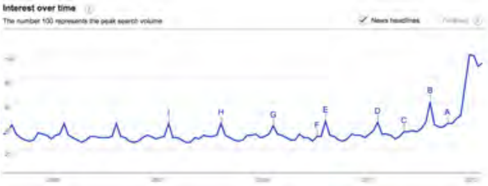

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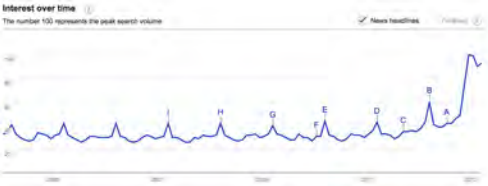

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Google Search for "Pi Day 2013" 345,000 hits (13-3-13)

1. Pi Day
www.timeanddate.com › Calendar › Holidays

Pi Day 2013. Thursday, March 14, 2013. Monday, July 22, 2013. Pi Day 2014. Friday, March 14, 2014. Tuesday, July 22, 2014. List of dates for other years **...**

2. News for **"Pi day 2013"**
3. **Celebrate Pi Day 2013 -- with Pie**
Patch.com - 8 hours ago

A perfect day for math geeks, Einstein lovers, and admirers of pie.

4. Celebrate **Pi Day 2013** with Fredericksburg Pizza

Patch.com - 22 hours ago

5. **Pi Day 2013**: A Celebration of the Mathematical Constant 3.1415926535...

Patch.com - 1 day ago

6. Celebrate **Pi Day 2013** -- with Pie - Millburn-Short Hills, NJ Patch
millburn.patch.com/.../celebrate-**pi-day-2013**-wit... - United States

9 hours ago – A perfect day for math geeks, Einstein lovers, and admirers of pie.

7. **Pi Day 2013**: A Celebration of the Mathematical Constant **...**
manassas.patch.com/.../**pi-day-2013**-a-celebration... - United States

2 days ago – March 14, or 3-14, is Pi Day – a day to celebrate the mathematical constant 3.14. What Pi Day activities do you have planned?

8. **"Pi" Day 2013** - FunCheapSF.com
sf.funcheap.com › City Guide

2 days ago – **Pi Day 2013** Any day can be a holiday, so why not look to math for some inspiration. Pi day (March 14th... 3/14... 3.14) seeks to celebrate π **...**

9. **Pi Day 2013** | Facebook
www.facebook.com/events/181240568664057/

Thu, 14 Mar - Everywhere, ,

Celebrate mathematics by celebrating Pi Day! Pi is the ratio of the circumference of a circle to its diameter (3.14159265...) For more info see: http://www.piday.org **...**

10. **Pi Day 2013**: Events, Activities, & History | Exploratorium
www.exploratorium.edu/learning_studio/pi/

Welcome to our twenty-fifth annual Pi Day! Help us celebrate this never-ending number (3.14159 . . .) and Einstein's birthday as well. On the afternoon of March ...

Crossword Pi — NYT March 14, 2007

- To solve the puzzle, first note that the clue for 28 `DOWN` is

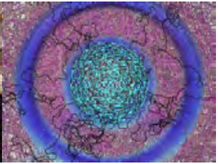
  `March 14, to Mathematicians`,

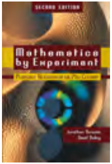
  to which the answer is `PIDAY`. Moreover, roughly a dozen other characters in the puzzle are π=**PI**.

- For example, the clue for 5 down was `More pleased` with the six character answer `HAP`π`ER`.

... — ...

Borweins and Plouffe | Pi Art | A Fine Book | Puzzle





(MSNBC Thanksgiving 1997)

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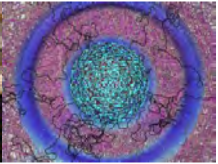

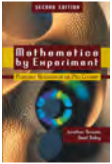

(**MSNBC** Thanksgiving 1997)

The Puzzle (**By Permission**)


The New York Times Crossword | Edited by Will Shortz | No. 0314

Across

Down

ANSWER TO PREVIOUS PUZZLE

The Puzzle Answered

ANSWER TO PREVIOUS PUZZLE

T	E	A	C	H		C	S	I	S		π	P	π	N
A	L	C	O	A		O	U	S	T		Z	O	N	E
R	E	T	O	P		N	L	E	R		Z	O	O	M
π	N	U	P	π	C	T	U	R	E		A	R	N	O
T	A	P		E	R	E		E	A	S	T			
			P	R	I	M	P		M	T	O	S	S	A
S	π	R	O		E	N	I	D		U	P	π	N	G
A	L	A	P		R	E	D	O	N		π	N	O	T
P	O	T	π	E		D	A	L	E		N	E	W	S
S	T	E	N	T	S		Y	O	U	N	G			
			G	A	T	O		M	R	I		S	π	N
O	K	A	π		O	π	N	I	O	N	π	E	C	E
P	U	π	L		W	A	I	T		J	E	R	K	S
U	R	A	L		E	T	T	E		A	T	I	L	T
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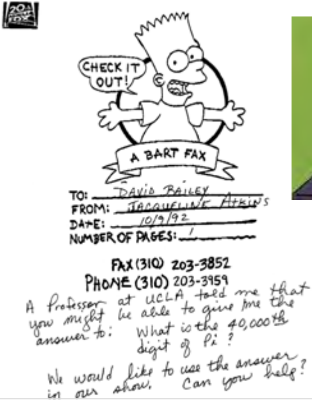
CARMA

The Simpsons (**Permission refused by Fox**)



CHECK IT OUT!

A BART FAX

TO: David Bailey
FROM: Jacqueline Atkins
DATE: 10/9/92
NUMBER OF PAGES: 1

FAX (310) 203-3852
PHONE (310) 203-3959

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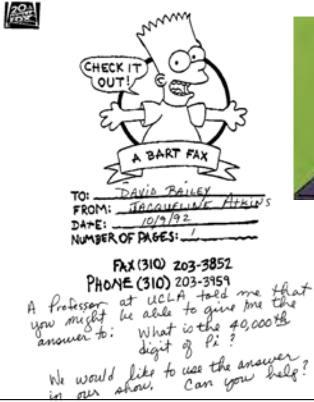
We would like to use the answer in our show. Can you help?

Apu: *I can recite pi to 40,000 places. The last digit is 1.* Homer: *Mmm... pie.* ("Marge in Chains," May 6, 1993)

– See also "Springfield Theory," (*Science News*, June 10, 2006) at www.aarms.math.ca/ACMN/links, Mouthful of Pi, http://tvtropes.org/pmwiki/pmwiki.php/Main/MouthfulOfPi and http://www.recordholders.org/en/list/memory.html#pi. The record is now over 80,000.

The Simpsons (**Permission refused by Fox**)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NUMBER OF PAGES: 1

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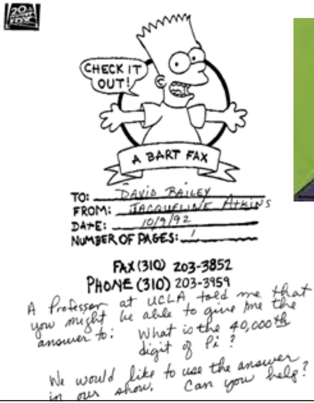

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National Pi Day **3.12.2009**: The first successful Pi Law

H.RES.224

Latest Title: Supporting the designation of Pi Day, and for other purposes.
Sponsor: Rep Gordon, Bart [TN-6] (introduced 3/9/2009) Cosponsors (15)
Latest Major Action: 3/12/2009 Passed/agreed to in House. Status: On motion to suspend the rules and agree to the resolution Agreed to by the Yeas and Nays: (2/3 required): 391 - 10 (Roll no. 124).

1985-2011. Gordon in Congress
2007- 2011. Chairman of House Committee on Science and Technology.

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National Pi Day? Congress makes it official

Caption: To celebrate Pi Day 2008, the San Francisco Exploratorium made a Pi string with more than 4,000 colored beads on it, each color representing a digit from 0 to 9.
(Credit: Daniel Terdiman/CNET)

Washington politicians took time from **bailouts** and **earmark-laden** spending packages on Wednesday for what might seem like an unusual act: officially designating a **National Pi Day**.

That's Pi as in ratio-of-a-circle's-circumference-to-diameter, better known as the mathematical constant beginning with 3.14159.

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Home > News > Politics and Law

March 17, 2009 5:01 PM PDT

National Pi Day? Congress makes it official

by Declan McCullagh

Print E-mail Share 20 comments

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Home > News > Politics and Law

March 11, 2009 5:01 PM PDT

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by Declan McCullagh

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CNN Pi Day **3.13.2010**: and Google (in North America)

EDITION: U.S. | INTERNATIONAL

CNN Tech

Home Video NewsPulse U.S. World Politics Justice Entertainment Tech Health L

On Pi Day, one number 'reeks of mystery'

By Elizabeth Landau, CNN
March 12, 2010 12:36 p.m. ESTMarch 12, 2010 12:36 p.m. EST

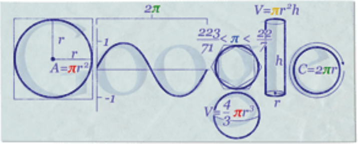
Sunday is Pi Day, on which math geeks celebrate the number representing the ratio of circumference to diameter of a circle.

STORY HIGHLIGHTS

- Pi Day falls on March 14, which is also Albert Einstein's birthday
- The true "randomness" of pi's digits -- 3.14 and so on -- has never been proven
- The U.S. House passed a resolution supporting Pi Day in March 2009

(CNN) -- The sound of meditation for some people is full of deep breaths or gentle humming. For Marc Umile, it's "3.14159265358979..."

Whether in the shower, driving to work, or walking down the street, he'll mentally rattle off digits of pi to pass the time. Holding 10th place in the world for pi memorization -- he typed out 15,314 digits from memory in 2007 -- Umile meditates through one of the most beloved and mysterious numbers in all of mathematics.



Google's homage to 3.14.10

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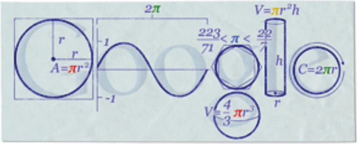

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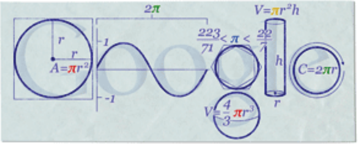
Sunday is Pi Day, on which math geeks celebrate the number representing the ratio of circumference to diameter of a circle.

STORY HIGHLIGHTS

- Pi Day falls on March 14, which is also Albert Einstein's birthday
- The true "randomness" of pi's digits -- 3.14 and so on -- has never been proven
- The U.S. House passed a resolution supporting Pi Day in March 2009

(CNN) -- The sound of meditation for some people is full of deep breaths or gentle humming. For Marc Umile, it's "3.14159265358979..."

Whether in the shower, driving to work, or walking down the street, he'll mentally rattle off digits of pi to pass the time. Holding 10th place in the world for pi memorization -- he typed out 15,314 digits from memory in 2007 -- Umile meditates through one of the most beloved and mysterious numbers in all of mathematics.



Google's homage to 3.14.10

Judge rules "Pi is a non-copyrightable fact" on **3.14.2012**

NewScientist Physics & Math

Home | Physics & Math | News

US judge rules that you can't copyright pi

18:15 16 March 2012 by Stephen Ornes

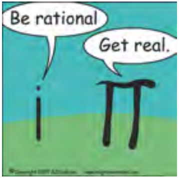
Video: What pi sounds like

The mathematical constant pi continues to infinity, but an extraordinary lawsuit that centred on this most beloved string of digits has come to an end. Appropriately, the decision was made on Pi Day.

On 14 March, which commemorates the constant that begins 3.14, US district court judge Michael H. Simon dismissed a claim of copyright infringement brought by one mathematical musician against another, who had also created music based on the digits of pi.

"Pi is a non-copyrightable fact, and the transcription of pi to music is a non-copyrightable idea," Simon wrote in his legal opinion dismissing the case. "The resulting pattern of notes is an expression that merges with the non-copyrightable idea of putting pi to music."

The bizarre tale began about a year ago, when Michael Blake of Portland, Oregon, released a song and YouTube video featuring an original musical composition, "What pi sounds like", translating the constant's first few dozen digits into musical notes. On Pi Day 2011, the number of page views skyrocketed as the video went viral. *New Scientist* was among those who



Two of many cartoons

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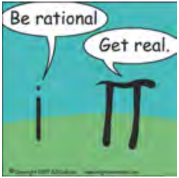

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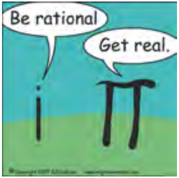


Two of many cartoons

Google (29-1-13) and US Gov't (14-8-12) still both love π

CSO

Hackers slug Aussies with trojans in ATO, air ticket spam

5 iPhone and iPad apps that invade your privacy, and 1 that doesn't

Google rounds up Pwnie prize to $π million for Chrome OS hacks

Google shoves Chrome OS in to the hacker spotlight

U.S. Population Reaches 314,159,265, Or Pi Times 100 Million: Census

The Huffington Post

The U.S. population has reached a nerdy and delightful milestone.

Shortly after 2:29 p.m. on Tuesday, August 14, 2012, the U.S. population was exactly 314,159,265, or Pi (π) times 100 million, the U.S. Census Bureau reports.

Pi (π) is a unique number in multiple ways. For one, it is the ratio of a circle's circumference to its diameter. It is also an irrational number, meaning it goes on forever without ever repeating itself. Are you remembering how much you loved geometry class? You can check out Pi to one million places here.

Contestants will be offered $110,000 for a successful exploit delivered by a web page that achieves a browser or system level compromise "in guest mode or as a logged-in user". A $150,000 prize will be offered for a "compromise with device persistence -- guest to guest with interim reboot, delivered via a web page".

CARMA

Each year brings more π-trivia and serious stuff

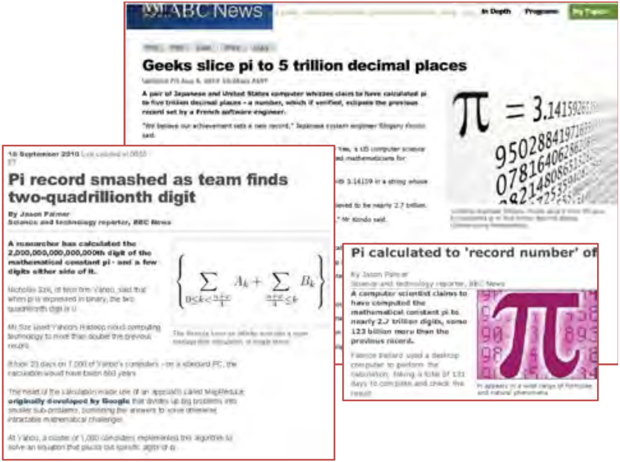
1. September 2014. *Pencil, Paper and Pi* or where Shanks computation went wrong

   http://www.americanscientist.org/issues/pub/2014/5/pencil-paper-and-pi
2. March 2015. J.M. Borwein and Scott T. Chapman, "I Prefer Pi: A Brief History and Anthology of Articles in the American Mathematical Monthly." **122** (2015), 195–216.
3. 22.10.14. A mile of Pi on one piece of paper

   http://www.youtube.com/watch?v=0r3cEKZiLmg&feature=youtu.be

π Records *Always* Make The News

More later



- By now you get the idea: π is everywhere ... also volumes, areas, lengths, probabilities, everywhere.

CARMA

26. Links and References

1. The Pi Digit site: `http://carma.newcastle.edu.au/bbp`
2. Dave Bailey's Pi Resources: `http://crd.lbl.gov/~dhbailey/pi/`
3. The Life of Pi: `http://carma.newcastle.edu.au/jon/pi-2012.pdf`.
4. Experimental Mathematics: `http://www.experimentalmath.info/`.
5. Dr Pi's brief Bio: `http://carma.newcastle.edu.au/jon/bio_short.html`.

. . . — . . .

1. D.H. Bailey and J.M. Borwein, "On Pi Day 2014, Pi's normality is still in question." *American Mathematical Monthly*, **121** March (2014), 191–204. (and H*uffington Post* 3.14.14 Blog)
2. D.H. Bailey, and J.M. Borwein, *Mathematics by Experiment: Plausible Reasoning in the 21st Century,* AK Peters Ltd, Ed 2, 2008, ISBN: 1-56881-136-5. See `http://www.experimentalmath.info/`
3. J.M. Borwein, "Pi: from Archimedes to ENIAC and beyond," in *Mathematics and Culture,* Einaudi, 2006. Updated 2012: `http://carma.newcastle.edu.au/jon/pi-2012.pdf`.
4. J.M. & P.B. Borwein, and D.A. Bailey, "Ramanujan, modular equations and pi or how to compute a billion digits of pi," *MAA Monthly*, **96** (1989), 201–219. Reprinted in *Organic Mathematics*, `www.cecm.sfu.ca/organics`, 1996, *CMS/AMS Conference Proceedings*, **20** (1997), ISSN: 0731-1036.
5. J.M. Borwein and P.B. Borwein, "Ramanujan and Pi," *Scientific American*, February 1988, 112–117. Also pp. 187-199 of *Ramanujan: Essays and Surveys,* Bruce C. Berndt and Robert A. Rankin Eds., AMS-LMS History of Mathematics, vol. 22, 2001.
6. Jonathan M. Borwein and Peter B. Borwein, *Selected Writings on Experimental and Computational Mathematics*, PsiPress. October 2010.[1]
7. L. Berggren, J.M. Borwein and P.B. Borwein, *Pi: a Source Book,* Springer-Verlag, (1997), (2000), (2004). Fourth Edition, in Press.

The Infancy of Pi: Babylon, Egypt and Israel

2000 BCE. Babylonians used the approximation $3\frac{1}{8} = 3.125$.

1650 BCE. Rhind papyrus: a circle of diameter *nine* has the area of a square of side *eight*:

$$\pi = \frac{256}{81} = 3.1604\ldots$$

– **Pi** is the only topic from the earliest strata of mathematics being actively researched today.

Some argue ancient Hebrews used $\pi = 3$:

Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about. (I Kings 7:23; 2 Chron. 4:2)

– More interesting is that Moses ben Maimon Maimonedes (the 'Rambam') (**1135-1204**) writes in *The true perplexity* that because of its nature "nor will it ever be possible to express it [π] exactly."

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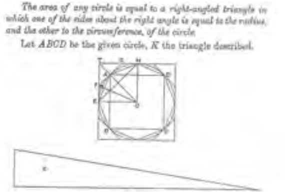
There are two Pi(es): Did they tell you?

Archimedes of Syracuse (c.**287 – 212 BCE**) was first to show that the "two Pi's" are one in *Measurement of the Circle* (c.**250 BCE**):

$$\text{Area} = \pi_1\, r^2 \text{ and Perimeter} = 2\, \pi_2\, r.$$



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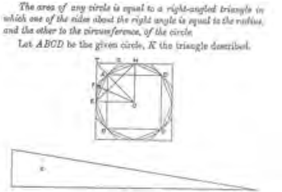
The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

Let $ABCD$ be the given circle, K the triangle described.

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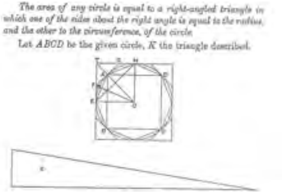
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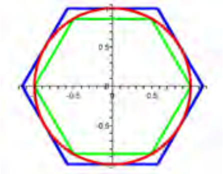
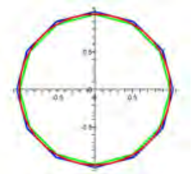
Archimedes Method circa 250 BCE

The first rigorous mathematical calculation of π was also due to Archimedes, who used a brilliant scheme based on doubling inscribed and circumscribed polygons

$$\mathbf{6} \mapsto \mathbf{12} \mapsto 24 \mapsto 48 \mapsto \mathbf{96}$$

to obtain the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$.





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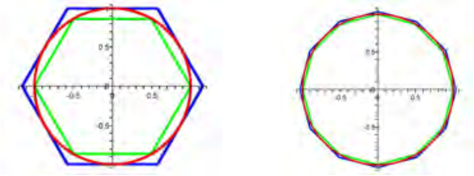

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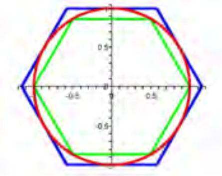

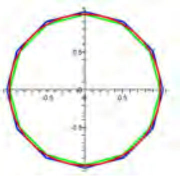

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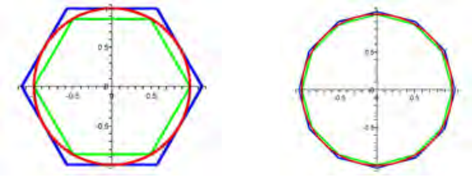

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Where Greece Was: Magna Graecia

▸ SKIP

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2. Troy
3. Byzantium
   Constantinople
4. Rhodes
   (Helios)
5. Hallicarnassus
   (Mausolus)
6. Ephesus
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7. Athens
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The others of the Seven Wonders: Lighthouse of Alexandria, Pyramids of Giza, Gardens of Babylon

CARMA

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CARMA

Archimedes Palimpsest (Codex C)

- **1906.** Discovery of a 10th-C **palimpsest** in Constantinople.
  - Sometime before April 14 **1229**, partially erased, cut up, and overwritten by religious text.
  - After **1929.** Painted over with gold icons and left in a wet bucket in a garden.
  - **1998.** Bought at auction for $2 million.
  - **1998-2008.** "Reconstructed" using very high-end mathematical imaging techniques.
  - Contained bits of 7 texts including Archimedes *On Floating Bodies* and *Method of Mechanical Theorems*, thought lost.

"Archimedes used knowledge of levers and centres of gravity to envision ways of balancing geometric figures against one another which allowed him to compare their areas or volumes. He then used rigorous geometric argument to prove *Method* discoveries."

- See Bernard Beauzamy, *Archimedes' modern works*, 2012.

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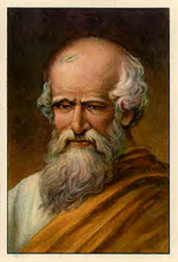
Archimedes from *The Method*

"… certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge."



Let's be Clear: π Really is not $\frac{22}{7}$

Even *Maple* or *Mathematica* 'knows' this since

$$0 \; < \; \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx \; = \; \frac{22}{7} - \pi, \tag{1}$$

though it would be prudent to ask 'why' it can perform the integral and 'whether' to trust it?

Assume we trust it. Then the integrand is strictly positive on $(0,1)$, and the answer in (1) is an area and so strictly positive, despite millennia of claims that π is $22/7$.

- Accidentally, $22/7$ is one of the early continued fraction approximation to π. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$

CARMA

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Archimedes Method circa **1800 CE**

As discovered — by Schwabb, Pfaff, Borchardt, Gauss — in the 19th century, this becomes a simple recursion:

Algorithm (Archimedes)

Set $a_0 := 2\sqrt{3}, b_0 := 3$. Compute

$$a_{n+1} = \frac{2a_nb_n}{a_n + b_n} \qquad (H)$$

$$b_{n+1} = \sqrt{a_{n+1}b_n} \qquad (G)$$

These tend to π, error decreasing by a *factor of four* at each step.

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$$\int_0^{t} \frac{x^4\,(1-x)^4}{1+x^2}\,dx \quad = \quad \frac{1}{7}\,t^7 - \frac{2}{3}\,t^6 + t^5 - \frac{4}{3}\,t^3 + 4\,t - 4\,\arctan(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus **proves** (1). **QED**

One can take this idea a bit further. Note that

$$\int_0^1 x^4\,(1-x)^4\,dx \quad = \quad \frac{1}{630}. \tag{2}$$

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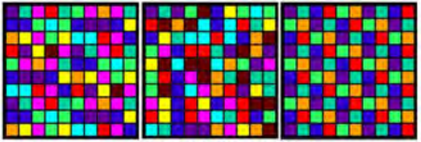
... Going Further

Hence

$$\frac{1}{2}\int_0^1 x^4(1-x)^4\,dx < \int_0^1 \frac{(1-x)^4x^4}{1+x^2}\,dx < \int_0^1 x^4(1-x)^4\,dx.$$



Archimedes: 223/71 < π < 22/7

Combine this with (1) and (2) to derive:

$$223/71 < 22/7 - 1/630 < \pi < 22/7 - 1/1260 < 22/7$$

and so re-obtain Archimedes' famous

$$\mathbf{3\frac{10}{71}} < \pi < \mathbf{3\frac{10}{70}}. \qquad (3)$$

CARMA

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Kuhnian 'Paradigm Shifts' and Normal Science

Variations of Archimedes' method were used for all calculations of π for **1800** years — well beyond its 'best before' date.

– **480CE**. In China Tsu Chung-Chih got π to *seven digits*.

1429. A millennium later, Al-Kashi in Samarkand — on the silk road — "*who could calculate as eagles can fly*" computed 2π in sexagecimal:

$$2\pi = 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9},$$

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Precalculus π Calculations

IBM

Name	Year	Digits
Babylonians	2000? BCE	1
Egyptians	2000? BCE	1
Hebrews (1 Kings 7:23)	550? BCE	1
Archimedes	250? BCE	3
Ptolemy	150	3
Liu Hui	263	5
Tsu Ch'ung Chi	480?	7
Al-Kashi	1429	14
Romanus	1593	15
Van Ceulen (**Ludolph's number***)	1615	35

* Used 2^{62}-gons for 39 places/35 correct — published posthumously.

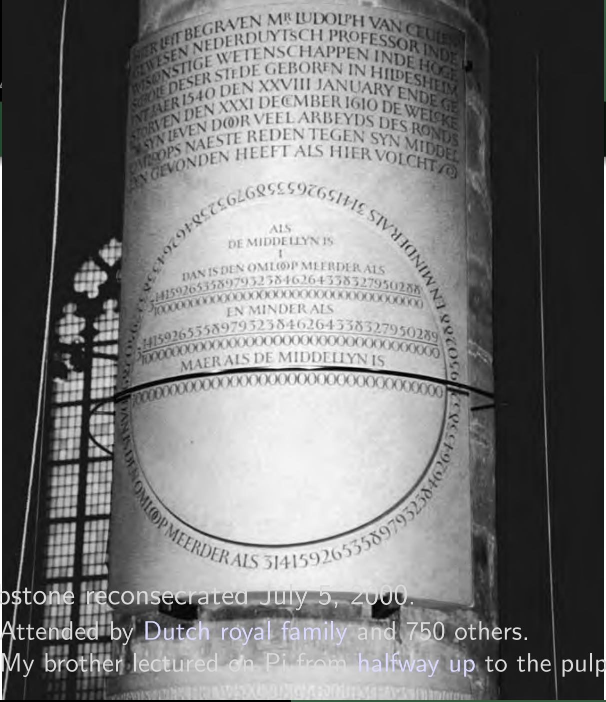
CARMA

Ludolph's Rebuilt Tombstone in Leiden

Ludolph van Ceulen (1540-1610)

- Destroyed several centuries ago; the plans remained.

Ludolph's



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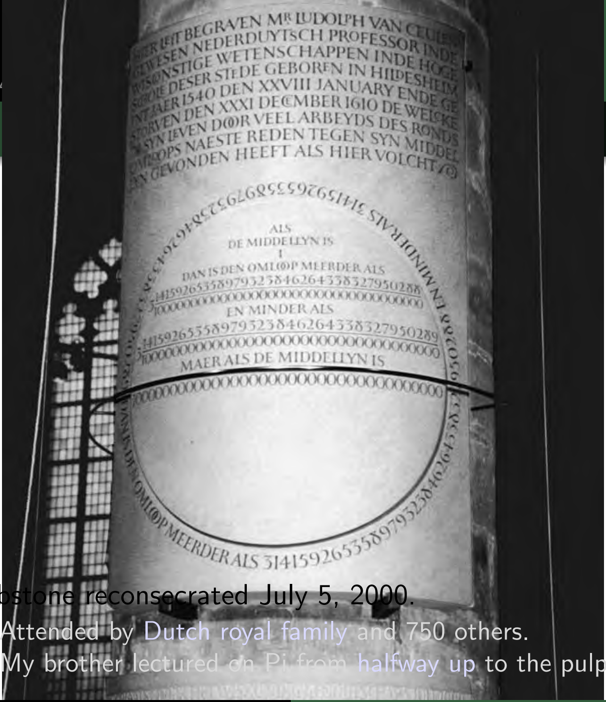

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- Still underestimated, this greatly enhanced arithmetic and mathematics in general, and computing π in particular.
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- See DHB & JMB, "Ancient Indian Square Roots: An Exercise in Forensic Paleo-Mathematics," *MAA Monthly*. 2012.
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Google Buys (Pi-3) $\times$ 100,000,000 Shares

Google

The New York Times
nytimes.com

August 19, 2005

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By JOHN MARKOFF

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CARMA

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CARMA

45. Pi's (troubled) Adolescence

1579. Modern mathematics dawns in *Viéte's product*

$$\frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots=\frac{2}{\pi} \tag{4}$$

— considered to be the *first truly infinite formula* — and in the *first continued fraction* given by Lord Brouncker (**1620-1684**):

$$\frac{2}{\pi}=\cfrac{1}{1+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cdots}}}}$$

Wallis Product

Eqn. (4) was based on John Wallis' (**1613-1706**) 'interpolated' product:

$$\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots=\prod_{k=1}^{\infty}\frac{4k^2-1}{4k^2}=\frac{2}{\pi} \tag{5}$$

which led to discovery of the *Gamma function* and much more.

- Christiaan Huygens (**1629-1695**) did not believe (5) before he checked it numerically.

> *It's a clue.*
> *A never repeating or ending chain, the total timeless catalogue, elusive sequences, sum of the universe.*
> *This riddle of nature begs:*
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Mathematical Interlude I: the Zeta Function

Formula (5) follows from Euler's product formula for π,

$$\frac{\sin(\pi x)}{x} = c \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \quad (6)$$

with $x = 1/2$, or by integrating $\int_0^{\pi/2} \sin^{2n}(t)\, dt$ by parts.

One may divine (6) — as Euler did — by *considering* $\sin(\pi x)$ *as an 'infinite' polynomial* and obtaining a product in terms of the roots $0, \{1/n^2\}$. Euler argued that, like a polynomial, $c = \pi$ is the value at 0.

The coefficient of x^2 in the Taylor series is the sum of the roots: $\zeta(2) := \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$. Hence, $\zeta(2n) =$ rational $\times \pi^{2n}$: so $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$ (using Bernoulli numbers)

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One may divine (6) — as Euler did — by *considering* $\sin(\pi x)$ *as an 'infinite' polynomial* and obtaining a product in terms of the roots $0, \{1/n^2\}$. Euler argued that, like a polynomial, $c = \pi$ is the value at 0.	The coefficient of x^2 in the Taylor series is the sum of the roots: $\zeta(2) := \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$. Hence, $\zeta(2n)$ = rational $\times \pi^{2n}$: so $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$ (using Bernoulli numbers)

1976. Apéry showed $\zeta(3)$ irrational; and Zudilin (CARMA) has shown *at least one of* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

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Arithmetic is absolutely as much science as geometry [is]. Rational magnitudes are conveniently designated by rational numbers, and irrational by irrational [numbers]. If someone measures magnitudes with numbers and by his calculation get them different from what they really are, it is not the reckoning's fault but the reckoner's.

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Pi's Adult Life with Calculus

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- **17C** Newton and Leibnitz discovered calculus ... and fought over priority (Machin adjudicated).
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One early use comes from the arctan integral and series:[3]

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt$$
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Formally $x := 1$ gives the Gregory–Leibniz formula (**1671–74**)

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- By contrast, Euler's (1738) trigonometric identity

$$\tan^{-1}(1) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \tag{7}$$

produces the geometrically convergent:

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An even faster formula, found earlier by John Machin — Brook Taylor's teacher — lies in the identity

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- Used in numerous computations of π (starting in **1706**) culminating with Shanks' computation of π to **707** decimals in **1873**.
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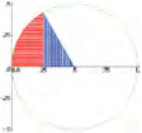
Isaac Newton's arcsin

Newton discovered a different (disguised arcsin) formula. He considered the area A of the red region to the right:



Now $A := \int_0^{1/4} \sqrt{x - x^2}\, dx$ equals the circular sector, $\pi/24$, less the triangle, $\sqrt{3}/32$. His new binomial theorem gave:

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Integrating term-by-term and combining the above:

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{2}{3 \cdot 8} - \frac{1}{5 \cdot 32} - \frac{1}{7 \cdot 512} - \frac{1}{9 \cdot 4096} \cdots\right).$$

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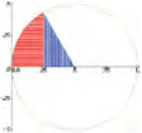

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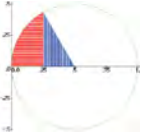

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Newton used his formula to find **15 digits** of π.

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Calculus π Calculations: and an IBM 7090

SKIP IBM

Name	Year	Digits
Sharp (and Halley)	1699	71
Machin	1706	100
Strassnitzky and Dase	1844	200
Rutherford	1853	440
W. Shanks	1874	(707) 527
Ferguson (**Calculator**)	1947	808
Reitwiesner et al. (**ENIAC**)	1949	2,037
Genuys	1958	10,000
D. Shanks and Wrench (**IBM**)	1961	100,265
Guilloud and Bouyer	1973	1,001,250

Why a Serial God Should Not Play Dice

Buffon (1707-78) & Ulam (1909-84)

Share the count to speed the process

An early vegetarian (who misused needles) next to the inventor of Monte Carlo methods.

1. Draw a unit square and inscribe a circle within: the area of the circle is $\frac{\pi}{4}$.

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Monte Carlo Methods

- This is a Monte Carlo estimate **(MC)** for π.
- **MC** simulation: slow ($\sqrt{n}$) convergence — but great in parallel on *Beowulf clusters*.
- Used in Manhattan project ... the atomic-bomb predates digital computers!

Gauss (1777-1855), Johan Dase and William Shanks



In his teens, Viennese *computer* and *'kopfrechner'* Dase (1824 -1861) publicly demonstrated his skill by multiplying

$$79532853 \times 93758479 = 7456879327810587$$

- in **54 seconds**; 20-digits in 6 min; 40-digits in 40 min; **100-digit** numbers in $8\frac{3}{4}$ **hours** etc.
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SKIP

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William Shanks (1812-82): "A Human Computer" (1853)

CONTRIBUTIONS TO MATHEMATICS,

COMPRISING CHIEFLY

THE

RECTIFICATION OF THE CIRCLE

TO 607 PLACES OF DECIMALS.

BY

WILLIAM SHANKS,

HOUGHTON-LE-SPRING, DURHAM.

LONDON:

G. BELL, 186, FLEET-STREET; MACMILLAN & Co., CAMBRIDGE; ANDREWS, DURHAM.

1853.

PRINTED FOR THE AUTHOR.

TOWARDS the close of the year 1850, the Author first formed the design of rectifying the Circle to upwards of 300 places of decimals. He was fully aware, at that time, that the accomplishment of his purpose would add little or nothing to his fame as a Mathematician, though it might as a Computer; nor would it be productive of anything in the shape of pecuniary recompense at all adequate to the labour of such lengthy computations. He was anxious to fill up scanty intervals of leisure with the achievement of something original, and which, at the same time, should not subject him either to great tension of thought, or to consult books. He is aware that works on nearly every branch of Mathematics are being published almost weekly, both in Europe and America; and that it has therefore become no easy task to ascertain what really is original matter, even in the pure science itself. Beautiful speculations, especially in both Plane and Curved

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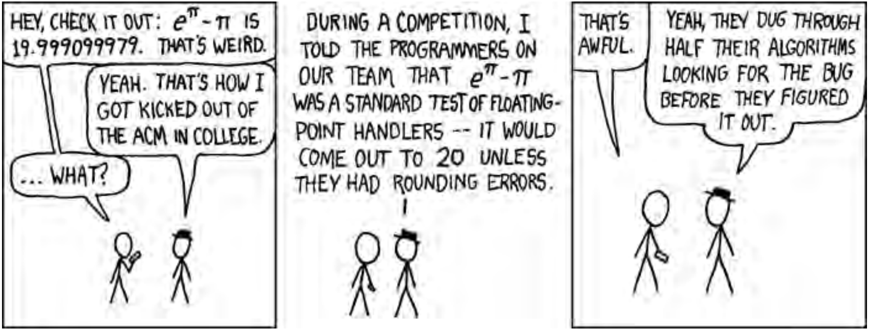

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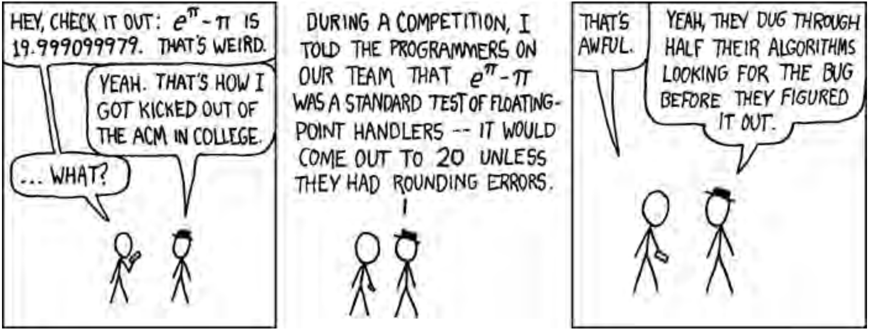

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Lambert (1728-77)

Legendre (1752-1833)

Lindemann (1852-1939)

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  Lambert showed $\arctan(x)$ is irrational when x is rational. Now set $x = 1/2$.

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The Three Construction Problems of Antiquity

The other two are doubling the cube and trisecting the angle

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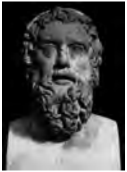

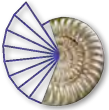

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CARMA

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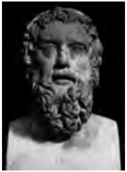

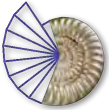

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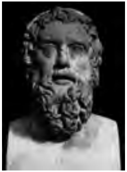

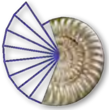

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τετραγωνιζειν

CARMA

The Irrationality of π, II

Ivan Niven's 1947 proof that π is irrational. Let $\pi = a/b$, the quotient of positive integers. We define the polynomials

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

the positive integer being specified later. Since $n!f(x)$ has integral coefficients and terms in x of degree not less than n, $f(x)$ and its derivatives $f^{(j)}(x)$ have integral values for $x = 0$; also for $x = \pi = a/b$, since $f(x) = f(a/b - x)$. By elementary calculus we have

$$\begin{aligned} & \frac{d}{dx}\{F'(x)\sin x - F(x)\cos x\} \\ = \quad & F''(x)\sin x + F(x)\sin x = f(x)\sin x \end{aligned}$$

The Irrationality of π, II

and

$$\int_0^\pi f(x)\sin x dx = [F'(x)\sin x - F(x)\cos x]_0^\pi = F(\pi) + F(0). \tag{10}$$

Now $F(\pi) + F(0)$ is an *integer*, since $f^{(j)}(0)$ and $f^{(j)}(\pi)$ are integers. But for $0 < x < \pi$,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (10) is *positive but arbitrarily small* for n sufficiently large. Thus (10) is false, and so is our assumption that π is rational. **QED**

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Richard Parker (L) and Pi Molitor
Ang Lee's 2012 film Life of Pi

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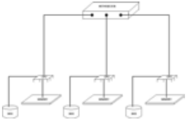
Summation. Why Pi? "Pi is Mount Everest."

What motivates modern computations of π — given that irrationality and transcendence of π were settled a century ago?

- One motivation is the raw challenge of harnessing the stupendous power of modern computer systems.



Programming is quite hard — especially on large, distributed memory computer systems: load balancing, communication needs, etc.

Substantial practical spin-offs accrue:

- Accelerating computations of π sped up the fast Fourier transform (FFT) — heavily used in science and engineering.
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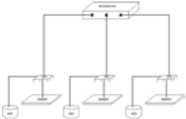

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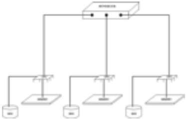

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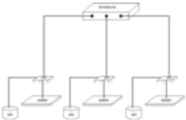

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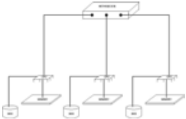

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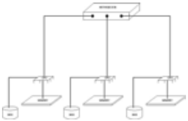

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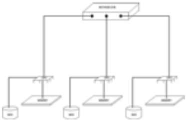

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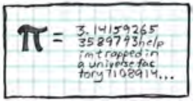

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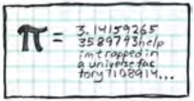

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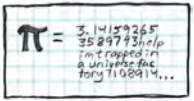

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  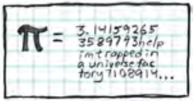

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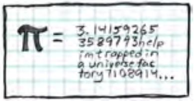

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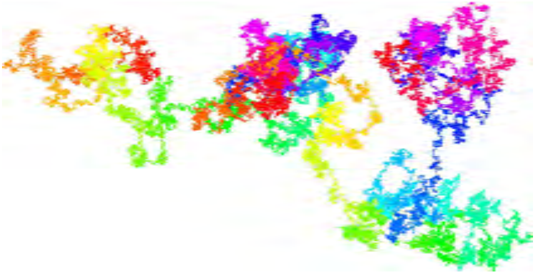
- We still know very little about the decimal expansion or continued fraction of π. We can not prove half of the bits of $\sqrt{2}$ are zero.

Pi Seems Normal: Things we sort of know about Pi

A walk on a billion hex digits of Pi with *box dimension* 1.85343...



- A 100Gb 100 billion step walk is at `http://carma.newcastle.edu.au/walks/`
- A Poisson inter-arrival time model applied to 15.925 trillion bits gives: probability Pi is not normal $< 1/10^{3600}$.

D. Bailey, J. Borwein, C. Calude, M. Dinneen, M. Dumitrescu, and A. Yee, "An empirical approach to the normality of pi." *Exp. Math.* **21**(4) (2012), 375–384. DOI 10.1080/10586458.2012.665333.

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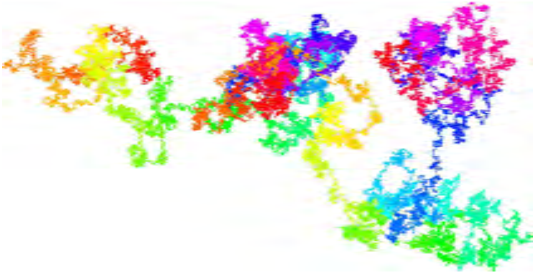

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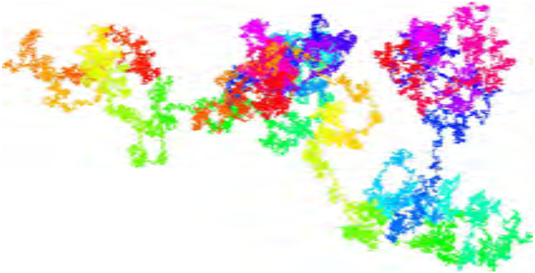

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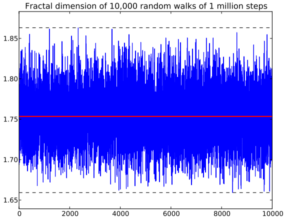
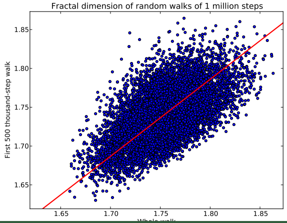
Pi Seems Normal: Some million bit comparisons

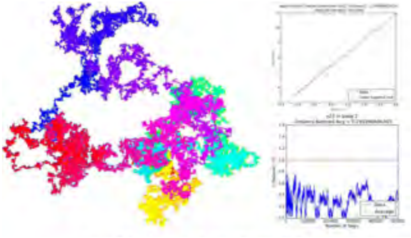
Euler's constant and a pseudo-random number

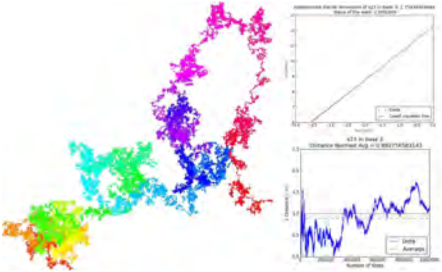




Pi Seems Normal: Comparisons to Stoneham's number $\sum_{k\geq 1} 1/(3^k 2^{3^k})$, I

In base 2 Stoneham's number is provably normal. It may be normal base 3.





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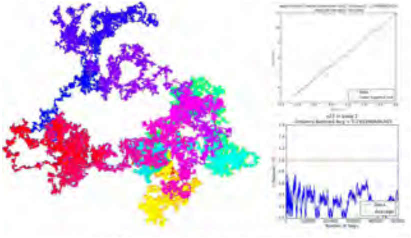
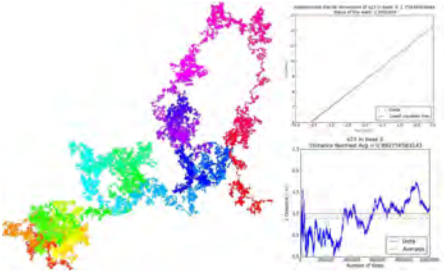
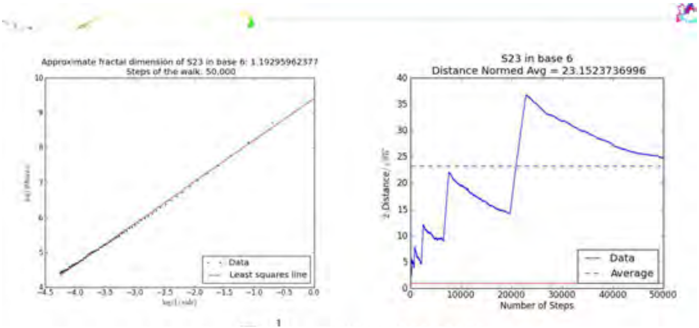




Pi Seems Normal: Comparisons to Stoneham's number, II

Stoneham's number is provably abnormal base 6 (too many zeros).



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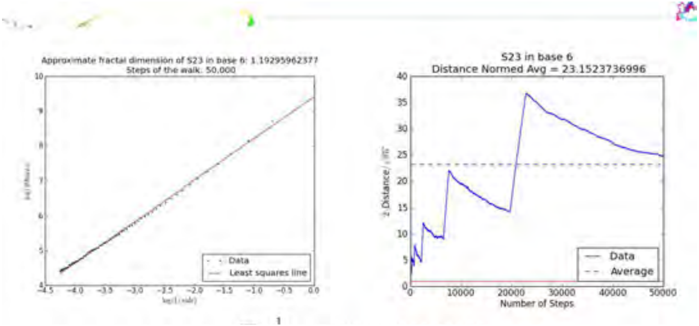
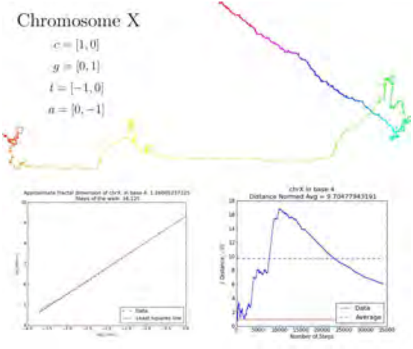
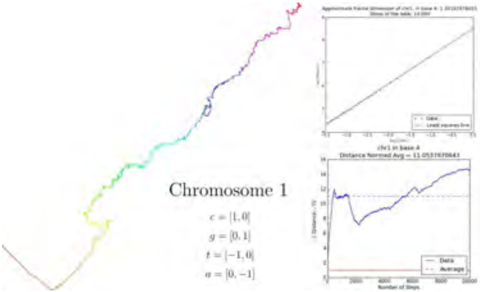


CARMA

Pi Seems Normal: Comparisons to Human Genomes — we are base 4 no's





The X Chromosome (34K) and Chromosome One (10K).

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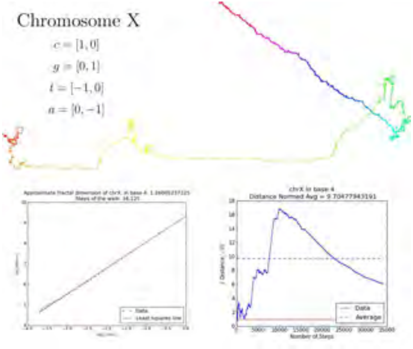

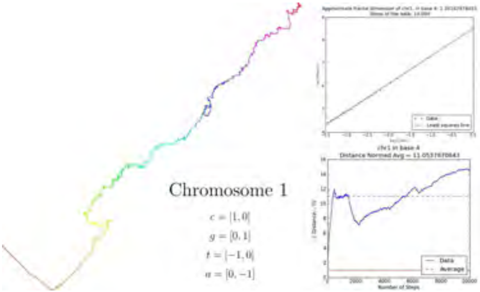

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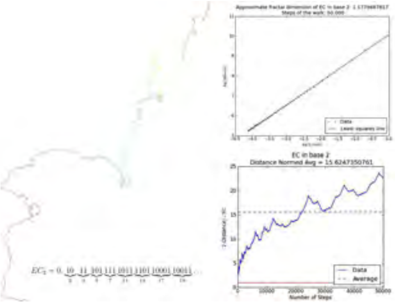

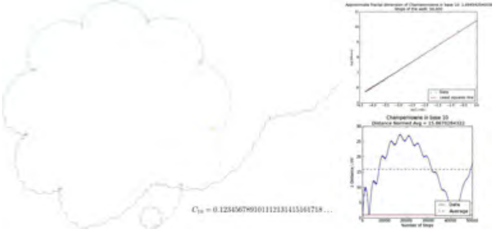

Erdös-Copeland number (base 2) and Champernowne number (base 10).

All pictures are thanks to Fran Aragon and Jake Fountain
http://www.carma.newcastle.edu.au/numberwalks.pdf

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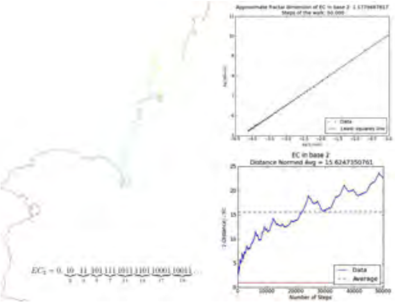

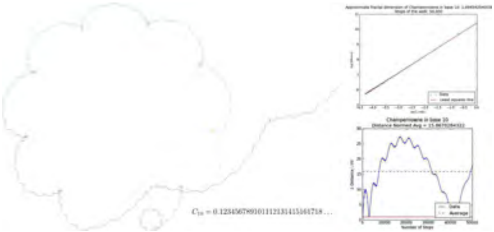

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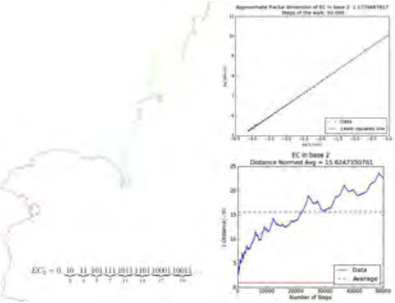

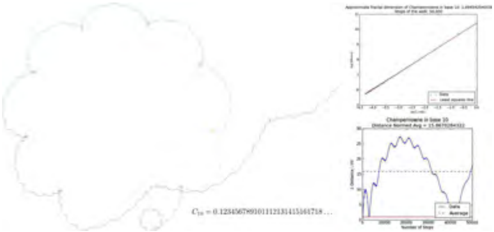

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Pi is Still Mysterious: Things we don't know about Pi

We do not 'know' (in the sense of being able to prove) whether ....

- The simple continued fraction for Pi is unbounded.
  - Euler found the **292**.
- There are infinitely many sevens in the decimal expansion of Pi.
- There are infinitely many ones in the ternary expansion of Pi.
- There are equally many zeroes and ones in the binary expansion of Pi.
- Or pretty much anything I have not told you.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}}}$$

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \dots}}}}}}}}}$$

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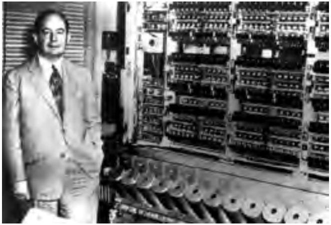
Decimal Digit Frequency: and "Johnny" von Neumann

IBM SKIP

1st von Neumann architecture machine



JvN (1903-57) at the Institute for Advanced Study

Decimal	Occurrences
0	99999485134
1	99999945664
2	100000480057
3	99999787805
4	100000357857
5	99999671008
6	99999807503
7	99999818723
8	100000791469
9	99999854780
Total	1000000000000

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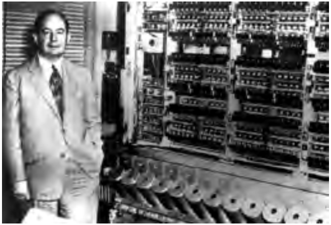

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Hexadecimal Digit Frequency: and Richard Crandall (Apple HPC)

0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
5	62500007205
6	62499925426
7	62499878794
8	62500216752
9	62500120671
A	62500266095
B	62499955595
C	62500188610
D	62499613666
E	62499875079
F	62499937801

(1947–2012)

Changing Cognitive Tastes

Why in antiquity π was not *measured* to greater accuracy than $22/7$ (with rope)?

It reflects not an inability, but rather a very different mindset to a modern (Baconian) experimental one — see Francis Bacon's *De augmentis scientiarum* (1623).

- Gauss and Ramanujan did not exploit their identities for π.
- An algorithm, as opposed to a closed form, was unsatisfactory to them — especially Ramanujan. He preferred

$$\frac{3}{\sqrt{163}} \log(640320) \approx \pi \qquad \text{and} \qquad \frac{3}{\sqrt{67}} \log(5280) \approx \pi$$

correct to 15 and 9 decimal places respectively.

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CARMA

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Changing Cognitive Tastes: Truth without Proof

Gourevich used integer relation computer methods to find the Ramanujan-type series — discussed below — in (11):

$$\frac{4}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8}\right)^{2n+1} \qquad (11)$$

where $r(n) := \frac{1}{2} \cdot \frac{3}{4} \cdot \cdots \cdot \frac{2n-1}{2n}$.

- I can "discover" it using **30**-digit arithmetic. and check it to **1,000** digits in **0.75** sec, **10,000** digits in **4.01** min with two naive command-line instructions in *Maple*.
  - No one has any inkling of how to prove it.
  - I "know" the beautiful identity is true — it would be more remarkable were it eventually to fail.
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Pi in High Culture (1993)

The admirable number pi:
three point one four one.
All the following digits are also initial,
five nine two because it never ends.
It can't be comprehended *six five three five* at a glance,
eight nine by calculation,
seven nine or imagination,
not even *three two three eight* by wit, that is, by comparison
four six to anything else
two six four three in the world.
The longest snake on earth calls it quits at about forty feet.
Likewise, snakes of myth and legend, though they may hold out a bit longer.
The pageant of digits comprising the number pi
doesn't stop at the page's edge.
It goes on across the table, through the air,
over a wall, a leaf, a bird's nest, clouds, straight into the sky,
through all the bottomless, bloated heavens.

Oh how brief - a mouse tail, a pigtail - is the tail of a comet!
How feeble the star's ray, bent by bumping up against space!
While here we have *two three fifteen three hundred nineteen*
my phone number your shirt size the year
nineteen hundred and seventy-three the sixth floor
the number of inhabitants sixty-five cents
hip measurement two fingers a charade, a code,
in which we find *hail to thee, blithe spirit, bird thou never wert*
alongside *ladies and gentlemen, no cause for alarm*,
as well as *heaven and earth shall pass away*,
but not the number pi, oh no, nothing doing,
it keeps right on with its rather remarkable *five*,
its uncommonly fine *eight*,
its far from final *seven*,
nudging, always nudging a sluggish eternity
to continue.

1996 Nobel Wislawa Szymborska (2-7-1923 1-2-2012)

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Computers Cease Being Human

TOC

1950s. Commercial computers — and discovery of advanced algorithms for arithmetic — unleashed π.

1965. The *new* fast Fourier transform (**FFT**) performed high-precision multiplications much faster than conventional methods — viewing numbers as polynomials in $\frac{1}{10}$.

- Newton methods helped reduce time for computing π to ultra-precision from millennia to weeks or days.

$$x \hookleftarrow x + x(1 - bx) \qquad\qquad x \hookleftarrow x + x(1 - ax^2)/2$$

converts $1/b$ to **4**$\times$ converts $1/\sqrt{a}$ to **6**$\times$ (**7** for $\sqrt{a}$)

∇ But until the **1980s** all computer evaluations of π employed classical formulas, usually of Machin-type.

Happily, **MRI** and **FFT** were discovered at the same time.

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> restart:Digits:=100:N:=x->x+x*(1-7*x);
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$$N := x \to x + x\,(1 - 7\,x)$$

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> Digits:=64:x:=.142;for k from 1 to 6 do x:=evalf(N(x),2^(k)+2); od;
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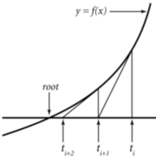

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1. Newton's method is self-correcting and quadratically convergent.
2. So we start close (to the left); and
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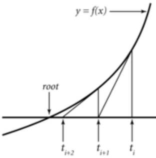

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Ramanujan Series for $1/\pi$ — See "Ramanujan at 125", *Notices* 2012-13

One of these series is the remarkable:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(\mathbf{1103} + 26390k)}{(k!)^4 396^{4k}} \qquad (12)$$

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SIZE/WEIGHT: ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons.

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$$x\sum_{n=0}^{\infty}\frac{(n!)^2\,4^n}{(2\,n+1)!\,(x^2+1)^{n+1}} \quad = \quad \arctan\left(\frac{1}{x}\right).$$

As $10\,(18^2+1) = 57^2+1 = 3250$ we may rewrite the formula

$$\frac{\pi}{4} = \arctan\left(\frac{1}{18}\right) + 8\arctan\left(\frac{1}{57}\right) - 5\arctan\left(\frac{1}{239}\right)$$

used by Shanks and Wrench in **1961** for **100,000** digits, and by Guilloud and Boyer in **1973** for a million digits of Pi in the efficient form

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5. A Million Decimals? Can π be computed to 1,000,000 decimals with the computers of today? From the remarks in the first section we see that the program which we have described would require times of the order of *months*. But since the memory of a 7090 is too small, by a factor of ten, a modified program, which writes and reads partial results, would take longer still. One would really want a computer 100 times as fast, 100 times as reliable, and with a memory 10 times as large. No such machine now exists.

There are, of course, many other formulas similar to (1), (2), and (5), and other programming devices are also possible, but it seems unlikely that any such modification can lead to more than a rather small improvement.

Are there *entirely different* procedures? This is, of course, possible. We cite the following: compute $1/\pi$ and then take its reciprocal. This sounds fantastic, but, in fact, it can be faster than the use of equation (2). One can compute $1/\pi$ by Ramanujan's formula [8]:

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The first factors here are given by $(-1)^k(1123 + 21460k)$. A binary value of $1/\pi$ equivalent to 100,000D, can be computed on a 7090 using equation (6) in 6 hours instead of the 8 hours required for the application of equation (2).* To reciprocate this value of $1/\pi$ would take about 1 hour. Thus, we can reduce the time required by (2) by an hour. But unfortunately we lose our overlapping check, and, in any case, this small gain is quite inadequate for the present question.

One could hope for a theoretical approach to this question of optimization—a theory of the "*depth*" of numbers—but no such theory now exists. One can guess that e is not as "deep" as π,† but try to prove it!

Such a theory would, of course, take years to develop. In the meantime—say, in 5 to 7 years—such a computer as we suggested above (100 times as fast, 100 times as reliable, and with 10 times the memory) will, no doubt, become a reality. At that time a computation of π to 1,000,000D will not be difficult.

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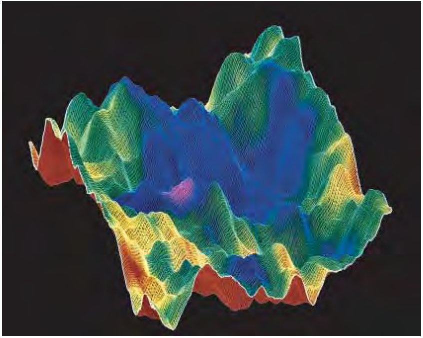
* We have computed $1/\pi$ by (6) to over 5000D in less than a min

† We have computed e on a 7090 to 100,265D by the obvious p
hours instead of the 8-hour run for π by (2).

CARMA

The First Million Digits of π



A *random walk* on π (courtesy David and Gregory Chudnovsky)

- See Richard Preston's: "The Mountains of Pi", *New Yorker*, March 2, **1992** (AAAS-Westinghouse Award for Science Journalism);

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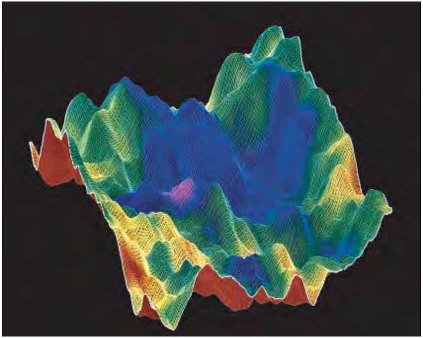

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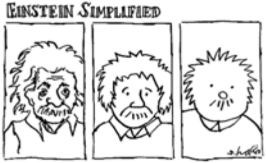
Reduced Complexity Methods

These series are much faster than classical ones, *but* the number of terms needed *still* increases linearly with the number of digits.

Twice as many digits correct requires twice as many terms of the series.



1976. Richard Brent of **ANU-CARMA** and Eugene Salamin independently found a reduced complexity algorithm for π.

- It takes $O(\log N)$ operations for N digits.
- Uses arithmetic-geometric mean iteration (AGM) and other elliptic integral ideas due to Gauss and Legendre circa **1800**.
  - Gauss — and others — missed connection to *computing* π.

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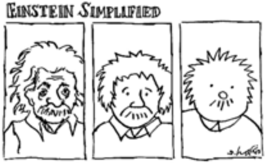

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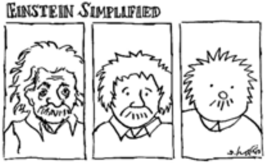

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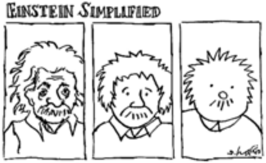

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A Reduced Complexity Algorithm

Algorithm (Brent-Salamin AGM iteration)

Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. Calculate

$$a_k = \frac{a_{k-1} + b_{k-1}}{2} \quad (A) \qquad b_k = \sqrt{a_{k-1} b_{k-1}} \quad (G)$$

$$c_k = a_k^2 - b_k^2, \qquad s_k = s_{k-1} - 2^k c_k$$

$$\text{and compute} \quad p_k = \frac{2a_k^2}{s_k}. \tag{15}$$

Then p_k converges quadratically to π.

- Each step doubles the correct digits — successive steps produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 digits of π.
  - 25 steps compute π to **45 million** digits. But, steps must be performed to the desired precision.

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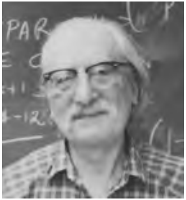
Four Famous Pi Guys: Salamin, Kanada, Bailey and Gosper in 1987

- To appear in Donald Knuth's book of mathematics pictures.

CARMA

And Some Others: Niven, Shanks (1917-96), Brent, Zudilin (◌)

CARMA

The Borwein Brothers

1985. Peter and I discovered algebraic algorithms of all orders:

Algorithm (Cubic Algorithm)

Set $a_0 = 1/3$ and $s_0 = (\sqrt{3}-1)/2$. Iterate

$$r_{k+1} = \frac{3}{1+2(1-s_k^3)^{1/3}}, \qquad s_{k+1} = \frac{r_{k+1}-1}{2}$$

and $$a_{k+1} = r_{k+1}^2 a_k - 3^k(r_{k+1}^2 - 1).$$

Then $1/a_k$ converges cubically to π.

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$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \quad \text{and}$$

$$a_{k+1} = a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2).$$

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- Using **4** $\times$ 'plus' **1** $\div$ 'plus' **2** $1/\sqrt{\cdot}$ = **19** full precision $\times$ per step. So **20 steps** costs out at around **400** full precision multiplications.
  (This assumes intermediate storage. Additions are cheap)

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Modern Calculation Records: and IBM Blue Gene/L at Argonne

IBM

Name	Year	Correct Digits
Miyoshi and Kanada	1981	2,000,036
Kanada-Yoshino-Tamura	1982	16,777,206
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Kanada and Takahashi	Oct. 1995	6,442,450,938
Kanada and Takahashi	Jul. 1997	51,539,600,000
Kanada and Takahashi	Sep. 1999	206,158,430,000
Kanada-Ushiro-Kuroda	Dec. 2002	1,241,100,000,000
Takahashi	Jan. 2009	1,649,000,000,000
Takahashi	April. 2009	2,576,980,377,524
Bellard	Dec. 2009	2,699,999,990,000
Kondo and Yee	Aug. 2010	**5,000,000,000,000**
Kondo and Yee	Oct. 2011	**10,000,000,000,000**
Kondo and Yee	Dec. 2013	**12,200,000,000,000**

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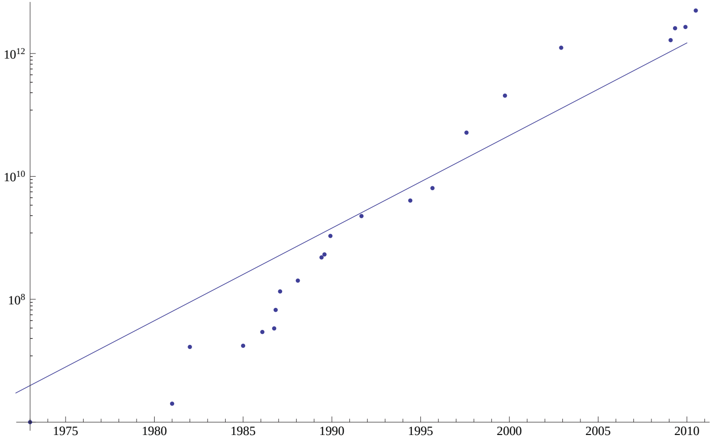
Moore's Law Marches On



Computation of π since 1975 plotted vs. Moore's law predicted increase

CARMA

An Amazing Algebraic Approximation to π

The **transcendental number** $\boxed{\pi}$ and the **algebraic number** $1/a_{20}$ actually agree for more than **1.5 trillion decimal places**.

- π and $1/a_{21}$ agree for more than **six trillion decimal places.**

1984. I found these on a **16K** upgrade of an 8K double-precision TRS80-100 Radio Shack portable.

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$$y_{12} = \frac{1 - \sqrt[4]{1 - {y_{11}}^4}}{1 + \sqrt[4]{1 - {y_{11}}^4}}, a_{12} = a_{11}\left(1 + y_{12}\right)^4 - 2^{25} y_{12}\left(1 + y_{12} + {y_{12}}^2\right)$$

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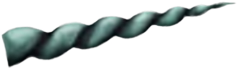

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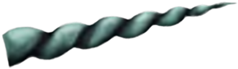

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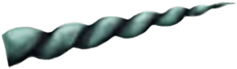

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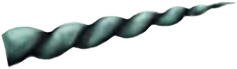

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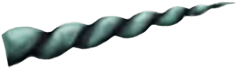

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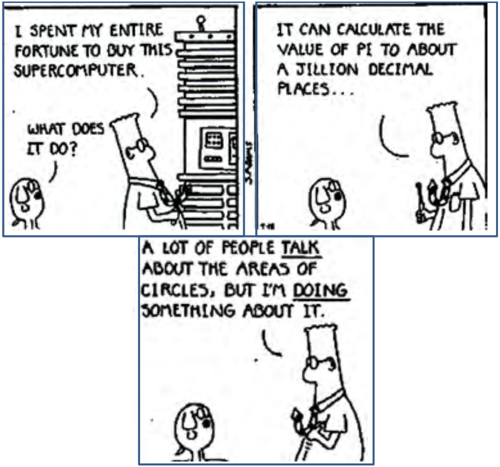
Billions and Billions



Star Trek

Kirk asks:

> *"Aren't there some mathematical problems that simply can't be solved?"*

And Spock 'fries the brains' of a rogue computer by telling it:

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Pi the Song: from the album Aerial

2005 Influential Singer-songwriter *Kate Bush* sings "Pi" on Aerial.

Sweet and gentle and sensitive man
With an obsessive nature and deep fascination
for numbers
And a complete infatuation
with the calculation of Pi
Chorus: Oh he love, he love, he love
He does love his numbers
And they run, they run, they run him
In a great big circle
In a circle of infinity

"a sentimental ode to a mathematician, audacious in both subject matter and treatment. The chorus is the number sung to many, many decimal places." [**150** – wrong after 50] — Observer Review

Back to the Future

2002. Kanada computed π to over **1.24 trillion decimal digits**. His team first computed π in **hex** (base 16) to **1,030,700, 000,000** places, using good old Machin type relations:

$$\pi = 48\tan^{-1}\frac{1}{49} + 128\tan^{-1}\frac{1}{\mathbf{57}} - 20\tan^{-1}\frac{1}{\mathbf{239}} + 48\tan^{-1}\frac{1}{110443}$$

(Takano, pop-song writer **1982**)

$$\pi = 176\tan^{-1}\frac{1}{\mathbf{57}} + 28\tan^{-1}\frac{1}{\mathbf{239}} - 48\tan^{-1}\frac{1}{682} + 96\tan^{-1}\frac{1}{12943}$$

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Yasumasa Kanada

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$\leftrightsquigarrow$

11.00100100001111110110101010001000100001011010001100001000110100110001001100011001100010100010111000

- The decimal expansion was checked by converting it back to hex.
  - Base conversion require pretty massive computation.
- **Six times** as many digits as before: hex and decimal ran **600** hrs on same 64-node Hitachi — at roughly **1 Tflop/sec** (2002).
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> *Advances in π-computation during the past decade have all involved sophisticated improvements in computational techniques and environments.*
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Two New Pi Guys: Alex Yee and his Elephant

- The elephant may have provided extra memory?

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♠ The elephant may have provided extra memory?

Two New Pi Guys: Mario Livio (JPL) in 01-31-2013 *HuffPost*

Mario Livio
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As Easy as Pi

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Figure 1. The computer used by Alexander Yee and Shigeru Kondo to calculate π to 10 trillion digits (reproduced by permission from Alexander Yee)

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Computing Individual Digits of π

TOC IBM

1971. One might think everything of interest about computing π has been discovered. This was Beckmann's view in *A History of* π

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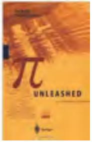

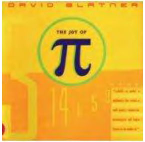

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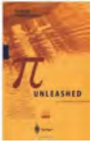

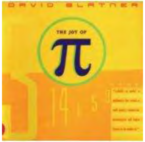

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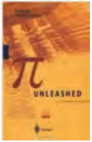

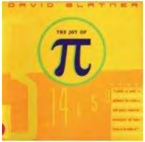

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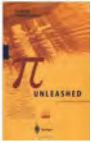

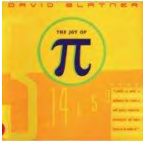

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What BBP Does?

Prior to **1996**, most folks thought to compute the d-th digit of π, you had to generate the (order of) the entire first d digits.

- **This is not true**, at least for hex (base 16) or binary (base 2) digits of π. In **1996**, P. Borwein, Plouffe, and Bailey found an algorithm for individual hex digits of π. It produces:

- a modest-length string hex or binary digits of π, beginning at an any position, *using no prior bits*;
  1. is implementable on any modern computer;
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What BBP Is? Reverse Engineered Mathematics

This is based on the following then new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \qquad (16)$$

- The millionth hex digit (four millionth binary digit) of π can be found in under **30** secs on a fairly new computer in Maple (not C++) and the billionth in **10** hrs.

Equation (16) was discovered numerically using integer relation methods over months in our Vancouver lab, **CECM**. It arrived in the coded form:

$$\pi = 4\,{}_2F_1\left(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}\right) + 2\tan^{-1}\left(\frac{1}{2}\right) - \log 5$$

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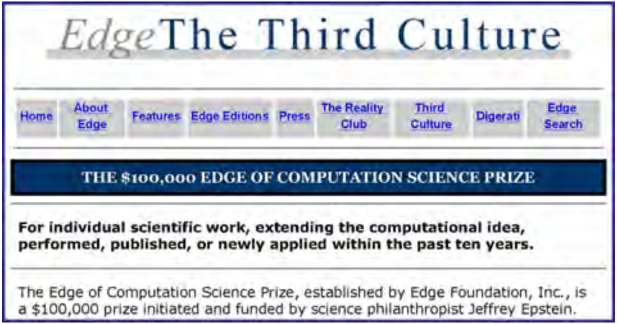

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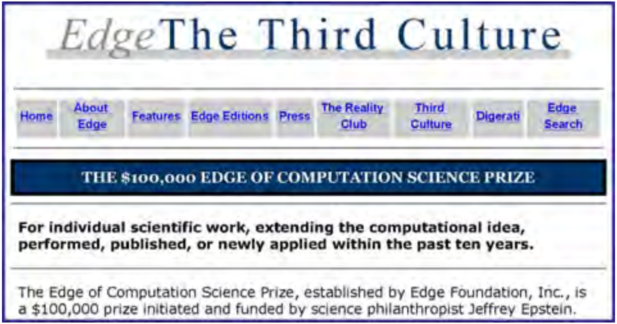

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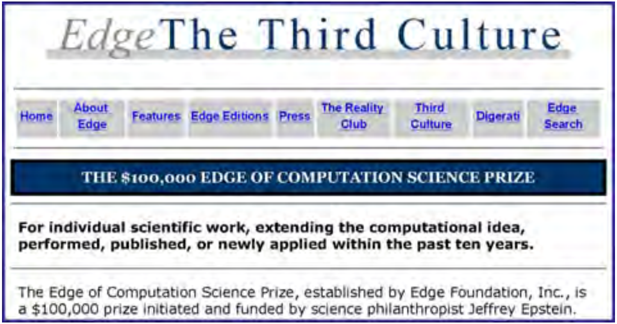

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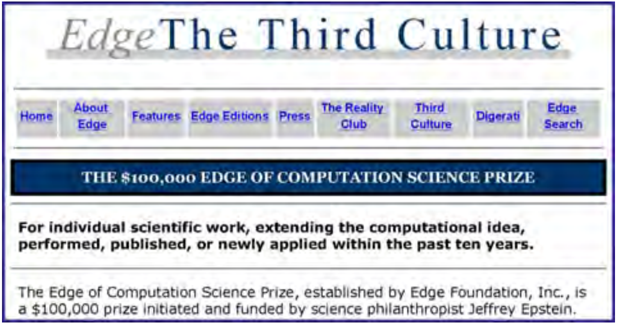

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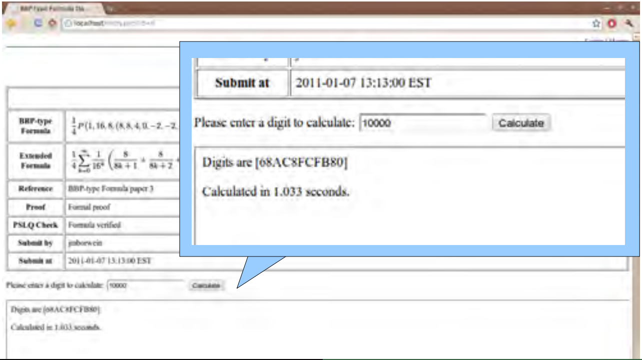

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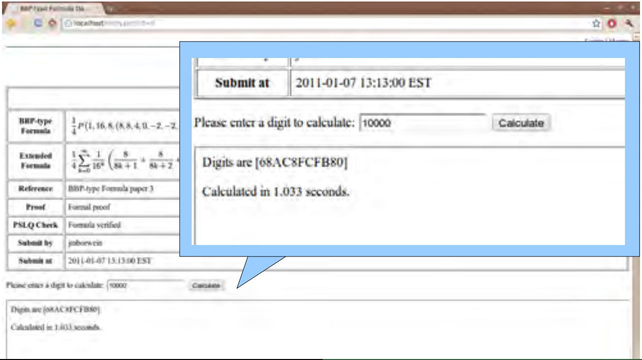

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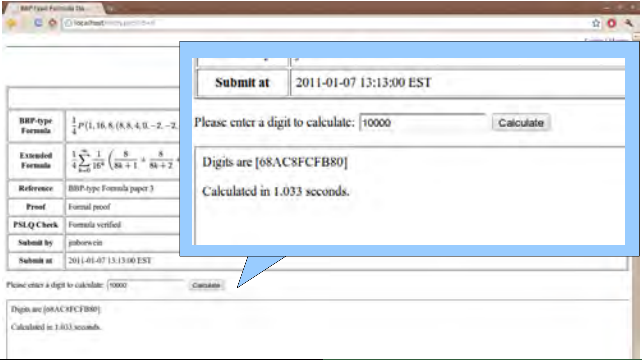

Mathematical Interlude: III. (Maple, Mathematica and Human)

Proof of (16). For $0 < k < 8$,

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\,dx = \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}.$$

Thus, one can write

$$\sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right)$$
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Tuning BBP Computation

- **1997**. Fabrice Bellard of INRIA computed 152 bits of π starting at the trillionth position;
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Bellard used the following variant of (16):

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We present a new record on computing specific bits of π, the mathematical constant, and discuss performing such computations on Apache Hadoop clusters. The new record represented in hexadecimal is

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- They are $353CB3F7F0C9ACCFA9AA215F2$
- Using BBP on *CUDA* (too 'hard' for Blue Gene)
- All processing done on four NVIDIA GTX 690 graphics cards (GPUs) installed in CUDA. Yahoo's run took 23 days; this took 37 days.

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where $p(k)$ and $q(k)$ are integer polynomials and $b = 2, 3, \ldots$.

- I illustrate why this works in binary for $\log 2$. We start with:

$$\log 2 = \sum_{k=0}^{\infty} \frac{1}{k2^k} \tag{19}$$

as discovered by Euler.

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BBP Formula for $\log 2$

We can write

$$\begin{aligned} \{2^d \log 2\} &= \left\{ \left\{ \sum_{k=0}^{d} \frac{2^{d-k}}{k} \right\} + \left\{ \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k} \right\} \right\} \\ &= \left\{ \left\{ \sum_{k=0}^{d} \frac{\mathbf{2^{d-k} \bmod k}}{k} \right\} + \left\{ \sum_{k=d+1}^{\infty} \frac{\mathbf{2^{d-k}}}{\mathbf{k}} \right\} \right\}. \end{aligned} \tag{20}$$

- The key: the numerator in (20), $2^{d-k} \bmod k$, can be found rapidly by binary exponentiation, performed modulo k. So,

$$3^{17} = ((((3^2)^2)^2)^2) \cdot 3$$

uses only **5** multiplications, not the usual **16**. Moreover, $3^{17} \bmod 10$ is done as $3^2 = 9; 9^2 = 1; 1^2 = 1; 1^2 = 1; 1 \times 3 = 3$

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Catalan's Constant G: and BBP for G in Binary

The simplest number not proven irrational is

$$G := 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots, \qquad \frac{\pi^2}{12} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

2009. G is calculated to **31.026** billion digits. Records often use:

$$G = \frac{3}{8}\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}(2n+1)^2} + \frac{\pi}{8}\log(2+\sqrt{3}) \text{ (Ramanujan)} \qquad (21)$$

– holds since $G = -T(\frac{\pi}{4}) = -\frac{3}{2}T(\frac{\pi}{12})$ where $T(\theta) := \int_0^\theta \log\tan\sigma\, d\sigma$.

– An **18** term binary BBP formula for $G = 0.9159655941772190\ldots$ is:

$$G = \sum_{k=0}^{\infty} \frac{1}{4^{6k+5}} \left(\frac{3072}{(24k+1)^2} - \frac{3072}{(24k+2)^2} - \frac{23040}{(24k+3)^2} + \frac{12288}{(24k+4)^2} - \frac{768}{(24k+5)^2} + \frac{9216}{(24k+6)^2} + \frac{10368}{(24k+8)^2} + \frac{2496}{(24k+9)^2} - \frac{192}{(24k+10)^2} + \frac{768}{(24k+12)^2} - \frac{48}{(24k+13)^2} + \frac{360}{(24k+15)^2} + \frac{648}{(24k+16)^2} + \frac{12}{(24k+17)^2} + \frac{168}{(24k+18)^2} + \frac{48}{(24k+20)^2} - \frac{39}{(24k+21)^2} \right)$$

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Eugene Catalan (**1818-94**)– a revolutionary

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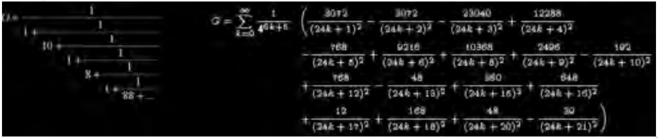

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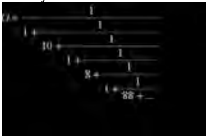

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A Better Formula for G

A **16** term formula in concise BBP notation is:

$$\begin{aligned} G &= P\left(2, \mathbf{4096}, 24, \vec{v}\right) \qquad \text{where} \\ \vec{v} &:= (6144, -6144, -6144, 0, -1536, -3072, -768, 0, -768, \\ &\quad -384, 192, 0, -96, 96, 96, 0, 24, 48, 12, 0, 12, 6, -3, 0) \end{aligned}$$

It takes almost exactly **8/9**th the time of **18** term formula for G.

- This makes for a very cool calculation
- Since we can not prove G is irrational, *Who can say what might turn up*?

What About Base Ten?

- The first integer logarithm with no known binary BBP formula is $\log 23$ (since $23 \times 89 = 2^{10} - 1$).

Searches conducted by numerous researchers for base-ten formulas have been unfruitful. Indeed:

2004. D. Borwein (my father), W. Gallway and I showed there are no BBP formulas of the *Machin-type* of (16) for π if base is not a power of **two**.

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Pi Photo-shopped: a 2010 PiDay Contest

"Noli Credere Pictis"

π^2 in Binary and Ternary

Bailey and Pi on a bus. Only in Berkeley?

Thanks to Dave Broadhurst, a ternary BBP formula exists for π^2 (unlike π):

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{3^{6k}} \times \left\{\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right\}$$

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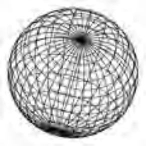
- We do not fully understand why π^2 allows BBP formulas in two distinct bases.



- $4\pi^2$ is the area of a sphere in three-space (L).
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  - So in binary we are computing these fundamental physical constants.

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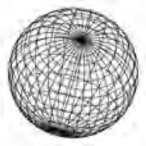

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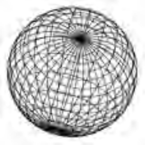

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IBM's New Record Results

Algorithm (What We Did)

Dave Bailey, Andrew Mattingly (L) and Glenn Wightwick (R) of IBM Australia, and I, have obtained and (nearly) confirmed:

1. **106** digits of π^2 base **2** at the **ten trillion**th place base **64**
2. **94** digits of π^2 base **3** at the **ten trillion**th place base **729**
3. **150** digits of G base **2** at the **ten trillion**th place base **4096**

on a 4-rack BlueGene/P system at IBM's Benchmarking Centre in Rochester, Minn, USA.

The 3 Records Use Over 1380 CPU Years (135 rack days)

An enormous amount of delicate computation: **1380 years** is a long time. Suppose a spanking new IBM single-core PC went back **1381 years**.

- It would find itself in **632 CE**.
- The year that Mohammed died, and the Caliphate was established. If it then calculated π nonstop:
  - Through the Crusades, black plague, Moguls, Renaissance, discovery of America, Gutenberg, Reformation, invention of steam, Napoleon, electricity, WW2, the transistor, fiber optics,...
- With no breaks or break-downs:
- It would have finished in **2012**.

August 2013, *Notices of the AMS*
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IBM's New Results: π^2 base **2**

Algorithm (10 trillionth digits of π^2 in base 64 — in **230** years)

1. The calculation took, on average, **253529** seconds per thread. It was broken into 7 "partitions" of **2048** threads each. For a total of $7 \cdot 2048 \cdot 253529 = 3.6 \cdot 10^9$ CPU seconds.

2. On a single Blue Gene/P CPU it *would* take **115 years**!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used $\frac{7 \cdot 2048 \cdot 253529}{4096 \cdot 60 \cdot 60 \cdot 24} =$ **10.3** "rack days".

3. The verification run took the same time (within a few minutes): **106 base 2 digits** are in agreement.

```
base-8 digits = 75|601145053032364757245000057432627545303630524163506 34|573227604
                   6011450530323647572450000574326275453036305241635063 4|22021056612
```

IBM's New Results: π^2 base **3**

Algorithm (10 trillionth digits of π^2 in base 729 — in **414** years)

1. The calculation took, on average, **795773** seconds per thread. It was broken into 4 "partitions" of **2048** threads each. For a total of $4 \cdot 2048 \cdot 795773 = 6.5 \cdot 10^9$ CPU seconds.
2. On a single Blue Gene/P CPU it *would* take **207 years**!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used $\frac{4 \cdot 2048 \cdot 795773}{4096 \cdot 60 \cdot 60 \cdot 24} =$ **18.4** "rack days".
3. The verification run took the same time (within a few minutes): **94 base 3 digits** are in agreement.

```
base-9 digits = 001|122644850645485831771111352101628560483234534681|0565567|635862
                    122644850645485831771111352101628560483234534680|4744867|134524345
```

CARMA

IBM's New Results: G base **2**

Algorithm (10 trillionth digits of G in base **4096** — in **735** years)

1. The calculation took, on average, **707857** seconds per thread. It was broken into 8 "partitions" of **2048** threads each. For a total of $8 \cdot 2048 \cdot 707857 = 1.2 \cdot 10^{10}$ CPU seconds.

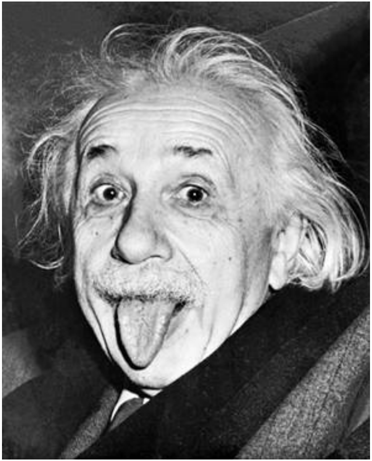
2. On a single Blue Gene/P CPU it *would* take **368 years**!

   Each rack of BG/P contains 4096 threads (or cores). Thus, we used $\frac{8 \cdot 2048 \cdot 707857}{4096 \cdot 60 \cdot 60 \cdot 24} =$ **32.8** "rack days".

3. The verification run will take the same time (within a few minutes): **xxx base 2 digits** will be in agreement.

```
base-8 digits = 0176|3470505377477705112261337162012525732721732452260001775457 27
                    xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx|xxxx
```

Thank You, One and All, and Happy Birthday, Albert



Albert Einstein 3.14.1879 – 18.04.1955

139. Links and References

1. The Pi Digit site: http://carma.newcastle.edu.au/bbp
2. Dave Bailey's Pi Resources: http://crd.lbl.gov/~dhbailey/pi/
3. The Life of Pi: http://carma.newcastle.edu.au/jon/pi-2010.pdf.
4. Experimental Mathematics: http://www.experimentalmath.info/.
5. Dr Pi's brief Bio: http://carma.newcastle.edu.au/jon/bio_short.html.

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2. D.H. Bailey, and J.M. Borwein, *Mathematics by Experiment: Plausible Reasoning in the 21st Century,* AK Peters Ltd, 2003, ISBN: 1-56881-136-5. See http://www.experimentalmath.info/
3. J.M. Borwein, "Pi: from Archimedes to ENIAC and beyond," in *Mathematics and Culture,* Einaudi, 2006. Updated 2012: http://carma.newcastle.edu.au/jon/pi-2012.pdf.
4. J.M. & P.B. Borwein, and D.A. Bailey, "Ramanujan, modular equations and pi or how to compute a billion digits of pi," *MAA Monthly*, **96** (1989), 201–219. Reprinted in *Organic Mathematics*, www.cecm.sfu.ca/organics, 1996, *CMS/AMS Conference Proceedings*, **20** (1997), ISSN: 0731-1036.
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6. Jonathan M. Borwein and Peter B. Borwein, *Selected Writings on Experimental and Computational Mathematics*, PsiPress. October 2010.[4]
7. L. Berggren, J.M. Borwein and P.B. Borwein, *Pi: a Source Book,* Springer-Verlag, (1997), (2000), (2004). Fourth Edition, in Press.