Monotonicity of Certain Riemann Sums

David Borwein * Jonathan M. Borwein † Brailey Sims ‡

January 14, 2015

Abstract

We consider conditions ensuring the monotonicity of right and left Riemann sums of a function $f : [0,1] \to \mathbb{R}$ with respect to uniform partitions. Experimentation suggests that symmetrization may be important and leads us to results such as: *if f is decreasing on $[0,1]$ and its symmetrization, $F(x) := \frac{1}{2}(f(x) + f(1-x))$ is concave then its right Riemann sums increase monotonically with partition size.* Applying our results to functions such as $f(x) = 1/\left(1+x^2\right)$ also leads to a nice application of Descartes' rule of signs.

1 Introduction

For a bounded function $f : [0,1] \to \mathbb{R}$ the *left* and *right Riemann sums* of f with respect to the uniform partition $\mathcal{U}_n$ of $[0,1]$ into n equal intervals are,

$$\sigma_n := \sigma_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right), \quad \text{and} \quad \tau_n := \tau_n(f) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

*Department of Mathematics, Western University, London, ON, Canada. Email: dborwein@uwo.ca.

†Centre for Computer-assisted Research Mathematics and its Applications (CARMA), School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia. Email: jonathan.borwein@newcastle.edu.au, jborwein@gmail.com

‡Centre for Computer-assisted Research Mathematics and its Applications (CARMA), School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia. Email: brailey.sims@newcastle.edu.au.

Both are linear functionals with $\sigma_n(1) = \tau_n(1) = 1$ and $\sigma_n(f) - \tau_n(f) = \dfrac{1}{n}(f(0) - f(1))$. If f is decreasing (increasing) on $[a, b]$ then σ_n is the upper (lower), and τ_n the lower (upper), Riemann sum of f with respect to $\mathcal{U}_n$. And, of course, if f is Riemann integrable (as it is in either of the above cases) then both σ_n and τ_n converge to $\int_0^1 f(x)\mathrm{dx}$ (see, for example [1]). Further, for all n, $\sigma_n(f(1-x)) = \tau_n(f(x))$, so if f is symmetric about the midpoint of $[0, 1]$; that is, $f(x) = f(1-x)$, then $\tau_n = \sigma_n$.

We seek conditions which will ensure the sequence (σ_n), (τ_n), or perhaps some other sequence of related Riemann sum, increases/decreases with n. If for example f is decreasing then $\tau_{2n} \geq \tau_n$, so τ_{2^n} increases monotonically to $\int_0^1 f$, but how does τ_{n+1} compare to τ_n?

In the process of producing [2] one of the current authors gave the following example.

Example 1 (Digital assistance, arctan(1) and a black-box)**.** Consider for integer $n > 0$ the sum

$$\sigma_n := \sum_{k=0}^{n-1} \frac{n}{n^2 + k^2}.$$

The definition of the Riemann sum means that

$$\begin{aligned} \lim_{n\to\infty} \sigma_n &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{1 + (k/n)^2} \frac{1}{n} \\ &= \int_0^1 \frac{1}{1+x^2} \mathrm{d}x \\ &= \arctan(1). \end{aligned} \tag{1}$$

Even without being able to do this *Maple* will quickly tell you that

$$\sigma_{10^{14}} = 0.78539816339746\ldots$$

Now if you ask for 100 billion terms of most slowly convergent series, a computer will take a long time. So this is only possible because *Maple* “knows”

$$\sigma_N = -\frac{i}{2}\Psi\left(N - iN\right) + \frac{i}{2}\Psi\left(N + iN\right) + \frac{i}{2}\Psi\left(-iN\right) - \frac{i}{2}\Psi\left(iN\right)$$

and has a fast algorithm for computing our new friend the psi function of a complex variable. Now `identify`(0.78539816339746) yields $\frac{\pi}{4}$.

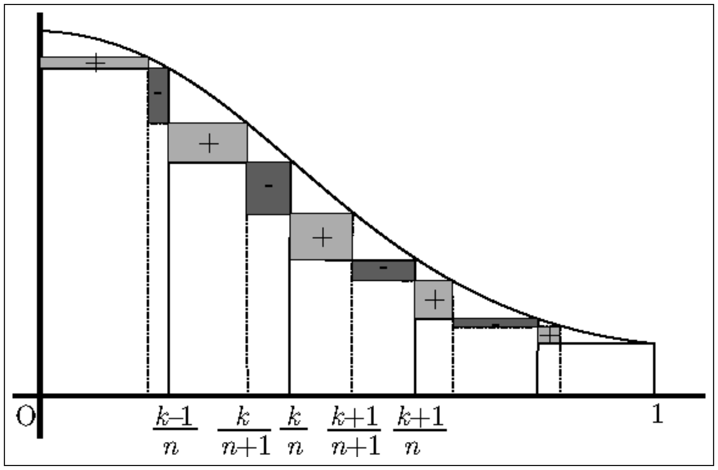

Figure 1: Difference in the lower Riemann sums for $\frac{1}{1+x^2}$

We can also note that

$$\tau_n := \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$$

is another (lower) Riemann sum converging to $\int_0^1 \frac{1}{1+x^2} \mathrm{d}x$. Indeed, $\sigma_n - \tau_n = \frac{1}{2n} > 0$. Moreover, experimentation *suggests* that σ_n decreases, and τ_n increases, to $\pi/4$. As we will see, the validity of this at least for τ_n is a consequence of our principal result.$\diamondsuit$

If we enter "monotonicity of Riemann sums" into Google, one of the first entries is http://elib.mi.sanu.ac.rs/files/journals/tm/29/tm1523.pdf which is a 2012 article by Szilárd[4] that purports to show the monotonicity of the two sums for the function

$$f(x) := \frac{1}{1+x^2}.$$

The paper goes on to prove that *if* $f\colon [0,1] \to R$ *is concave, or convex, and decreasing then* $\tau_n := \frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n})$ *increases and* $\sigma_n := \frac{1}{n}\sum_{k=0}^{n-1} f(\frac{k}{n})$ *decreases to* $\int_0^1 f(x)\,\mathrm{d}x$*, as* $n \to \infty$. Related results for a concave, or convex, and increasing function follow by applying these results to $-f$.

All proofs in [4] are based on looking at the rectangles which comprise the difference between τ_{n+1} and τ_n as in Figure 1 (or the corresponding sums for σ_n). This yields

$$\tau_{n+1}(f) - \tau_n(f) \;=\; \frac{1}{n}\sum_{k=1}^{n}\left\{\frac{(n+1-k)}{n+1} f\left(\frac{k}{n+1}\right) + \frac{k}{n+1} f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n}\right)\right\}. \tag{2}$$

In the easiest case, each bracketed term

$$\delta_n(k) := \frac{(n+1-k)}{n+1} f\left(\frac{k}{n+1}\right) + \frac{k}{n+1} f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n}\right)$$

has the same sign for all n and $1 \le k \le n$ as happens for a function which is concave, or convex, and decreasing.

But in [4] the author mistakenly asserts this applies for $1/(1+x^2)$ which has an inflection point at $1/\sqrt{3}$. Indeed, the proffered proof flounders at the inequality in the last line of [4, p. 115] which fails for instance when $n = 5$ and $k = 1$. This same error invalidates the assertion in [4] that the monotonicity of the corresponding σ_n can be proved by by similar reasoning (left to the reader). Below in Corollary 3 and Example 2 we supply a correct proof that $\tau_n = \sum_{k=1}^{n} n/(n^2+k^2)$ increases, but we are unable as of yet to prove that the corresponding σ_n decreases.

It appears, however, on checking in a computer algebra system (CAS), that $\delta_n(k) + \delta_n(n-k) \ge 0$ which if rigourously established would repair the hole in the proof, it also suggests that symmetry may have a role to play.

In our opinion all of this provides a fine instance of digital assistance in action.

For the convenience of the reader we supply the following proofs of Szilárd's theorems. The proofs are basically his but a bit cleaner. The proofs use telescoping and do not need consideration of the + and - rectangles of Figure 1.

2 Szilárd's Theorems

Theorem 1. *If the function $f : [0,1] \to \mathbb{R}$ is concave and decreasing on the interval $[0,1]$, then $\tau_n(f)$ increases and $\sigma_n(f)$ decreases as n increases.*

Theorem 2. *If the function $f : [0,1] \to \mathbb{R}$ is convex and decreasing on the interval $[0,1]$, then $\tau_n(f)$ increases and $\sigma_n(f)$ decreases as n increases.*

Before proceeding to the proofs of Theorems 1 and 2 we first give two lemmas.

Lemma 1. *If* $f : [0,1] \to \mathbb{R}$ *is concave and decreasing on the interval* $[0,1]$, *then*

$$f\left(\frac{k+1}{n+1}\right) \geq \frac{n-k}{n} f\left(\frac{k+1}{n}\right) + \frac{k}{n} f\left(\frac{k}{n}\right). \tag{3}$$

Proof. Since f is concave on $[0,1]$ we have

$$\begin{aligned} \frac{n-k}{n} f\left(\frac{k+1}{n}\right) + \frac{k}{n} f\left(\frac{k}{n}\right) &\leq f\left(\frac{n-k}{n} \cdot \frac{k+1}{n+1} + \frac{k}{n} \cdot \frac{k}{n}\right) \\ &= f\left(\frac{nk+n-k}{n^2}\right). \end{aligned} \tag{4}$$

Due to the monotonicity of f on $[0,1]$ and the readily verified inequality

$$\frac{nk+n-k}{n^2} \geq \frac{k+1}{n+1}, \tag{5}$$

we have

$$f\left(\frac{k+1}{n+1}\right) \geq f\left(\frac{nk+n-k}{n^2}\right). \tag{6}$$

Together, inequalities (4) and (6) imply inequality (3). This completes the proof of the lemma. □

Lemma 2. *If* $f : [0,1] \to \mathbb{R}$ *is convex and decreasing on the interval* $[0,1]$, *then*

$$f\left(\frac{k}{n}\right) \leq \frac{n+1-k}{n+1} f\left(\frac{k}{n+1}\right) + \frac{k}{n+1} f\left(\frac{k+1}{n+1}\right). \tag{7}$$

Proof. Since f is convex on $[0,1]$ we have

$$\begin{aligned} \frac{n+1-k}{n+1} f\left(\frac{k}{n+1}\right) + \frac{k}{n+1} f\left(\frac{k+1}{n+1}\right) &\geq f\left(\frac{n+1-k}{n+1} \cdot \frac{k}{n+1} + \frac{k}{n+1} \cdot \frac{k+1}{n+1}\right) \\ &= f\left(\frac{(n+2)k}{(n+1)^2}\right). \end{aligned} \tag{8}$$

Due to the monotonicity of f on $[0,1]$ and the inequality

$$\frac{(n+2)k}{(n+1)^2} \leq \frac{k}{n}, \tag{9}$$

we have

$$f\left(\frac{(n+2)k}{(n+1)^2}\right) \geq f\left(\frac{k}{n}\right). \tag{10}$$

Together, inequalities (8) and (10) imply inequality (7). This completes the proof of the lemma. □

Proof of Theorem 1. Since for any constant K we have $\tau_n(f+K) = \tau_n(f) + K$ (and the same for σ_n), we may suppose without loss in generality that $f(1) = 0$. Observe that inequality (3) is equivalent to

$$\frac{1}{n+1} f\left(\frac{k+1}{n+1}\right) \geq \frac{1}{n} f\left(\frac{k+1}{n}\right) + \frac{1}{n(n+1)}\left(kf\left(\frac{k}{n}\right) - (k+1) f\left(\frac{k+1}{n}\right)\right), \tag{11}$$

from which it follows that

$$\frac{1}{n+1}\sum_{k=0}^{n-1} f\left(\frac{k+1}{n+1}\right) \geq \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k+1}{n}\right),$$

or equivalently

$$\tau_{n+1}(f) = \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) \geq \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \tau_n(f). \tag{12}$$

This completes the proof of the first part of Theorem 1. The second part can be obtained by applying the first part of Theorem 2 (established below) to $-f(1-x)$. □

Proof of Theorem 2. We again suppose without loss in generality that $f(1) = 0$. Observe that inequality (7) is equivalent to

$$\frac{1}{n} f\left(\frac{k}{n}\right) \leq \frac{1}{n+1} f\left(\frac{k}{n+1}\right) + \frac{1}{n(n+1)}\left(kf\left(\frac{k+1}{n+1}\right) - (k-1) f\left(\frac{k}{n+1}\right)\right), \tag{13}$$

from which it follows that

$$\tau_n(f) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq \frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n+1}\right) = \tau_{n+1}(f). \tag{14}$$

This completes the first part of the proof of Theorem 2. The second part can be obtained by applying the first part of Theorem 1 to $-f(1-x)$. □

3 Extensions of Szilárd's Theorems

Theorem 3. *If the function* $f : [0,1] \to \mathbb{R}$ *is convex on the interval* $[0,c]$, *concave on* $[c,1]$, *and decreasing on* $[0,1]$, *then* $\tau_n(f)$ *increases and* $\sigma_n(f)$ *decreases as* n *increases.*

Proof. Define

$$f_1(x) := \begin{cases} f(x) & \text{for } 0 \leq x \leq c \\ f(c) & \text{for } c < x \leq 1, \end{cases}$$

$$f_2(x) := \begin{cases} f(c) & \text{for } 0 \leq x < c \\ f(x) & \text{for } c \leq x \leq 1. \end{cases}$$

Observe first that $f_1(x)$ is convex and decreasing on $[0,1]$. It is convex on $[0,1]$ since if $0 \leq x_1 < c < x_2 \leq 1$, $0 < \alpha < 1$ then $\alpha f_1(x_1) + (1-\alpha) f_1(x_2) = \alpha f(x_1) + (1-\alpha) f(c) \geq f(\alpha x_1 + (1-\alpha)c) = f_1(\alpha x_1 + (1-\alpha)c) \geq f_1(\alpha x_1 + (1-\alpha)x_2)$. Likewise, $f_2(x)$ is concave and decreasing on $[0,1]$. Observe next that $f(x) + f(c) = f_1(x) + f_2(x)$. It follows from Theorems 2 and 1 that $\tau_n(f_1)$ and $\tau_n(f_2)$ increase while $\sigma_n(f_1)$ and $\sigma_n(f_2)$ decrease. Since $\tau_n(f) + f(c) = \tau_n(f_1) + \tau_n(f_2)$ and $\sigma_n(f) + f(c) = \sigma_n(f_1) + \sigma_n(f_2)$, this yields the desired conclusion. □

Note that we cannot hope to have a version of Theorem 3 with convex and concave interchanged, since for $\chi_{[0,\frac{1}{2}]}$, the characteristic function of the interval $[0, \frac{1}{2}]$, which is concave on $[0, \frac{1}{2}]$ and convex on $[\frac{1}{2}, 1]$, we have $\tau_{2m-1} + \frac{1}{2(m-1)} = \tau_{2m} = \tau_{2m+1} + \frac{1}{2m}$. However, applying Theorem 3 to $-f$ yields:

Theorem 4. *If the function* $f : [0,1] \to \mathbb{R}$ *is concave on the interval* $[0,c]$, *convex on* $[c,1]$, *and increasing on* $[0,1]$, *then* $\tau_n(f)$ *decreases and* $\sigma_n(f)$ *increases as* n *increases.*

Next, we prove:

Theorem 5. *If the function* $f : [0,1] \to \mathbb{R}$ *is concave on the interval* $[0,1]$, *with maximum* $f(c)$, $0 < c < 1$, *then*

$$\tau_n(f) - \frac{f(c) - f(0)}{n}$$

increases as n *increases.*

Proof. Define f_1 and f_2 as in the proof of Theorem 3, and note that $f_1(x)$ is concave and increasing on $[0,1]$ while $f_2(x)$ is concave and decreasing on $[0,1]$. The concavity of f_1 and f_2 can be verified by the method used in the proof of Theorem 3. It follows from Theorem 2 that $-\sigma_n(f_1)$ decreases and from Theorem 1 that $\tau_n(f_2)$ increases, and hence that

$$\begin{aligned}\sigma_n(f_1)+\tau_n(f_2) &= \tau_n(f_1)-\frac{f(c)-f(0)}{n}+\tau_n(f_2)\\ &= \tau_n(f)-\frac{f(c)-f(0)}{n}+f(c)\end{aligned}$$

increases as n increases. □

Corollary 1. *If the function $f:[0,1]\to\mathbb{R}$ is concave on the interval $[0,1]$ and symmetric about its midpoint, then*

$$\tau_n(f)-\frac{f(1/2)-f(0)}{n}$$

increases as n increases.

3.1 Symmetrisation

The *symmetrization* of $f:[0,1]\to\mathbb{R}$ about $x=\frac{1}{2}$ is defined to be

$$F(x):=F_f(x)=\frac{1}{2}\left(f(x)+f(1-x)\right). \tag{15}$$

We will make use of F_f throughout the rest of this note and start by observing that such a symmetrization never destroys convexity or concavity and often improves it.

Example 2 (Concavity of the symmetrization of $1/(1+x^2)$)**.** Although the function

$$f(x)=\frac{1}{1+x^2} \tag{16}$$

is neither convex or concave on $[0,1]$ its symmetrization,

$$F_f(x)=\frac{x^2-x+3/2}{(x^2+1)\,(x^2-2\,x+2)} \tag{17}$$

is concave.

To establish this we show that $F_f''(x) \leq 0$ on $[0,1]$. Since F_f and hence F_f'' are symmetric about $\frac{1}{2}$ we need only show this on $[\frac{1}{2}, 1]$. Moreover, using the change of variable $x := \frac{1}{2}(y+1)$ this is equivalent to showing

$$F_f''\left(\frac{1}{2}(y+1)\right) \leq 0 \text{ for } 0 \leq y \leq 1. \tag{18}$$

Now,

$$F_f''\left(\frac{1}{2}(y+1)\right) = \frac{8(y^8 + 44y^6 - 30y^4 - 660y^2 - 125)}{(y^2+2y+5)^3(y^2-2y+5)^3}. \tag{19}$$

The denominator of (19) is always positive while the numerator is a polynomial, say $p(y)$, that is negative both at $y = 0$ and $y = 1$. To show that it is negative throughout $[0, 1]$ we invoke *Descartes' rule of signs*, see http://mathworld.wolfram.com/DescartesSignRule.html, which tells us that:

> *for a real polynomial p, the number, $n(p)$, of zeros on the positive axis does not exceed the number of sign changes, $s(p)$, in the nonzero coefficients (in order) and that $2|(n(p) - s(p))$.*

The coefficients of $p(y)$ change signs only once so Descartes' rule of signs tells us that $p(y)$ has at most one positive zero. It follows that $p(y) \leq 0$ for all $y \in (0, 1)$, indeed if $p(c) > 0$ for some $0 < c < 1$, then $p(y)$ must have a zero in $(0, c)$ and another zero in $(c, 1)$. This establishes (18) thus proving that $F_f(x)$ is concave on $[0, 1]$. $\Diamond$

Another example of a class of functions with a concave symmetrization is $e_a := x \mapsto e^{-ax^2}$, for $a > 0$. The functions are themselves only concave on $[0,1]$ for $a \leq 1$ since $e_a''(x) = 2\,a\mathrm{e}^{-ax^2}\left(2\,ax^2 - 1\right)$. The concavity of the symmetrization for $a > 1$ is left for the reader to verify.

4 Monotonicity and symmetrization

Numerical experiments suggest it is very common for f to be such that τ_n and σ_n exhibit monotonicity but it is harder to find applicable conditions that assure this. Thus, we seek verifiable conditions that in particular will apply to $f(x) = 1/(1+x^2)$.

As will soon become apparent, calculations involving symmetric (concave) functions lead us naturally to the introduction of the following *symmetric Riemann sum*.

For $f : [0,1] \to \mathbb{R}$ we define:

$$\lambda_n := \lambda_n(f) = \frac{1}{n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right) - \frac{1}{n} f\left(\frac{1}{2}\right). \tag{20}$$

For all $n \in \mathbb{N}$, $\lambda_n(f)$ is linear and symmetric in that $\lambda_n(f) = \lambda_n(f(1-\cdot))$ and so $\lambda_n(f) = \lambda_n(F_f)$ where F_f is the symmetrization of f; namely, as above $F_f(x) := \frac{1}{2}\left(f(x) + f(1-x)\right)$.

The term involving $f\left(\frac{1}{2}\right)$ ensures that $\lambda_n(1) = 1$ by making a correction to the central term(s) of $\frac{1}{n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)$; if n is even we simply omit the central term, $\frac{1}{n} f(\frac{1}{2})$, while if n is odd we replace the two central terms by $\frac{1}{n}\left(f(\frac{1}{2} - \frac{1}{2n}) - f(\frac{1}{2}) + f(\frac{1}{2} + \frac{1}{2n})\right)$.

Further,

$$\lambda_n(f) = \frac{\tau_n + \sigma_n}{2} + \frac{1}{2n}\left(f(0) + f(1) - 2f\left(\frac{1}{2}\right)\right) \tag{21}$$

$$= \sigma_n + \frac{1}{n}\left(f(1) - f\left(\frac{1}{2}\right)\right) \tag{22}$$

$$= \tau_n + \frac{1}{n}\left(f(0) - f\left(\frac{1}{2}\right)\right). \tag{23}$$

As an immediate consequence of (23) and Corollary1 we get:

Theorem 6 (Monotonicity for symmetric concave functions)**.** *If the function $f : [0,1] \to \mathbb{R}$ is concave on the interval $[0,1]$ and symmetric about its midpoint, then $\lambda_n(f)$ increases with n.*

Corollary 2. *If the function $f : [0,1] \to \mathbb{R}$ has a concave symmetrization and $f(0) > f(1/2)$, then τ_n increases with n.*

Proof. Theorem 6 applies to F_f to show that $\lambda_n(f) = \lambda_n(F_f)$ is increasing and the conclusion follows from (23). □

In particular we have:

Corollary 3 (Monotonicity for decreasing functions with a concave symmetrization)**.** *If the function* $f : [0,1] \to \mathbb{R}$ *is decreasing on the interval* $[0,1]$ *and its symmetrization;* $F_f(x) = \frac{1}{2}(f(x) + f(1-x))$*, is concave, then* τ_n *increases with* n*, necessarily to* $\int_0^1 f$*.*

Example 3 (Monotonicity of τ_n for $1/(1+x^2)$)**.** Consider the function $f(x) := 1/(1+x^2)$ for which

$$\tau_n := \sum_{k=1}^{n} \frac{n}{n^2 + k^2}.$$

Clearly f is decreasing on $[0,1]$ and we already observed in Example 2 that its symmetrization $F_f(x) := \frac{1}{2}(f(x) + f(1-x))$ is concave, so Corollary 3 applies to show that τ_n is increasing. ♢

Similarly, for $a > 0$ and $f_a(x) := e^{-ax^2}$, we see by calculating f_a' and F_{f_a}'' that $\tau_n(f_a)$ increases with n.

Remark 1 (Variations on the theme)**.** Let $f : [0,1] \to \mathbb{R}$. Noting from their linearity that $\tau_n(-f) = -\tau_n(f)$ and similarly for σ_n, and also observing that $\sigma_n(f(x)) = \tau_n(f(1-x))$, we can deduce the following variants of the results above.

(i) If f is symmetric and convex, then λ_n is decreasing. [Apply Theorem 6 to $-f$.]

(ii) If $f(0) < f(1/2)$ (in particular, if f is increasing) and has a convex symmetrization, then τ_n is decreasing. [Apply Corollary 2 to $-f$.]

(iii) If $f(1/2) < f(1)$ (in particular, if f increasing) and has a concave symmetrization, then σ_n is increasing. [Apply Corollary 2 to $f(1-x)$.]

(iv) If $f(1/2) > f(1)$ (in particular if f is decreasing) and has a convex symmetrization, then σ_n is decreasing. [Apply Corollary 2 to $-f(1-x)$.]

Since the symmetrization of f is concave (convex) if f is concave (convex) we observe that Corollary 2 and part (iv) extend the final two theorems in [4]; our theorems 1 and 2. ♢

5 Analysis of the function $\frac{1}{1-bx+x^2}$

As a way of highlighting the subtleties in a seemingly innocent question, we finish by analyzing a one-parameter class of functions to which our results sometimes apply.

We consider the the family of functions

$$f_b : [0,1] \to \mathbb{R}, \quad \text{where } f_b(x) := \frac{1}{x^2 - bx + 1} \tag{24}$$

in the parameter range $|b| < 2$ so that each f_b assumes only positive values.

The symmetrization of f_b about $1/2$ is

$$F_b(x) := \frac{x^2 - x + (3-b)/2}{(x^2 - bx + 1)\left(x^2 - (2-b)x + (2-b)\right)}. \tag{25}$$

Then $f_0(x) = 1/(1+x^2)$ while $f_1(x) = F_1(x) = 1/(x^2 - x + 1)$. Now F_0, F_1 and $F_{3/2}$ are concave on $[0,1]$, while F_{-1} is convex and

$$F_2(x) = \frac{(1-x)x + 1/2}{(1-x)^2\, x^2}$$

is convex as an extended value function from $[0,1]$ into $(-\infty, \infty]$. By contrast $F_{5/4}, F_{7/4}$ are neither convex nor concave on the unit interval (for more details see Remark 2 below).

In passing we compute for $|b| < 2$ that

$$\int_0^1 \frac{\mathrm{d}x}{x^2 - bx + 1} = \frac{2}{\sqrt{4-b^2}} \left(\arctan\left(\frac{b}{\sqrt{4-b^2}} \right) + \arctan\left(\frac{2-b}{\sqrt{4-b^2}} \right) \right).$$

When $b \to -2$ we arrive at $\int_0^1 \frac{\mathrm{d}x}{x^2+2x+1} = \frac{1}{2}$.

With a view to applying Corollary 2 or Corollary 3, we begin by noting that $f_b(x)$ is decreasing on $[0,1]$ for $b \leq 0$ and increasing only for $b \geq 2$, however $f_b(0) > f_b(1/2)$ whenever $b < 1/2$.

We next prove that F_b is concave for $0 \leq b \leq 1$; again we employ Descartes' rule of signs.

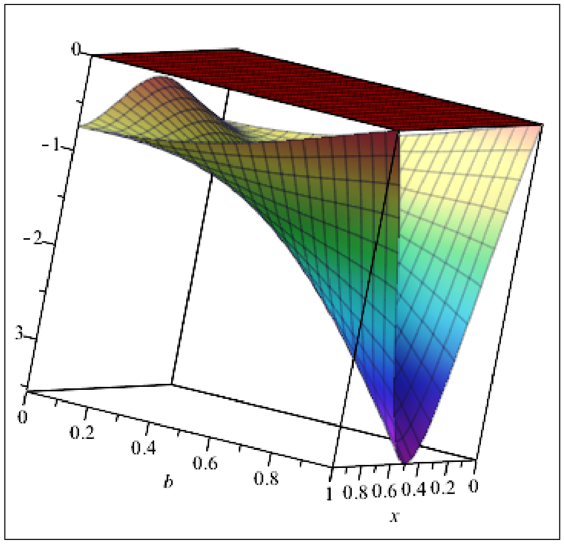

Figure 2: The second derivative of F_b for $0 \leq b, x \leq 1$.

Theorem 7 (Concavity of F_b). *The function F_b given by (25) is concave on $[0, 1]$ for $0 \leq b \leq 1$.*

Proof. To establish concavity of F_b we show that F_b'' is negative on $[0, 1]$, see Figure 2 and to do this we need only show its the numerator polynomial, n_b, is negative, as the denominator is always positive.[1]

Further, since F_b and hence F_b'' are symmetric about $\frac{1}{2}$ we need only show this on $[1/2, 1]$. Moreover, using the change of variable $x := (y + 1)/2$ allows us to use Descartes' rule of signs to detect roots of $n_b(x)$ for $x \geq 1/2$ (that is, for $y \geq 0$).

Now, the numerator of $F_b''((y + 1)/2)$ is

$$\begin{aligned} n_b(y) := {} & 24\, y^8 + 32\, \left(b^2 - 6\, b + 11\right) y^6 + 48\, (2\, b - 5) \left(6\, b^2 - 10\, b + 1\right) y^4 \\ & - 96\, (2\, b - 5) \left(4\, b^2 - 2\, b - 11\right) (b - 1)^2\, y^2 - 8\, \left(4\, b^2 - 6\, b - 1\right) (2\, b - 5)^3 . \quad (26) \end{aligned}$$

For $0 < b < 1$ the first two terms in (26) are always positive and the final two are negative, so that irrespective of the sign of the coefficient of y^4 (it in fact has three zeroes, at $5/2$ and $(5 \pm \sqrt{19})/6$) Descartes' rule of signs applies to show the numerator has one positive real zero (including multiplicity). This zero must lie to the right of

[1]Note in Figure 2 how much clearer the situation is made by also plotting the horizontal plane.

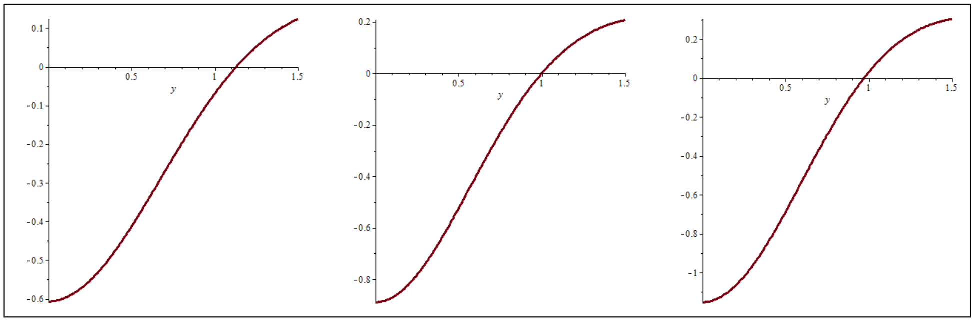

Figure 3: Graph of $n_b(y)$ on $[0, 3/2]$ for $b = 3/4$ (L), $b = 1$ (M), and $b = 5/4$ (R)

the point 1 except for $b = 1$ when it equals 1, as illustrated in Figure 3. (Note how close to one the inflection point is for $b = 5/4$.)

For $0 \le b < 1$ we have

$$n_b(0) = 8\,\left(4\,b^2 - 6\,b - 1\right)(5 - 2\,b)^3 < 0$$

and

$$n_b(1) = -1024\,(b-2)\,(b-1)\left(b^3 - 3\,b^2 + 3\right) < 0.$$

Thus, when $0 \le b \le 1$ the numerator is non-positive for $y \in [-1, 1)$ and so $F_b(x)$ is concave on $[0, 1]$. □

This proof of concavity for F_b was discovered by examining animations of the behaviour of n_b and then getting a computer algebra system to provide the requisite expressions after shifting the symmetry to zero so that Descartes' rule was applicable. Some snapshots of the animation are illustrated in Figure 3. The animation makes it clear that the solution of $n_b(y) = 1$ decreases monotonically with b.

Remark 2 (Convexity properties throughout the range $|b| < 2$**.** In this range the function provides further interesting applications of Descartes' rule.

A careful analysis of the coefficients a_k of y^{2k} for $k = 0, 1, 2, 3$ in (26) and of the signs of $n_b(0)$ and $n_b(1)$ [see Figure 3 where we plot $n_b(0)$ and $n_b(1)$ with $n_0(b)$ a dashed line], coupled with reasoning similar to that in the proof of Theorem 7 allows us to extend the results of that theorem to the whole parameter range $|b| < 2$.

The analysis and conclusions are summarized in Table 1, wherein we denote

α_- $= $ the negative root of $b^3 - 3b^2 + 3 \approx -0.8794$

α $= $ the smallest positive root of $b^3 - 3b^2 + 3 = 1 + \sqrt{3}\sin(2\pi/9) \approx 1.3473$

α_+ $= $ the largest root of $b^3 - 3b^2 + 3 \approx 2.5231$

$\beta_-,\ \beta_+$ $= $ the roots of $4b^2 - 6b - 1 = (3 \pm \sqrt{13})/3 \approx -0.1539,\ 1.6514$

$\gamma_-,\ \gamma_+$ $= $ the roots of $6b^2 - 10b + 1 = (5 \pm \sqrt{19})/6 \approx -0.1069,\ 1.5598$

$\delta_-,\ \delta_+$ $= $ the roots of $4b^2 - 2b - 11 = (1 \pm \sqrt{45})/4 \approx -1.4271,\ 1.9271$

and

$= $ the number of positive roots of $n_b((y+1)/2)$

Table 1: Table of signs

b	$[-2,\delta_-]$	$[\delta_-,\alpha_-]$	$[\alpha_-,\beta_-]$	$[\beta_-,\gamma_-]$	$[\gamma_-,1]$	$[1,\alpha]$	$[\alpha,\gamma_+]$	$[\gamma_+,\beta_+]$	$[\beta_+,\delta_+]$	$[\delta_+,2)$
a_4	+	+	+	+	+	+	+	+	+	+
a_3	+	+	+	+	+	+	+	+	+	+
a_2	-	-	-	-	+	+	+	-	-	-
a_1	+	-	-	-	-	-	-	-	-	+
a_0	+	+	+	-	-	-	-	-	+	+
$n_b(0)$	+	+	+	-	-	-	-	-	+	+
$n_b(1)$	+	+	-	-	-	+	-	-	-	-
#	2	2	2	1	1	1	1	1	2	2
$F_b(x)$	conv	conv	infl	conc	conc	infl	conc	conc	infl	infl

The conclusion that F_b is convex for $-2 < b \leq \alpha_-$ requires the observation that in this range $n_b(x)$ is negative for values of $x > 1$, so neither positive root can lie within the interval $[0,1]$.

Putting all this together we are able to conclude that the sequence $\tau_n(f_b)$ is increasing for $b \in [\beta_-, 1/2]$ and $\sigma_n(f_b)$ is decreasing for $b \in [-2, \alpha_-]$.

A similar analysis in the cases $|b| > 2$ is left to the interested reader. $\Diamond$

6 Concluding Remarks

The role of symmetrization in mathematics is rich and various and includes the first proofs of the isoperimetric problem. For a recent survey of techniques and applications in analysis we refer the reader to [3].

The story we have told highlights the many accessible ways that the computer and the internet can enrich mathematical research and instruction. The story would be even more complete if we could also deduce that $\sigma_n(1/(1+x^2))$ was decreasing.

References

[1] Tom Apostol, *Mathematical Analysis*, Addison-Wesley, 1974.

[2] J.M. Borwein, "The Life of Modern Homo Habilis Mathematicus: Experimental Computation and Visual Theorems. Chapter in John Monaghan, Luc Troche and Jonathan Borwein, *Tools and mathematics: Instruments for learning*, Spring-Verlag, 2015.

[3] Jonathan M. Borwein and Qiji (Jim) Zhu, "Variational methods in the presence of symmetry." *Advances in Nonlinear Analysis*, **2**(3) (2013), 271–307. DOI: http://dx.doi.org/10.1515/anona-2013-1001.

[4] András Szilárd, "Monotonicity of certain Riemann type sums." *Teaching of Mathematics*, **15**(2) (2012), 113–120.