On formulas for π experimentally conjectured by Jauregui–Tsallis

To the fine memories of Herbert Wilf (6/13/1931–1/7/2012)

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Abstract

In a recent study of representing Dirac's delta distribution using q-exponentials, M. Jauregui and C. Tsallis experimentally discovered formulae for π as hypergeometric series as well as certain integrals. Herein, we offer rigorous proofs of these identities using various methods and our primary intent is to lay down an illustration of the many technical underpinnings of such evaluations. This includes an explicit discussion of creative telescoping and Carlson's Theorem. We also generalize the Jauregui–Tsallis identities to integrals involving Chebyshev polynomials. In our pursuit, we provide an interesting tour through various topics from classical analysis to the theory of special functions.

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1 Introduction

In exploring applications of the q-exponential function and formal representations of the Dirac function, Jauregui and Tsallis conjectured [6] that

$$T(r) := \int_{-\infty}^{\infty} \frac{\sin(2r\arctan(t))}{t(1+t^2)^r}\,\mathrm{d}t = \pi \tag{1}$$

for all $r > 0$. When r is a half-integer, they rewrite (1) as the finite sum

$$S(n) := n\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \frac{\Gamma(n-k-\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(2k+2)\Gamma(n-2k)} = \pi \tag{2}$$

which they confirmed, up to $n = 5000$, by a symbolic algebra system. The connection $S(n) = T(n/2)$ is given in Section 2.4.

In the present work, we show the validity of the Jauregui–Tsallis conjecture (1) — along with the special case (2) — and present natural generalizations of these identities. Earlier proofs of these conjectures, in the more general (physical) setting of [6], appear in [4], by recourse to the notion of superstatistics, and in [10], by appealing to the theory of tempered ultradistributions. In offering several proofs of (1) and (2), the primary intent of the present work is to lay down an illustration of the many technical underpinnings of such integral evaluations. In particular, we hope that the explicit discussion of creative telescoping and Carlson's Theorem — which, we believe, deserve to be better known — is useful to the reader.

The paper is organized as follows. Section 2 is devoted to the proof of the above conjectures: in Section 2.1 we deduce the special case (2) from the classical Pfaff–Saalschütz identity of hypergeometric function theory and give a short proof of (1) when r is a half-integer (that is, $2r$ is an integer) using complex analysis in Section 2.2. Section 2.3 lifts this result to complex r with the help of Carlson's theorem on discrete analytic continuation. The relation between (1) and (2) is shown in Section 2.4 with an added illustration of Carlson's theorem and a cautionary example that, despite its success on (1), exhibits failure of application in the case of (2). In Section 3 we provide an entirely different computer algebra proof of (2) and the half-integer case of (1) via the method of creative telescoping [9]. Finally, in Sections 4 and 5 we revisit and then generalize our earlier results in the language of orthogonal polynomials. This actually has its bases on the following observations.

The integral in (1) has several equivalent representations highlighting its different aspects. The change of variables $t = \tan(\theta)$ shows that

$$T(r) = \int_{-\pi/2}^{\pi/2} \frac{\sin(2r\theta)}{\sin(\theta)} \cos^{2r-1}(\theta)\,\mathrm{d}\theta. \tag{3}$$

Let U_n denote the Chebyshev polynomial of the second kind, given by

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}. \tag{4}$$

For positive integers n,

$$T(\tfrac{n+1}{2}) = 2\int_0^1 x^n U_n(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}}. \tag{5}$$

Motivated by a corresponding identity, given in (35), for the Chebyshev polynomial T_n of the first kind, we generalize our considerations (see Section 4) to the integral

$$M_U(n,s) := \int_0^1 x^{s-1} U_n(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}}. \tag{6}$$

Our main result will be the evaluation, proven in Theorem 4.11,

$$M_U(n,s) = \frac{\pi}{2^s}\binom{s-1}{\frac{s-n-1}{2}} {}_2F_1\left(\left.\begin{matrix} 1, s \\ \frac{s-n+1}{2} \end{matrix}\right| \frac{1}{2}\right) - \frac{\pi}{2},$$

valid for complex n and Re $s > 0$, which has the Jauregui–Tsallis evaluation (1) as a direct consequence.

2 The Jauregui–Tsallis conjectures

2.1 The sum

Recall that the (generalized) hypergeometric function is defined by

$${}_pF_q\left(\left.\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{7}$$

for $|z| < 1$ (assuming $p \leqslant q+1$), and by analytic continuation elsewhere. Here $(a)_n := a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the rising factorial.

We need the following classical theorem given, for instance, in [2, Thm. 2.2.6]:

Theorem 2.1 (Pfaff–Saalschütz)**.** *For $n = 1, 2, 3, \ldots$ one has*

$${}_3F_2\left(\left.\begin{matrix} a, b, -n \\ 1+a+b-c-n, c \end{matrix}\right| 1\right) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}. \tag{8}$$

Letting n increase to infinity recovers Gauss's formula

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{9}$$

valid when $\operatorname{Re}(c-a-b) > 0$.

We now turn the Jauregui–Tsallis conjecture (2) into a theorem.

Theorem 2.2. *For $n = 1, 2, 3, \ldots$ one has*

$$ S(n) = n \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(n-k-\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(2k+2)\Gamma(n-2k)} = \pi. \tag{10}$$

Proof. For the first equality we only need to observe that every term in the sum with $k > (n-1)/2$ vanishes. Now, using the Legendre duplication formula

$$ \Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) 2^{2z-1} = \Gamma(2z)\sqrt{\pi} $$

in (10), it is easily seen that proving $S(n) = \pi$ is equivalent to showing

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, -\frac{n}{2}+1, -\frac{n}{2}+\frac{1}{2} \\ \frac{3}{2}, -n+\frac{3}{2} \end{matrix}\middle| 1\right) = \frac{\Gamma(n)\sqrt{\pi}}{n\,\Gamma\left(n-\frac{1}{2}\right)}. \tag{11}$$

We shall apply Theorem 2.1 to (11) separately for even and odd n.

Suppose that $n = 2m+2$ is even. Then the left-hand side of (11) is

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, -m-\frac{1}{2}, -m \\ \frac{3}{2}, -2m-\frac{1}{2} \end{matrix}\middle| 1\right) $$

which by Theorem 2.1, with $a = \frac{1}{2}$, $b = -m-\frac{1}{2}$ and $c = \frac{3}{2}$, evaluates to the right-hand side of (11).

The case when $n = 2m+1$ is odd follows analogously. □

2.2 The integral – half-integer case

We embark on proving the Jauregui–Tsallis conjecture (1) for half-integers r. In Section 2.3 this will be extended to the analytic case, thus validating the conjecture for all complex $\operatorname{Re} r > 0$.

Theorem 2.3. *For positive half-integers r one has*

$$ T(r) = \int_{-\pi/2}^{\pi/2} \frac{\sin(2r\theta)}{\sin(\theta)} \cos^{2r-1}(\theta)\, \mathrm{d}\theta = \pi. \tag{12}$$

Proof. Writing $z = e^{2i\theta}$ in (12) we have

$$T(r) = \frac{1}{2^{2r-1}} \int_{-\pi/2}^{\pi/2} \frac{z^{2r}-1}{z-1} \frac{(z+1)^{2r-1}}{z^{2r-1}} \mathrm{d}\theta = \frac{\pi}{2^{2r-1}} \cdot \frac{1}{2\pi i} \oint \frac{z^{2r}-1}{z-1} \frac{(z+1)^{2r-1}}{z^{2r}} \mathrm{d}z$$

where the last integral is a contour integral along the positively oriented unit circle. Hence, by the residue theorem

$$T(r) = \frac{\pi}{2^{2r-1}} \cdot \left[z^{2r-1}\right] \left(\frac{z^{2r}-1}{z-1} (z+1)^{2r-1} \right).$$

Here, $[z^n]\, f(z)$ denotes the coefficient of z^n in the Taylor expansion of $f(z)$. Assume that r is a half-integer. Since the coefficient z^n in

$$\frac{z^{n+1}-1}{z-1} (z+1)^n = (1 + z + \ldots + z^n)(1+z)^n$$

is $\sum_k \binom{n}{k} = 2^n$ we find $T(r) = \pi$, as claimed. □

2.3 The analytic case

In the previous section, we have shown that $T(r) = \pi$ for all positive half-integers r. We now extend this result to all complex $\operatorname{Re} r > 0$. To do so, appeal is made to a classical result due to Fritz D. Carlson (from his 1914 dissertation [3]). The first published proof was given in [12, §5.81]. An accessible proof of a special case, due to Selberg, is presented in [2, p. 112].

We recall that a function f is of *exponential type* in a region if $|f(z)| \leqslant Me^{c|z|}$ for some constants M and c.

Theorem 2.4 (Carlson)**.** *Let f be analytic in the right half-plane $\operatorname{Re} z \geqslant 0$ and of exponential type with the additional requirement that*

$$|f(z)| \leqslant Me^{d|z|}$$

for some $d < \pi$ on the imaginary axis $\operatorname{Re} z = 0$. If $f(k) = 0$ for $k = 0, 1, 2, \ldots$ then $f(z) = 0$ identically in the right half-plane.

Note that the example $f(z) = \sin(\pi z)$ shows that the growth condition on the imaginary axis can not be relaxed.

To establish analyticity, we use the following criterion discussed in [7]. As it is very convenient yet not commonly found in textbooks, we include a specialization of it here.

Theorem 2.5 (Analyticity)**.** *Let $G \subset \mathbb{C}$, $W \subset \mathbb{R}$ be open, and let $f : G \times W \to \mathbb{C}$ be a function such that $f(z, \cdot)$ is Lebesgue measurable for all $z \in G$, and $f(\cdot, w)$ is analytic for all $w \in W$. If for each $z_0 \in G$ there is a $\delta > 0$ such that*

$$\sup_{z \in G,\, |z-z_0|<\delta} \int_W |f(z,w)| \, \mathrm{d}w < \infty, \tag{13}$$

which is a local boundedness requirement, then $\int_W f(\cdot, w)\, \mathrm{d}w$ is analytic in G.

Proposition 2.6. *The integral*

$$T(r) = \int_{-\pi/2}^{\pi/2} \frac{\sin(2r\theta)}{\sin(\theta)} \cos^{2r-1}(\theta)\, \mathrm{d}\theta \tag{14}$$

is analytic on the half-plane $\operatorname{Re} r > 0$*. Moreover, let $\varepsilon > 0$. Then, for all* $\operatorname{Re} r > \varepsilon$*,*

$$|T(r)| \leqslant C|r|\, \mathrm{e}^{\pi|\operatorname{Im} r|} \tag{15}$$

for some constant $C = C(\varepsilon)$.

Proof. The following estimate is valid for all complex r:

$$|\sin(2r\theta)| \leqslant \frac{1}{2}\left(|\mathrm{e}^{2ir\theta}| + |\mathrm{e}^{-2ir\theta}|\right) \leqslant \mathrm{e}^{2\theta|\operatorname{Im} r|} \leqslant \mathrm{e}^{\pi|\operatorname{Im} r|}; \tag{16}$$

where in the last step we assumed $|\theta| \leqslant \pi/2$. Therefore, if additionally $|\theta| \geqslant 1/|r|$, then

$$\left|\frac{\sin(2r\theta)}{\sin(\theta)}\right| \leqslant \frac{\mathrm{e}^{\pi|\operatorname{Im} r|}}{|\sin(\theta)|} \leqslant \frac{\pi}{2}|r|\, \mathrm{e}^{\pi|\operatorname{Im} r|}. \tag{17}$$

On the other hand, assuming instead $|r\theta| < 1$,

$$\left|\frac{\sin(2r\theta)}{\sin(\theta)}\right| = 2|r| \left|\frac{\sin(2r\theta)}{2r\theta}\right| \left|\frac{\theta}{\sin(\theta)}\right| \leqslant \pi|r| \left|\frac{\sin(2r\theta)}{2r\theta}\right| \leqslant C_1\pi|r| \tag{18}$$

with $C_1 = \max_{|x|=2} \frac{\sin(x)}{x} = \sinh(2)/2$. Combining (17) and (18) we find, for $\operatorname{Re} r > \varepsilon$,

$$\begin{aligned} |T(r)| &\leqslant C_2|r|\, \mathrm{e}^{\pi|\operatorname{Im} r|} \int_{-\pi/2}^{\pi/2} |\cos^{2r-1}(\theta)|\, \mathrm{d}\theta \\ &= C_2|r|\, \mathrm{e}^{\pi|\operatorname{Im} r|}\, \frac{1}{2} B(\operatorname{Re} r, \tfrac{1}{2}) \\ &< C_2|r|\, \mathrm{e}^{\pi|\operatorname{Im} r|}\, \frac{1}{2} B(\varepsilon, \tfrac{1}{2}). \end{aligned} \tag{19}$$

It follows from (19) that the local boundedness condition and all other conditions of Theorem 2.5, with $W = (-\pi/2, \pi/2)$, are satisfied. Hence $T(r)$ is analytic for $\operatorname{Re} r > 0$. The estimate (15) is a consequence of (19). □

The stage is set to prove the Jauregui–Tsallis conjecture (1) in its full generality.

Theorem 2.7. *For* Re $r > 0$ *one has*

$$T(r) = \int_{-\infty}^{\infty} \frac{\sin\left(2r \arctan(t)\right)}{t(1+t^2)^r}\,\mathrm{d}t = \pi. \tag{20}$$

Proof. We have shown that $T(r) = \pi$ for all positive half-integers r. Put $f(z) = T(\frac{z+1}{2}) - \pi$ so that $f(0) = f(1) = \ldots = 0$. Then, by Proposition 2.6, the function $f(z)$ is analytic for Re $z > -1/2$ and satisfies

$$f(z) \leqslant C_f |z+1|\, \mathrm{e}^{\frac{\pi}{2}|\,\mathrm{Im}\, z|} \tag{21}$$

for some constant C_f. In particular, f meets the assumptions of Carlson's Theorem 2.4 for any $d > \frac{\pi}{2}$. Therefore $f(z) = 0$, identically, in the right half-plane. This shows $T(r) = \pi$ for all Re $r > 1/2$, and the extension to Re $r > 0$ results from analytic continuation. □

2.4 Revisiting the sum

Recall first that

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the Euler *beta function*.

Using the representation (3), Theorem 2.7 shows that

$$T(r) = \int_{-\pi/2}^{\pi/2} \frac{\sin(2r\theta)}{\sin(\theta)} \cos^{2r-1}(\theta)\,\mathrm{d}\theta = \pi \tag{22}$$

for all Re $r > 0$ (this integral, as observed in [4], appears as Entry 3.638.3 in [5]). In particular, when $n = 2r$ is a positive integer, we have

$$\begin{aligned}
T(\tfrac{n}{2}) &= 2\int_0^{\pi/2} \frac{\mathrm{Im}\left(\mathrm{e}^{in\theta}\right)}{\sin(\theta)} \cos^{n-1}(\theta)\,\mathrm{d}\theta \\
&= 2\sum_{k=0}^{\infty} (-1)^k \binom{n}{2k+1} \int_0^{\pi/2} (\cos\theta)^{2n-2\,k-2} (\sin\theta)^{2\,k}\,\mathrm{d}\theta \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{n}{2k+1} B\left(n-k-\frac{1}{2}, k+\frac{1}{2}\right) \\
&= n\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(n-k-\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(2k+2\right)\Gamma\left(n-2k\right)} \\
&= S(n).
\end{aligned} \tag{23}$$

Thus the sum $S(n)$, introduced in (2), is a special case of the integral $T(r)$. At the same time, Theorem 2.7 recovers the evaluation $S(n) = \pi$ for $n = 1, 2, 3 \ldots$ which we initially proved in Theorem 2.2 by different means.

Observe that the argument in (23) cannot be extended to non-integral $n = 2r$, since in that case we would have to apply the generalized binomial theorem [11, Theorem 7.46], and then exchange summation and integration. The step involving the beta function requires that $n = 2r$ be integral.

Remark 2.8 (Failure of Carlson's theorem)**.** By (10), the sum

$$Q(r) := r \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(r-k-\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(2k+2)\Gamma(r-2k)}$$

equals π when $r > 0$ is an integer. In general, for real $r > 0$, we have $Q(r) \neq \pi$ unless r is an integer. This demonstrates such an interesting resistance to the above applications of Carlson's theorem. An illustration is depicted in Figure 1 with a plot of $Q(r) - \pi$. Indeed, Q being analytic along vertical strips it can only coincide with π sporadically. $\Diamond$

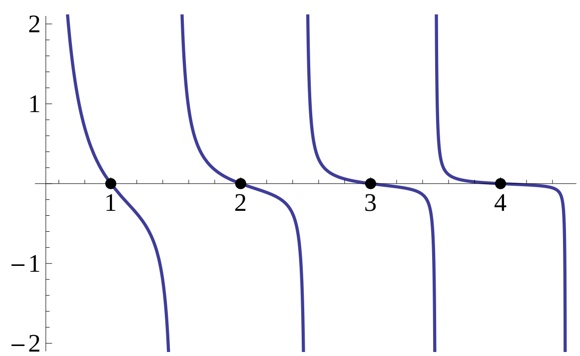

Figure 1: A plot of $Q(r) - \pi$

2.5 À la Fubini

We here give an alternative direct proof of (1) by using the identity

$$\frac{\sin(s \arctan(t/a))}{(a^2+t^2)^{s/2}} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} \sin(tx) \, \mathrm{d}x \tag{24}$$

to write the integral (1), in light analogy with the standard evaluation of the Gaussian integral, as a double integral. Equation (24) is recorded as Entry 3.944.5 in [5] and briefly proved next. For our purposes and simplicity we will assume that $a > 0$, $t \in \mathbb{R}$, and Re $s > 0$. Under these assumptions,

$$\begin{aligned}\frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sin(tx) \, \mathrm{d}x &= \frac{1}{2i} \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \left[e^{-(a-it)x} - e^{-(a+it)x} \right] \mathrm{d}x \\ &= \operatorname{Im} \frac{1}{(a-it)^s} \\ &= \frac{\sin(s \arctan(t/a))}{(a^2+t^2)^{s/2}}\end{aligned}$$

where, in the final step, we used that $a - it = \sqrt{a^2+t^2}\, e^{-i \arctan(t/a)}$.

Limiting special cases of (24) include

$$\int_0^\infty e^{-ax} \frac{\sin(x)}{x} \, \mathrm{d}x = \arctan(1/a), \qquad \int_0^\infty \frac{\sin(x)}{x} \, \mathrm{d}x = \frac{\pi}{2}$$

where the last integral converges only conditionally. Let $\varepsilon > 0$. Using these identities, we obtain

$$\begin{aligned}\Gamma(s) \arctan(1/\varepsilon) &= \int_0^\infty z^{s-1} e^{-z} \, \mathrm{d}z \int_0^\infty e^{-\varepsilon x} \frac{\sin(x)}{x} \, \mathrm{d}x \\ &= \int_0^\infty \int_0^\infty z^{s-1} e^{-z-\varepsilon x} \frac{\sin(x)}{x} \, \mathrm{d}x \, \mathrm{d}z \\ &= \int_0^\infty \int_0^\infty z^{s-1} e^{-(1+\varepsilon t)z} \frac{\sin(tz)}{t} \, \mathrm{d}t \, \mathrm{d}z \\ &= \Gamma(s) \int_0^\infty \frac{\sin(s \arctan(t/(1+\varepsilon t)))}{t((1+\varepsilon t)^2+t^2)^{s/2}} \, \mathrm{d}t\end{aligned}$$

where, in the last step, we changed the order of integration by Fubini and employed (24). Note that $\sin(s \arctan(T))$ is uniformly bounded, for $T \geqslant 0$, by some C_s, so that the final integrand is dominated by $\frac{C_s}{t(1+t^2)^{\operatorname{Re} s/2}}$. Also, for the $0 \leqslant t \leqslant 1$ the integrand is uniformly bounded. Hence we can let $\varepsilon \to 0$ and use dominated convergence to obtain

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin(s \arctan(t))}{t(1+t^2)^{s/2}} \, \mathrm{d}t.$$

This proves (1). We remark that, as in Theorem 2.7, the proof is valid for complex s with Re $s > 0$.

2.6 À la Fourier

We conclude this section by highlighting a connection between the evaluation of the Jauregui–Tsallis integral and basic results in Fourier analysis.

Let f be integrable on $(-\pi, \pi)$ having Fourier coefficients $\widehat{f}(k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-ikt}\,\mathrm{d}t$. One of the fundamental properties of the Dirichlet kernel [11, Theorem 8.27 (ii), p. 534] is that, for an integer n,

$$\sum_{k=-n}^{n} \widehat{f}(k)\,\mathrm{e}^{ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left((n+\frac{1}{2})\theta\right)}{\sin\left(\frac{\theta}{2}\right)} f(x-\theta)\,\mathrm{d}\theta. \tag{25}$$

In particular, if f is symmetric,

$$\sum_{k=-n}^{n} \widehat{f}(k) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sin((2n+1)\theta)}{\sin\theta} f(2\theta)\,\mathrm{d}\theta. \tag{26}$$

Assume n is a non-negative integer. Let's proceed in evaluating the integral

$$T\left(n+\frac{1}{2}\right) = \int_{-\pi/2}^{\pi/2} \frac{\sin((2n+1)\theta)}{\sin\theta}(\cos\theta)^{2n}\,\mathrm{d}\theta. \tag{27}$$

In light of (26), consider

$$g(\theta) := (\cos(\theta/2))^{2n} = \frac{1}{2^{2n}}\sum_{k=-n}^{n}\binom{2n}{n+k}\mathrm{e}^{ik\theta}.$$

Then, again by (26),

$$T\left(n+\frac{1}{2}\right) = \pi\sum_{k=-n}^{n}\widehat{g}(k) = \pi g(0) = \pi,$$

which is an apparent agreement with the previous evaluations.

The result $T(n) = \pi$ can be achieved by an inductive application of the recursions employed in the proof of Corollary 4.5.

3 Creative telescoping

In this section we wish to provide alternative proofs of the Jauregui–Tsallis conjectures using the *method of creative telescoping*. A very nice introduction to the underlying ideas of creative telescoping is [9] while a fine brief discussion is to be found in [2, Ch. 3]. Our aim is to illustrate and advertise the utility of this method. An obvious advantage of this approach is that it can be automated, to a large degree, in a computer algebra system such as *Maple* or *Mathematica*. A downside of this aspect is that proofs employing creative telescoping usually provide less insight into the problem than classical proofs. Also, we need to point out that we only prove the half-integral case in this section. The complete analytic case needs further consideration such as presented in Section 2.3.

First, we reprove Theorem 2.2.

Theorem 3.1. *For positive integers n,*

$$S(n) = n\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(n-k-\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(2k+2)\Gamma(n-2k)} = \pi. \tag{28}$$

Proof. Denote the summand as

$$f(n,k) := n(-1)^k \frac{\Gamma(n-k-\frac{1}{2})\Gamma(k+\frac{1}{2})}{\Gamma(2k+2)\Gamma(n-2k)}.$$

Then, using creative telescoping, we find that

$$\left[(N-1)+(K-1)\cdot\frac{k\,(2k+1)\,(2k+1-2n)}{(2k-n)\,n^2}\right]\cdot f\,(n,k)=0 \tag{29}$$

where N and K are the shift operators in n and k. That is, $N \cdot g(n,k) = g(n+1,k)$ and $K \cdot g(n,k) = g(n,k+1)$. We remark that verifying (29) is easy while the difficult part lies in discovering it algorithmically.

Since n is a positive integer. the sum has finite support and hence summing (29) over $k = 1, 2, 3, \ldots$ telescopes to

$$(N-1)\cdot S(n) = 0.$$

In other words, $S(n+1) = S(n)$ and the claim follows from $S(1) = \pi$. □

As a second demonstration, we reprove Theorem 2.3. The proof, in contrast to the previous example, employs creative telescoping with both discrete and continuous parameters. In hindsight, that is in light of (23), the two statements proven in Theorems 3.1 and 3.2 are equivalent.

Theorem 3.2. *For positive half-integers r,*

$$T(r) = \int_{-\infty}^{\infty} \frac{\sin(2r\arctan(t))}{t(1+t^2)^r}\,\mathrm{d}t = \pi. \tag{30}$$

Proof. Denote the integrand by

$$g(r,t) = \frac{\sin(2r\arctan(t))}{t(1+t^2)^r}.$$

Again, using creative telescoping, we find

$$\left[(R-1) - D_t\left(\frac{t\,(t^2+1)}{2\,(2r+1)}R + \frac{t\,(3r+1)}{2r\,(2r+1)}\right)\right]\cdot g\,(r,t) = 0 \tag{31}$$

where R is the shift in r and D_t the derivative with respect to t. Integrating (31) with respect to t over the real line, it follows that $(R-1)\cdot T(r) = 0$ whenever the integrals converge. That means, $T(r+1) = T(r)$ when $\operatorname{Re} r > 0$.

The boundary cases $T(1/2) = \pi$ and $T(1) = \pi$ are readily verified and they lead to $T\,(n/2) = \pi$ for all positive integers n. □

4 Chebyshev polynomials

Let T_n and U_n denote the *Chebyshev polynomials* of the first and second kind, respectively. Namely, T_n is defined by $T_n(\cos(\theta)) = \cos(n\theta)$ and U_n is as defined in (4). In this section we focus on the following two integrals

$$M_T(n,s) := \int_0^1 x^{s-1}T_n(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^{\pi/2}\cos(n\theta)\cos^{s-1}(\theta)\,\mathrm{d}\theta, \tag{32}$$

$$M_U(n,s) := \int_0^1 x^{s-1}U_n(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^{\pi/2}\frac{\sin((n+1)\theta)}{\sin(\theta)}\cos^{s-1}(\theta)\,\mathrm{d}\theta, \tag{33}$$

which are well-defined for $\operatorname{Re} s > 0$ and nonnegative integers n. In fact, using the trigonometric integrals, M_T and M_U are well-defined for arbitrary complex n. This case will be studied in Section 4.1. Observe that these integrals naturally generalize the Jauregui–Tsallis integral because

$$T(r) = 2M_U(2r-1, 2r). \tag{34}$$

For instance, the original conjecture (1) translates to $M_U(n, n+1) = \frac{\pi}{2}$. For an integer $n > 0$, an independent proof of this fact will be furnished by Corollary 4.5. Another motivation for our interest in the integral $M_U(n,s)$ is the following counterpart of $M_T(n,s)$ listed as Entry 7.346 in [5].

Proposition 4.1. *For* $n = 0, 1, 2, \ldots$ *and* $\mathrm{Re}\ s > 0$ *we have*

$$M_T(n,s) = \frac{\pi}{s 2^s B(\frac{s+n+1}{2}, \frac{s-n+1}{2})} = \frac{\pi}{2^s}\binom{s-1}{\frac{s-n-1}{2}}. \tag{35}$$

We therefore investigate the corresponding integral $M_U(n,s)$ for the Chebyshev polynomial of the second kind. The authors are not aware of a previous treatment of this integral.

Remark 4.2. For a fixed integer $n > 0$, expand $U_n(x)$ in terms of powers of x. Thus, for $\mathrm{Re}\ s > 0$, one derives the finite sum representations

$$M_U(n,s) = \frac{1}{2}\sum_{k\geqslant 0}(-1)^k \binom{n+1}{2k+1} B\left(\frac{n+s}{2} - k, k + \frac{1}{2}\right) \tag{36}$$

$$= \frac{1}{2}\sum_{k\geqslant 0}(-1)^k 2^{n-2k}\binom{n-k}{k} B\left(\frac{n+s}{2} - k, \frac{1}{2}\right). \tag{37}$$

We note that (36) directly generalizes (2) which, given (34), is the special case $n = 2r - 1$, $s = 2r$. ♢

The recursive relation between Chebyshev polynomials of first and second kind leads to the following link between the integrals.

Lemma 4.3. *Suppose* $\mathrm{Re}\,(s) > 0$. *Then*

$$M_U(n+2,s) - M_U(n,s) = 2M_T(n+2,s). \tag{38}$$

Proof. Use the identity $U_{n+2} - U_n = 2T_{n+2}$. □

Combining Lemma 4.3 and the evaluation (35) we find:

Corollary 4.4 (Finite series for $M_U(n,s)$)**.** *For positive integers* n *and* $\mathrm{Re}\ s > 0$ *we have the parity-dependent identities:*

$$M_U(2n-1,s) = \frac{2\pi}{2^s}\sum_{j=1}^{n}\binom{s-1}{\frac{s}{2} - 1 + j}$$

$$M_U(2n,s) = \frac{\pi}{2^s}\binom{s-1}{\frac{s-1}{2}} + \frac{2\pi}{2^s}\sum_{j=1}^{n}\binom{s-1}{\frac{s-1}{2} + j}$$

These identities allow yet another proof of the Jauregui–Tsallis integral evaluation $T(r) = 2M_U(2r-1, 2r) = \pi$, in the case of half-integral r.

Corollary 4.5. *For positive integers r we have*

$$M_U(r, r+1) = \frac{\pi}{2}.$$

Proof. Using Corollary 4.4 we find

$$M_U(2n-1, 2n) = \frac{2\pi}{2^{2n}} \sum_{j=1}^{n} \binom{2n-1}{n-1+j} = \frac{2\pi}{2^{2n}} \frac{2^{2n-1}}{2} = \frac{\pi}{2}$$

and, likewise, $M_U(2n, 2n+1) = \frac{\pi}{2}$. □

On the other hand:

Lemma 4.6. *Suppose* $\mathrm{Re}\,(s) > 0$. *Then*

$$M_U(n, s) - M_U(n-1, s+1) = M_T(n, s). \tag{39}$$

Proof. Applying the addition formula

$$\sin((n+1)\theta) = \sin(n\theta)\cos(\theta) + \cos(n\theta)\sin(\theta)$$

in the definition (33) yields the claim. □

This result enables us to avail another sum representation of $M_U(n, s)$ when n is an integer. The statement below complements the parity-dependent sums of Corollary 4.4.

Lemma 4.7. *For positive integers n and* $\mathrm{Re}\,(s) > 0$*,*

$$M_U(n, s) = \frac{\pi}{2^s} \sum_{j=0}^{n-1} \frac{1}{2^j} \binom{s+j-1}{\frac{s+n-1}{2}} + \frac{\pi}{2(s+n)} \binom{\frac{s+n}{2}}{\frac{1}{2}}. \tag{40}$$

Proof. Using the functional equation from Lemma 4.6 repeatedly, we find

$$\begin{aligned} M_U(n, s) &= M_U(n-1, s+1) + M_T(n, s) \qquad (41)\\ &= M_U(0, s+n) + \sum_{j=0}^{n-1} M_T(n-j, s+j) \\ &= M_U(0, s+n) + \sum_{j=0}^{n-1} \frac{\pi}{2^{s+j}} \binom{s+j-1}{\frac{s+n-1}{2}} \end{aligned}$$

where we used the evaluation (35) of M_T. Then (40) follows from

$$M_U(0,s) = \int_0^1 \frac{x^{s-1}}{\sqrt{1-x^2}}\, \mathrm{d}x = \frac{1}{2} B\left(\tfrac{s}{2}, \tfrac{1}{2}\right) = \frac{\pi}{2s} \binom{\frac{s}{2}}{\frac{1}{2}} \tag{42}$$

which is a consequence of Euler's integral representation of the beta function. □

Remark 4.8. The recurrence in (41) is equivalent to

$$2t \left[{}_2F_1 \left(\left. \begin{matrix} 1, s \\ t \end{matrix} \right| \frac{1}{2} \right) - 1 \right] = s\, {}_2F_1 \left(\left. \begin{matrix} 1, s+1 \\ t+1 \end{matrix} \right| \frac{1}{2} \right).$$

Thus for all $n = 0, 1, 2, \ldots$ we obtain the evaluation

$${}_2F_1 \left(\left. \begin{matrix} 1, s+n \\ t+n \end{matrix} \right| \frac{1}{2} \right) = 2^n \frac{(t)_n}{(s)_n} \left[{}_2F_1 \left(\left. \begin{matrix} 1, s \\ t \end{matrix} \right| \frac{1}{2} \right) - \sum_{k=0}^{n-1} 2^{-k} \frac{(s)_k}{(t)_k} \right],$$

valid for all s, t. ◊

Observe that for integers m, n formula (40) shows that $M_U(n,m)$ is rational when m and n have the same parity, and is a rational multiple of π otherwise. This is illustrated in Figure 2 in which we have written $\tau := \frac{\pi}{2}$. The apparent pattern for when $M_U(n,m)$ evaluates as τ will be proved and explained in (58). A rather convenient alternative is to apply the same reasoning as for (27) with the exponent $2n$ of the cosine replaced by $2m$ for $m \leqslant n$.

$$\begin{bmatrix}
2 & \tau & \frac{4}{3} & \frac{3}{8}\pi & \frac{16}{15} & \frac{5}{16}\pi & \frac{32}{35} & \frac{35}{128}\pi & \frac{256}{315} & \frac{63}{256}\pi & \frac{512}{693} & \frac{231}{1024}\pi \\
\tau & \frac{5}{3} & \tau & \frac{22}{15} & \frac{7}{16}\pi & \frac{136}{105} & \frac{25}{64}\pi & \frac{368}{315} & \frac{91}{256}\pi & \frac{3712}{3465} & \frac{21}{64}\pi & \frac{1280}{1287} \\
\frac{4}{3} & \tau & \frac{8}{5} & \tau & \frac{32}{21} & \frac{15}{32}\pi & \frac{64}{45} & \frac{7}{16}\pi & \frac{512}{385} & \frac{105}{256}\pi & \frac{1024}{819} & \frac{99}{256}\pi \\
\tau & \frac{23}{15} & \tau & \frac{166}{105} & \tau & \frac{488}{315} & \frac{31}{64}\pi & \frac{5168}{3465} & \frac{119}{256}\pi & \frac{64384}{45045} & \frac{57}{128}\pi & \frac{61696}{45045} \\
\frac{26}{15} & \tau & \frac{164}{105} & \tau & \frac{496}{315} & \tau & \frac{5408}{3465} & \frac{63}{128}\pi & \frac{68864}{45045} & \frac{123}{256}\pi & \frac{67072}{45045} & \frac{957}{2048}\pi \\
\tau & \frac{167}{105} & \tau & \frac{494}{315} & \tau & \frac{1816}{1155} & \tau & \frac{70544}{45045} & \frac{127}{256}\pi & \frac{13952}{9009} & \frac{501}{1024}\pi & \frac{1167104}{765765}
\end{bmatrix}$$

Figure 2: A matrix of $M_U(n,m)$ values for $1 \leqslant n \leqslant 6$ and $1 \leqslant m \leqslant 12$

4.1 Hypergeometric evaluations in the complex case

We now extend the results on $M_U(n,s)$ to a complex parameter n. The outcome will be expressed in terms of hypergeometric functions; see (7) for the definition of the latter. The following evaluation will be utilized.

Proposition 4.9. *For* Re $s>0$ *we have*

$$ {}_2F_1\left(\begin{matrix}1, s\\ \frac{s+1}{2}\end{matrix}\middle|\frac{1}{2}\right) = 1 + \frac{\sqrt{\pi}\,\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = 1 + \frac{\pi}{2}\binom{\frac{s-1}{2}}{\frac{1}{2}}. \tag{43}$$

Proof. We rewrite the terms of the hypergeometric series as

$$\frac{(s)_n}{\left(\frac{s+1}{2}\right)_n}\left(\frac{1}{2}\right)^n = \frac{s}{s+1}\frac{(s+1)_{n-1}}{\left(\frac{s+3}{2}\right)_{n-1}}\left(\frac{1}{2}\right)^{n-1} \tag{44}$$

to obtain

$$ {}_2F_1\left(\begin{matrix}1, s\\ \frac{s+1}{2}\end{matrix}\middle|\frac{1}{2}\right) - 1 = \frac{s}{s+1}\,{}_2F_1\left(\begin{matrix}1, s+1\\ \frac{s+3}{2}\end{matrix}\middle|\frac{1}{2}\right). \tag{45}$$

The right-hand side hypergeometric series now allows an implementation of Gauss's second summation theorem [8, Eqn (15.4.28)]

$$ {}_2F_1\left(\begin{matrix}a, b\\ \frac{a+b+1}{2}\end{matrix}\middle|\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} \tag{46}$$

which yields the value seen in (43). □

Remark 4.10. Proposition 4.9 may also be proved mechanically using the methods of Section 3. Converting the hypergeometric series to an integral and after some manipulation, (43) becomes equivalent to

$$2^s\int_0^1 \frac{x^{(s-3)/2}}{(x+1)^s}\,\mathrm{d}x = B\left(\frac{1}{2}, \frac{s-1}{2}\right) + \frac{2}{s-1} \tag{47}$$

and holding for Re $s>0$. Now, both sides of (47) remain bounded on vertical lines in the positive half-plane. Hence, to apply Carlson's Theorem 2.4, we need only confirm (47) at sufficiently large positive odd integers. To this end we discover and verify that both sides of (47) satisfy the recursive relation

$$(s+2)(s+3)f(s+4) - (s+1)(2s+3)f(s+2) + (s^2-1)f(s) = 0,$$

together with initial conditions $f(3)=3,\ f(5)=\frac{11}{6}$. ◇

Here comes an exact counterpart to Proposition 4.1.

Theorem 4.11. *For complex n and* $\operatorname{Re} s > 0$, *we have*

$$M_U(n,s) = \frac{\pi}{2^s}\binom{s-1}{\frac{s-n-1}{2}}\,{}_2F_1\left(\left.\begin{matrix}1, s\\ \frac{s-n+1}{2}\end{matrix}\right|\frac{1}{2}\right) - \frac{\pi}{2}. \tag{48}$$

Proof. Suppose n is a positive integer. In this case $M_U(n,s)$ is given by (40), namely that

$$M_U(n,s) = \frac{\pi}{2^s}\sum_{j=0}^{n-1}\frac{1}{2^j}\binom{s+j-1}{\frac{s+n-1}{2}} + \frac{\pi}{2(s+n)}\binom{\frac{s+n}{2}}{\frac{1}{2}}. \tag{49}$$

In order to write the right-hand side in hypergeometric terms, we note that

$$\sum_{j=0}^{\infty}\binom{a+j}{b}x^j = \binom{a}{b}\,{}_2F_1\left(\left.\begin{matrix}1, a+1\\ a-b+1\end{matrix}\right|x\right) \tag{50}$$

and this is immediate from the definition (7). Consequently,

$$\sum_{j=0}^{n-1}\binom{a+j}{b}x^j = \binom{a}{b}\,{}_2F_1\left(\left.\begin{matrix}1, a+1\\ a-b+1\end{matrix}\right|x\right) - x^n\binom{a+n}{b}\,{}_2F_1\left(\left.\begin{matrix}1, a+n+1\\ a+n-b+1\end{matrix}\right|x\right)$$

which, applied to (49), shows that

$$\begin{aligned} M_U(n,s) = {} & \frac{\pi}{2^s}\binom{s-1}{\frac{s-n-1}{2}}\,{}_2F_1\left(\left.\begin{matrix}1, s\\ \frac{s-n+1}{2}\end{matrix}\right|\frac{1}{2}\right) - \frac{\pi}{2^{n+s}}\binom{s+n-1}{\frac{s+n-1}{2}}\,{}_2F_1\left(\left.\begin{matrix}1, s+n\\ \frac{s+n+1}{2}\end{matrix}\right|\frac{1}{2}\right) \\ & + \frac{\pi}{2(s+n)}\binom{\frac{s+n}{2}}{\frac{1}{2}}. \end{aligned} \tag{51}$$

The final form in (48) is inferred by substituting the result from Proposition 4.9 that

$${}_2F_1\left(\left.\begin{matrix}1, s+n\\ \frac{s+n+1}{2}\end{matrix}\right|\frac{1}{2}\right) = 1 + \frac{\pi}{2}\binom{\frac{s+n-1}{2}}{\frac{1}{2}},$$

and after simplifying the residual binomial terms.

Suppose n is a complex number. Since, as in the proof of Theorem 2.7, the main ingredient is Carlson's Theorem we will be a bit sketchy here. Let $\varepsilon > 0$. Using the inequalities employed in the proof of Proposition 2.6, one finds that, for all $\operatorname{Re} s > \varepsilon$,

$$|M_U(n,s)| \leqslant C|n+1|\, e^{\frac{\pi}{2}|\operatorname{Im} n|} \tag{52}$$

where $C = C(\varepsilon)$ is some constant. It follows from Theorem 2.5 that $M_U(n,s)$ is an analytic function of n for fixed $\text{Re } s > 0$. The claim will therefore follow from Carlson's Theorem 2.4 once we are able to appropriately bound the right-hand side of (48) for each fixed s. Using Euler's integral representation of the hypergeometric series, we find

$$\binom{s-1}{\frac{s-n-1}{2}} {}_2F_1\left(\left.{1, s \atop \frac{s-n+1}{2}}\right| \frac{1}{2}\right) = -\frac{1}{\pi} \sin\left(\tfrac{(s+n-1)\pi}{2}\right) \int_0^1 \frac{x^{s-1}}{(1-\frac{x}{2})(1-x)^{(s+n+1)/2}} \, \mathrm{d}x.$$

This is valid so long as $\text{Re } s > 0$ and $\text{Re}\,(s+n) < 1$. Observe that the integral remains bounded on vertical lines $\text{Re } n = c$. On each such line, the sine term may be bounded by some constant multiple of $\mathrm{e}^{d|n|}$ with $d = \frac{\pi}{2} < \pi$. For fixed s with $0 < \text{Re } s < 1$, we therefore have, on each fixed line $\text{Re } n = c \leqslant 0$,

$$\left|\binom{s-1}{\frac{s-n-1}{2}} {}_2F_1\left(\left.{1, s \atop \frac{s-n+1}{2}}\right| \frac{1}{2}\right)\right| \leqslant C \mathrm{e}^{\frac{\pi}{2}|n|} \tag{53}$$

where $C = C(c)$ does not depend on the imaginary part of n. In light of the recurrence

$$(n+s+3) f(n+4,s) - 2s f(n+2,s) - (n-s+3) f(n,s) = 0, \tag{54}$$

which is readily checked to be satisfied by both sides of (48), one finds that the growth condition (53) is also fulfilled on each vertical line $\text{Re } n = c > 0$ (assuming the growth condition for $\text{Re } n = c$ and $\text{Re } n = c+2$ the recurrence yields an inequality for $\text{Re } n = c+4$). Moreover, one discerns from (53) and (54) that the right-hand side of (48) necessarily is of exponential type. Under the restriction that $0 < \text{Re } s < 1$, the equality (48) therefore follows from Carlson's Theorem 2.4. Since both sides of (48) are analytic in s, the restriction $\text{Re } s < 1$ may be removed by analytic continuation. □

The following alternative representation is a consequence of Pfaff's hypergeometric transformation [8, Eqn (15.8.1)] applied to (48).

Corollary 4.12. *For complex n and $\text{Re } s > 0$, we have*

$$M_U(n,s) = \pi \binom{s-1}{\frac{s-n-1}{2}} {}_2F_1\left(\left.{s, \frac{s-n-1}{2} \atop \frac{s-n+1}{2}}\right| -1\right) - \frac{\pi}{2}. \tag{55}$$

As an immediate consequence of (55) we obtain, for Re $r > 0$,

$$M_U(r, r+1) = \pi \, {}_2F_1\left(\begin{matrix} r+1, 0 \\ 1 \end{matrix} \middle| -1\right) - \frac{\pi}{2} = \frac{\pi}{2}. \tag{56}$$

This is equivalent to the evaluation (1), previously conjectured by Jauregui and Tsallis and already proven in Theorem 2.7, which is a motivation for this paper.

Moreover, we can now explain the 'τ's' in Figure 2. We have from (55) and since

$$M_U(n, s) = -M_U(-n-2, s)$$

that

$$\begin{aligned} M_U(n, n+1-2m) &= -M_U(-n-2, n+1-2m) \\ &= \frac{\pi}{2} - \pi \binom{n-2m}{m-n+1} \, {}_2F_1\left(\begin{matrix} n-m+1, n-2m+1 \\ n-m+2 \end{matrix} \middle| -1\right). \end{aligned} \tag{57}$$

Thus

$$M_U(n, n+1-2m) = \frac{\pi}{2}, \tag{58}$$

for n a nonnegative integer and $m = 0, 1, \ldots \lfloor \frac{n}{2} \rfloor$ — because in this case the binomial term vanishes while the hypergeometric term is finite.

Remark 4.13 (On the imaginary axis)**.** The integral $M_U(n, s)$, as defined by (33), converges for Re $s > 0$. If Re $s = 0$, then it diverges unless n is an odd integer. One readily checks that, for $s = 0$,

$$M_U(2n+1, 0) = \frac{\pi}{2} + (-1)^n \frac{\pi}{2}. \tag{59}$$

Some care, however, should be exercised when using various of the other representations derived for M_U. For instance, an attempt in using the binomial sum obtained in Corollary 4.4 results in doubling the correct values. The reason comes down to the fact that the quantity

$$\binom{s-1}{\frac{s}{2}} = \frac{\Gamma(s)}{\Gamma(\frac{s}{2}+1)\Gamma(\frac{s}{2})}$$

evaluates as $\binom{-1}{0} = 1$ if $s = 0$; on the other hand, the limiting value ($s \to 0$) is $\frac{1}{2}$. $\diamondsuit$

5 Ultraspherical polynomials

Herein, following a suggestion of the referee, we offer a further generalization of the integrals from the preceding sections. To this end, let us recall the *ultraspherical* (or Gegenbauer) polynomials which may be obtained from the explicit [1, Chapter 22] summation representation

$$C_n^\lambda(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n-k+\lambda)}{k!(n-2k)!\Gamma(\lambda)} (2x)^{n-2k}. \tag{60}$$

As in Section 2.1, the summation in (60) may be extended to ∞. The choice of $\lambda = 1$ gives the Chebyshev polynomial of the second kind: $C_n^1 = U_n$.

In the sequel, we consider the integral

$$Z_n^\lambda(s,\mu) := \int_0^1 x^{s-1} C_n^\lambda(x)(1-x^2)^\mu \,\mathrm{d}x \tag{61}$$

which generalizes $M_U(n,s)$ as defined in (33). Indeed, $M_U(n,s) = Z_n^1(s,-\frac{1}{2})$. For convergence of the integral, we will assume that $\operatorname{Re} s > 0$ and $\operatorname{Re} \mu > -1$.

Integrating termwise using (60) and invoking the Euler integral representation for the beta function, it follows that

$$Z_n^\lambda(s,\mu) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n-k+\lambda)}{k!(n-2k)!\Gamma(\lambda)} B\left(\mu+1, \tfrac{1}{2}(s+n-2k)\right) 2^{n-2k-1}. \tag{62}$$

Proceeding as in Section 2.1 and using the Legendre duplication formula, the integral takes on the hypergeometric form

$$Z_n^\lambda(s,\mu) = \frac{2^{n-1}\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)} B\left(\mu+1, \tfrac{s+n}{2}\right) {}_3F_2\left(\left. \begin{matrix} -\frac{n}{2}, -\frac{n-1}{2}, -\frac{s+n+2\mu}{2} \\ -\frac{s+n-2}{2}, -n-\lambda+1 \end{matrix} \right| 1 \right). \tag{63}$$

Example 5.1. In the case of $\lambda = 1$, $\mu = -\frac{1}{2}$, this results in

$$M_U(n,s) = 2^{n-1} B\left(\tfrac{1}{2}, \tfrac{s+n}{2}\right) {}_3F_2\left(\left. \begin{matrix} -\frac{n}{2}, -\frac{n-1}{2}, -\frac{s+n-1}{2} \\ -\frac{s+n-2}{2}, -n \end{matrix} \right| 1 \right). \tag{64}$$

Note that the right-hand side has to be treated with some care: the parameter $-n$ of the hypergeometric function is a negative integer which, by itself, would result in a division by zero in the hypergeometric series; here, one of the parameters $-n/2$

and $-(n-1)/2$ cancels this zero. Note also that the sum of the top and bottom parameters of the ${}_3F_2$ agree, so that the hypergeometric series converges only because it terminates. We return to this point in Example 5.2 but remark that the integrality of n is crucial for the convergence of the hypergeometric series.

In light of the evaluation of M_U, from Theorem 4.11, we have the hypergeometric identity

$$2^{n-1}B\left(\tfrac{1}{2},\tfrac{s+n}{2}\right){}_3F_2\left(\left.\begin{matrix}-\frac{n}{2},-\frac{n-1}{2},-\frac{s+n-1}{2}\\ -\frac{s+n-2}{2},-n\end{matrix}\right|1\right)=\frac{\pi}{2^s}\binom{s-1}{\frac{s-n-1}{2}}{}_2F_1\left(\left.\begin{matrix}1,s\\ \frac{s-n+1}{2}\end{matrix}\right|\frac{1}{2}\right)-\frac{\pi}{2}$$

where both sides equal $M_U(n,s)$. Indeed this identity may be proven automatically using the tools of Section 3 by showing that both sides satisfy the fourth order recurrence $(s+n+3)N^4-2sN^2+(s-n-3)=0$ and checking initial values.

However, we wish to stress that the ${}_2F_1$ representation of M_U established in Theorem 4.11, beside being simpler and easier to apply (as demonstrated in the next example), holds for complex n (with the Chebyshev integral naturally interpreted as on the right-hand side of (33)). It is this case which required separate attention in Section 4. $\Diamond$

Example 5.2. In particular, let us revisit the evaluation $M_U(n,n+1)=\frac{\pi}{2}$, given in (56), which is equivalent to the Jauregui–Tsallis integral evaluation (1). In terms of the integral Z_n^λ, we have

$$M_U(n,n+1)=Z_n^1(n+1,-\tfrac{1}{2})=2^{n-1}B\left(\tfrac{1}{2},n+\tfrac{1}{2}\right){}_2F_1\left(\left.\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2}\\ -n+\frac{1}{2}\end{matrix}\right|1\right). \qquad (65)$$

Observe that the parameters $a=-\frac{n}{2}$ and $b=-\frac{n}{2}+\frac{1}{2}$ add up to the third parameter $c=-n+\frac{1}{2}=a+b$. The hypergeometric series therefore does not converge when n is not an integer (if $n\geqslant 0$ is an integer then the sum terminates). In particular, the classical theorem of Gauss, for summing a ${}_2F_1$ at 1, does not apply (it needs $\operatorname{Re}(c-a-b)>0$). Instead, when n is a nonnegative integer, we may rewrite the hypergeometric term as

$$2^{n-1}B\left(\tfrac{1}{2},n+\tfrac{1}{2}\right)\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{2k}\binom{n}{k}}{\binom{2n}{2k}}=\frac{\pi}{2}.$$

Here, in contrast to the previous evaluations of M_U, the equality with $\frac{\pi}{2}$ is not entirely obvious; yet, once observed, it may be proven automatically using the methods of Section 3.

Moreover, the above argument only applies to the case that n is an integer. That the right-hand side of (65) does not converge when n is not an integer demonstrates that the evaluation of $Z_n^\lambda(s,\mu)$ as a ${}_3F_2$, given in (63), cannot be used directly to recover the results from Section 4. Those were, to a large extent, concerned with the case of complex n. $\diamondsuit$

Without pursuing this path further, we remark that the present treatment may be further generalized to the Jacobi polynomials $P_n^{(\alpha,\beta)}$ of which the ultraspherical polynomials are the special case $C_n^\lambda = P_n^{(\lambda,\lambda)}$.

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