Variational Methods in the Presence of Symmetry

Ongoing research with Jim Zhu (WMU)
Optimization of Planet Earth, AustMS 2013, Sydney

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February 7, 2014

Abstract

This talk and associated paper [1] aim to survey and to provide a unified framework to connect a diverse group of results, currently scattered in the literature, that can be aided by applying variational methods to problems involving symmetry.

Variational methods refer to mathematical treatment by construction of an appropriate action function whose critical points—or saddle points—correspond to or contain the desired solutions.

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Symmetry: is invariance with respect to some appropriate group or more usually a semigroup action

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Invariance

Let G be a *semigroup* acting on a complete metric space (X,d)

Definition: Invariance of a function

We say a lsc function $f : X \to R \cup \{+\infty\}$:
is G-subinvariant if

$$f(gx) \leq f(x) \ \forall g \in G, x \in X,$$

is G-superinvariant if

$$f(gx) \geq f(x) \ \forall g \in G, x \in X,$$

and is G-invariant if f is both sub and super invariant.

When G is a group these are all the same

Symmetrization

Definition: $S : X \to X$ is a (G,f)-symmetrization if

(i) for any $g \in G, x \in X$, $S(gx) = gS(x) = S(x)$;

(ii) for any $x \in X$, $S^2(x) = S(x)$;

(iii) for any $x \in X$, $f(S(x)) \leq f(x)$

If $S(x) \in \mathrm{cl}\ (G \cdot x)$ then (iii) always holds but:

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A simple extremal principle involving symmetry

The following idea captures the essence of variational methods in the presence of symmetry

Simple Extremal Principle (SEP)

Let $f : X \to R \cup \{+\infty\}$ be a G-subinvariant function and S be a (G, f)-symmetrization. Then

$$S(\mathrm{argmin}(f)) \subseteq \mathrm{argmin}(f).$$

Proof of SEP. One can not properly minorize the minimum! QED

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Compatible metrics

Q. What if the existence of the extremum is not guaranteed?

A. We need symmetric versions of "variational principles". This requires a compatible metric.

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(i) For any $x \in X$, $g \in G$, $d(x,y) \geq d(gx,gy)$; and

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Variational principles in the presence of symmetry

Symmetric Variational Principle (SymVP)

Let (X,d) be a complete metric space. Let $f : X \to R \cup \{+\infty\}$ be an G-invariant lsc function bounded below and let S be a (G,f)-symmetrization such that d is (G,S)-compatible.

Then, for any $\varepsilon, \lambda > 0$ there exist y, z such that

(i) $f(S(z)) < \inf_X f(x) + \varepsilon$;

(ii) $d(S(y), S(z)) \le \lambda$;

(iii) $f(S(y)) + (\varepsilon/\lambda) d(S(y), S(z)) \le f(S(z))$; and

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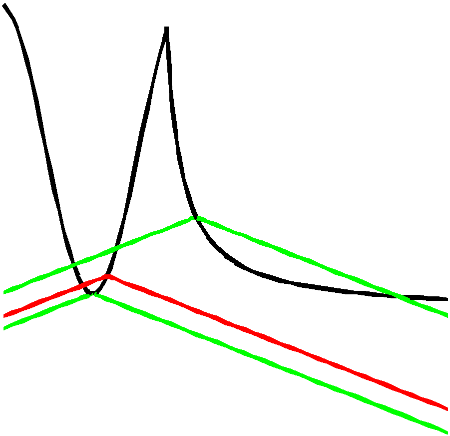
Variational Principle in Pictures



Producing a (local) non-dominated point

Proof of SymVP

Since f is invariant we can find $S(z)$ satisfying (i), that is:

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Finally, we check that $S(y)$ does what we need. **QED**

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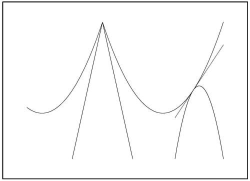

Ekeland VP and Smooth VP

Two other forms of SymVP use approximation of *Schwarz symmetry* via *polarization* (discussed below)

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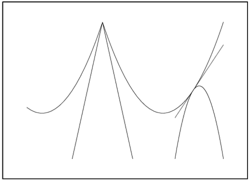

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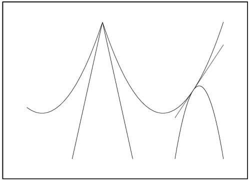

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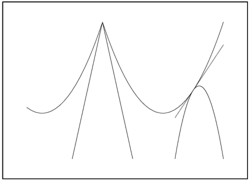

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Proof of AG inequality by using symmetry

Consider

$$\min f(x) := -\sum_{n=1}^{N} \log(x_n) + \iota_C(x),$$

where $C := \{x : \langle x, \vec{1}\rangle = K, x \geq 0\}$, while vector $\vec{1}$ has all components 1, and $\iota_C(x) = 0, x \in C$ and $+\infty$ otherwise

- Then f is permutation (P(N)) invariant
- $S(x) = \bar{x}\vec{1}$ is a $(P(N), f)$-symmetrization[1]

[1] $\bar{x}$ is the average of components of x

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The subdifferential of a convex function f on R^N is

$$\partial f(x) = \{y \in R^N : x \in \operatorname{argmin}(f - y)\}$$

Subdifferential of Spectral Functions

(Lewis 1999) Let $f : R^N \to R \cup \{+\infty\}$ be a convex $P(N)$-invariant function. Then

$$y \in \partial f(x)$$

iff

$$y^{\downarrow} \in \partial f(x^{\downarrow}) \text{ and } \langle x, y\rangle = \langle x^{\downarrow}, y^{\downarrow}\rangle,$$

where $x^{\downarrow}$ is a decreasing rearrangement of the components

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Example 3: Key steps of Proof

- u_{ij} – switch components x_i, x_j of x if $(x_i - x_j)(i - j) < 0$
- $G^{\downarrow} \subset P(N)$ – the semigroup of finite compositions of u_{ij}
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- $S(x) = x^{\downarrow}$ is a (G, f)-symmetrization

[2] $\langle A, B \rangle \leq \langle \lambda(A), \lambda(B) \rangle$ for symmetric matrices.

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Example 4: Spectral Functions (l^2)

Notation. For functions of (symmetric) nuclear equivalently Hilbert-Schmidt operators we use:

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Example 4: Symmetry of Spectral Subdifferential

Define $S(x) = x^*$ to be a rearrangement such that

1. nonnegative components decrease with nonnegative indices,
2. negative components increase as negative indices increase.

Example. if

$$x = (\ldots\ldots, -2, 3, -1, -5, -4, 7, 4, 5, 2, 0, 0, \ldots\ldots)$$

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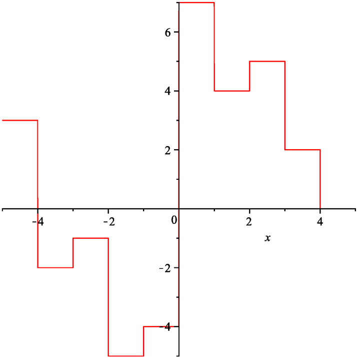
Example of the $*$-rearrangement in l^2



Before and after

Symmetry of Spectral Subdifferential

Spectral Subdifferential (Borwein, Lewis, Read & Zhu 2000)

Let $f : l^2 \to R \cup \{+\infty\}$ be a convex rearrangement invariant function. Then

$$y \in \partial f(x)$$

iff

$$y^* \in \partial f(x^*) \text{ and } \langle x, y\rangle = \langle x^*, y^*\rangle.$$

Can be done for c_0 and all Shatten p-class operators $(1 \leq p < \infty)$
[Conjugation: $c_0 \to \ell^1 \to \ell^\infty$ and $C_s(H) \to B_1(H) \to B_s(H)$]

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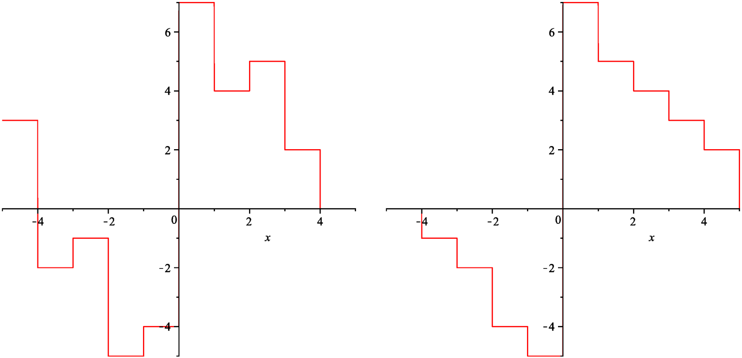
Visualizing Switch and Move



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Definition of Switch and Move operators

Switch Operator

$$s_{nm}x := x - x_n e^n - x_m e^m + \max(x_n, x_m)e^n + \min(x_n, x_m)e^m$$

Move Operator

$$m_n x := \begin{cases} x \circ 1_{-\infty}^{k-1} - x_n e^n + x_n e^k + R_S(x \circ 1_k^{\infty}) & n < 0, x_n > 0 \\ x \circ 1_{l+1}^{\infty} - x_n e^n + x_n e^l + L_S(x \circ 1_{-\infty}^{l}) & n \geq 0, x_n < 0 \\ x & \text{otherwise,} \end{cases}$$

where $k := \min\{m \geq 0 : \sup_{i \geq m} |x_i| < x_n\}$
and $l := \max\{m < 0 : \sup_{i \leq m} |x_i| < -x_n\}$

Example 4: Switch and Move Inequalities

Switch and Move Inequalities. Let $x, y \in l^2$. Then

$$\langle y^*, x\rangle \leq \langle y^*, s_{nm}x\rangle,$$

and

$$\langle y^*, x\rangle \leq \langle y^*, m_n x\rangle.$$

"IT'S UNIFIED AND IT'S A THEORY, BUT IT'S NOT THE UNIFIED THEORY WE'VE ALL BEEN LOOKING FOR."

Example 4: The missing semigroup

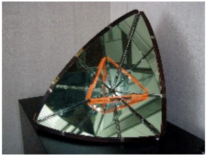
Definition: The semigroup H

Define H to be the semigroup of self-mappings on l^2 which (i) add or delete an arbitrary number of zeros and (ii) permute components

Though H is not a group, for $y \in l^2$ there exists $h_y, h^y \in H$ with

$$h_y y^* = y \text{ and } y^* = h^y y.$$

Moreover, $G \subseteq H$.



Coxeter's 1927 kaleidoscope

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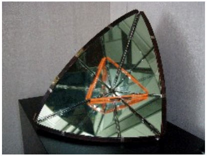

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1. Represent $G := \cup_{N=1}^{\infty} G_N$ where

$$G_N := \{ \text{ finite compositions of } s_{nm}, m_n \ \forall |n|, |m| \leq N \ \}$$

2. By Switch and Move Ineq. $\varphi(x) = -\langle y^*, x\rangle$ is G-subinvariant
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Since f is H-invariant and $*$ is an (H,f)-symmetrization,

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Example 5: Laplace equation

Laplace Equation

The solutions of

$$\Delta u = f \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0 \tag{1}$$

correspond to critical points of

$$F(u) := \int_{\Omega} \left(\frac{|\nabla u|^2}{2} + fu \right) \mu(dx), \tag{2}$$

in the Sobolev space $H_0^1(\Omega)$.

Example 5: Schwarz symmetry

We seek symmetric solution of Laplace's equation as follows:

Schwarz symmetrization (Decreasing rearrangement)

The symmetrization $*$ on $L^2(R^n, \mathscr{M}, \mu)^+$ for a measurable $M \in \mathscr{M}$ is

$$M^* = B_r(0) \text{ where } \mu(M) = \mu(B_r(0))$$

and for any $u \in L^2$ we then define u^* by

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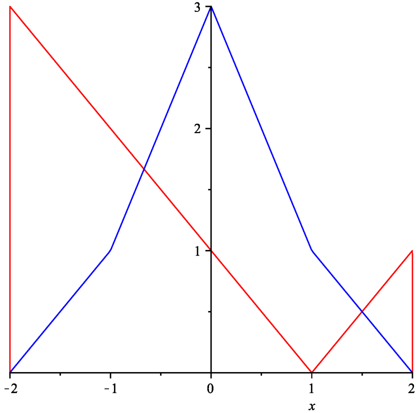
$|x-1|$ and its Schwarz symmetrization on $[-2,2]$



$|x-1|$ with blue symmetrization

Example 5: Polarization-building semigroup G

1. Let $0 \notin H_0$ be a hyperplane dividing R^N into two closed half-spaces $0 \in H_+$ and its complement H_-
2. Let σ be the reflection exchanging the two half-spaces

Definition: The polarization of f at H_0

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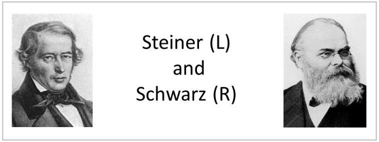

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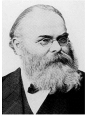

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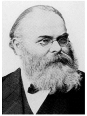

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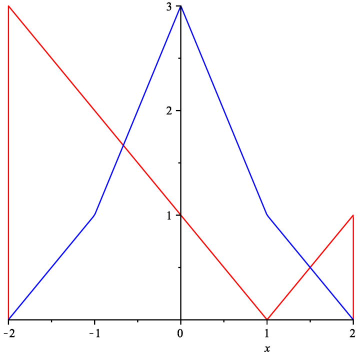
Picture of $|x-1|$ on $[-2,2]$



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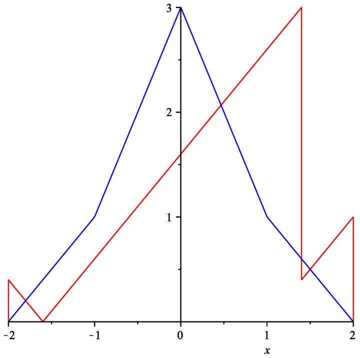

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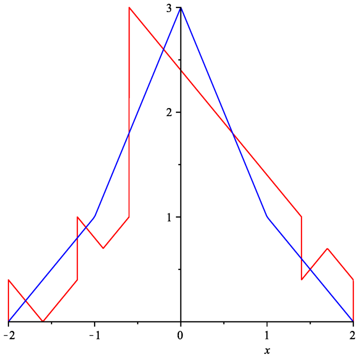

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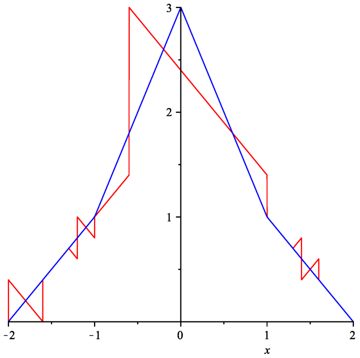

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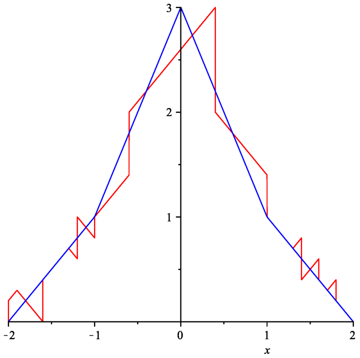
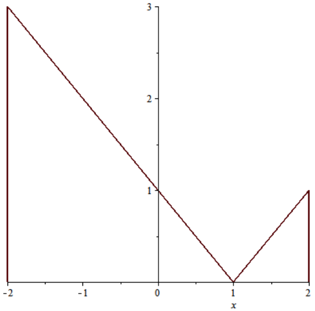


$$H_0 = (x = -0.1)$$

Example 5: Symmetrization Movie

The sequence of polarizations revisited



Properties of polarization: Brock and Solynin (1999)

Let G be semigroup of finite compositions of polarizations. Then

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6. Preservation of the norm: $\|f^{\sigma}\|_{H^1} = \|f\|_{H^1}$

[3] **4** illustrates *the curse of Sobolev*. It uses weak integration by parts.

Properties of polarization: Brock and Solynin (1999)

Let G be semigroup of finite compositions of polarizations. Then

1. Hardy-Littlewood inequality:
$$\int fg \leq \int f^\sigma g^\sigma \qquad \forall \sigma \in G$$
2. Decreasing L^2 norm:
$$\|f-g\|_2 \geq \|f^\sigma - g^\sigma\|_2 \qquad \forall \sigma \in G$$
3. Strong approximation of Schwarz symmetrization in L^2: there exists a sequence $g_k \in G \cdot f$ such that $\|g_k - f^*\|_2 \to 0$
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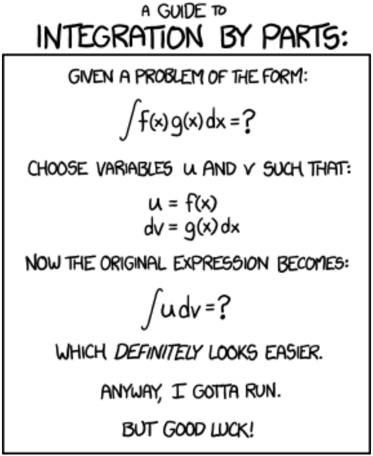
A GUIDE TO
INTEGRATION BY PARTS:
GIVEN A PROBLEM OF THE FORM:
$\int f(x)g(x)dx = ?$
CHOOSE VARIABLES u AND v SUCH THAT:
$u = f(x)$
$dv = g(x)dx$
NOW THE ORIGINAL EXPRESSION BECOMES:
$\int u\,dv = ?$
WHICH DEFINITELY LOOKS EASIER.
ANYWAY, I GOTTA RUN.
BUT GOOD LUCK!

Example 5: Putting everything together for the Laplacian

Recall

$$F(u) := \int_\Omega \left(\frac{|\nabla u|^2}{2} + fu \right) \mu(dx)$$

Then

1. F is convex in H^1 and, therefore, weakly lower continuous,
2. when $f^* = f$, F is G-subinvariant, and
3. $*$ is a (G,F)-symmetrization.

Thus, F has a symmetric minimum $u = u^*$. **QED**

- The use of approximate polarization is essential and nontrivial
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Example 6: Planar motion

The planar motion of two bodies

Mathematical formulation: minimize the action functional

$$F(x) := \int_0^P \left[\frac{\|x'(t)\|^2}{2} + \frac{1}{\|x(t)\|}\right] dt$$

in space of periodic orbits $\{x \in H^1([0,P],R^2) : x(0) = x(P)\}$

- Clearly F is rotation invariant
- Kepler first 'showed' the solution is a circle
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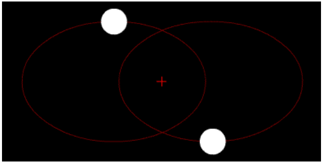

Example 7: Simple saddle points

Simple saddle point behavior

The function $F(x,y) := x^2 - y^2$ is rather typical:

- F has a saddle point at $(0,0)$
- F is reflection symmetric with respect to both x and y axis
- F has no local extremum, and is unbounded

We will use F to illustrate two different ideas:

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Palais principle of symmetric criticality

Here is a simplified but effective version to illustrate the idea:

Principle of Symmetric Criticality (PSC)

Let X be a Hilbert space with an isometric linear group action G and let $F \in C^1(X)$ be G-invariant.
Denote

$$\Sigma := \{x \in X : gx = x,\ \forall g \in G\}.$$

Then any critical point of $F|_\Sigma$ is also a critical point for F.

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Proof of Principle of Symmetric Criticality

For any $g \in G$, $v \in X$ and $x \in \Sigma$, $F \circ g = F$ implies that $dF_x(v) = dF_{gx}(g(v))$. Since g is an isometry

$$\langle g\nabla F(x), g(v)\rangle = \langle \nabla F(x), v\rangle = dF_x(v).$$

On the other hand $gx = x$ implies

$$dF_{gx}(g(v)) = \langle \nabla F(gx), g(v)\rangle = \langle \nabla F(x), g(v)\rangle.$$

Thus, for all $v \in X$ we have $\langle g\nabla F(x), g(v)\rangle = \langle \nabla F(x), g(v)\rangle$ and so

$$g\nabla F(x) = \nabla F(x).$$

It follows that $\nabla F(x) \in \Sigma$. Hence $\nabla F(x) \in T\Sigma|_x$. Thus, if x is a critical point of $F|_\Sigma$ – namely $\nabla F(x)$ restricted to $T\Sigma|_x$ is 0 – then

$$\nabla F(x) \in \Sigma^\perp \cap \Sigma = \{0\}$$

as claimed. **QED**

Example 7: Applying Palais principle to $x^2 - y^2$

- Consider the *reflection*

$$r(x,y) := (-x,y),$$

which is a linear isometry

- The *invariant* set of r is

$$\Sigma = \{(0,y) : y \in R\}$$

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- $G :=$ rotations around the origin is a group of isometries
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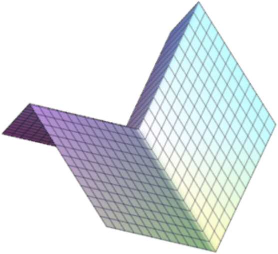
Nonsmooth Saddle Points

By mollification or regularization, we can relax somewhat the smoothness requirement in the Principle of Symmetric Criticality so that it can be applied to, say, the nonsmooth critical point of

$$F(x,y) = |x| - |y|$$



The Mountain Pass idea

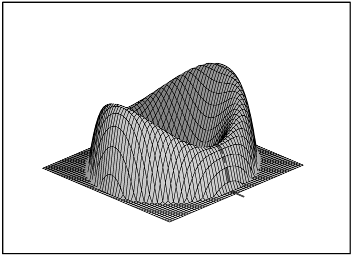
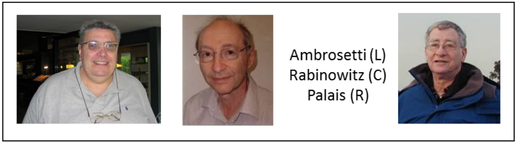


Figure : A typical mountain pass



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We now illustrate the use the Mountain pass lemma to deal with the saddle point of $F(x,y) := x^2 - y^2$

- Define

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$$\widehat{F}(\gamma) := \max_{t\in[0,1]} F(\gamma(t))$$

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For $a(x) \le c < 0$ and $2 < p < 2^* = 2N/(N-2)$, consider

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Then solution corresponds to a critical point of

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in the Sobolev space $H_0^1(\Omega)$.
It turns out F has a nontrivial saddle point. **QED**

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