QUADRATIC CONVERGENCE OF THE TANH-SINH QUADRATURE RULE

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ABSTRACT. In [5] and [2] the authors describe the remarkable effectiveness of the doubly exponential 'tanh-sinh' transformation for numerical integration —even for quite unruly integrands. Our intention in this note is to provide a theoretical underpinning when the integrand is analytic for the observed superlinear convergence of the method. Our analysis rests on the corresponding but somewhat easier analysis by Haber [1] of the less numerically effective 'tanh' substitution.

1. PRELIMINARIES

The standard *trapezoidal rule* for numerical integration when the integrand is defined on the interval $(-\infty, \infty)$ is:

$$\int_{-\infty}^{\infty} f(t)\, dt \approx h \sum_{n=-\infty}^{\infty} f(nh) \tag{1.1}$$

On changing variables, we can approximate the definite integral via:

$$\int_{-1}^{1} f(t)\, dt = \int_{-\infty}^{\infty} f(\psi(x))\psi'(x)\, dx \approx h \sum_{n=-N}^{N} \psi'(nh) f(\psi(nh)). \tag{1.2}$$

Here, ψ is any absolutely continuous monotonic increasing function mapping $(-\infty, \infty)$ onto $(-1, 1)$, and without loss of generality, we assume the integrand is defined over $(-1, 1)$.

The *tanh-sinh rule* use the doubly-exponential transformation

$$\psi(x) = \tanh\left(\frac{\pi}{2}\sinh(x)\right) \tag{1.3}$$

with

$$\psi'(x) = \frac{\pi \cosh(x)}{2\cosh^2\left(\frac{\pi}{2}\sinh(x)\right)}.$$

Correspondingly, $\psi(x) := \tanh(x)$ gives rise to the scheme analyzed by Haber in [1].
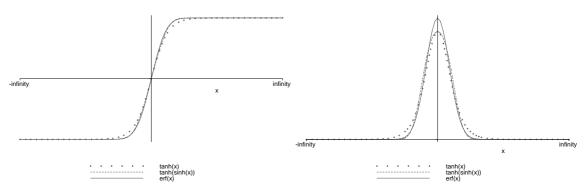
FIGURE 1. $\tanh(x), \mathrm{erf(x)}, \tanh(\sinh(x))$ and their derivatives

The tanh-sinh scheme is based on the frequent observation, rooted in the *Euler-Maclaurin summation formula* [6], that for certain bell-shaped integrands, a simple block-function approximation to the integral is much more accurate than one would normally expect. Various other efficient rules are described in [2, 5] (such as $\psi(x) :=$ $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, which gives rise to "error function" or *erf quadrature*) but since our experience is that the 'tanh-sinh' is almost always as-or-more effective, see [5, 6], we do not consider their analysis further herein. In practice tanh-sinh is almost invariably the best rule and is often the only effective rule when more that 50 or 100 digits are required. Figure 1 shows the three schemes we have introduced— erf and tanh(sinh) are visually very close while tanh is the outlier. One of the primary reasons why we present this detailed analysis is this remarkable efficacy of 'tanh-sinh'. A similar analysis may be undertaken for the 'erf' rule, but not as explicitly since the zeros of the error function will be estimated numerically. In addition, various of the summations required appear more delicate.

2. Hardy space

We will perform our analysis of the convergence of tanh-sinh rule in the Hardy space H^2, (see [3]).

Definition 2.1. ([3], p.2) For $0 < p < \infty$, **Hardy space** H^p consists of the functions f, which are analytic on the unit disk and satisfies the growth condition

$$\|f\|_{H^p} := \sup_{0 \le r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^p < \infty$$

where the quantity $\|f\|_{H^p}$ is called the *Hardy norm*.

Thus H^2 is the class of functions which are analytic on the unit disk ($\{$ z : $|z| < 1\}$) and whose Taylor coefficients satisfy

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

A function f is said to have *nontangential limit* L at $e^{i\theta}$, if $f(z) \to L$ as $z \to e^{i\theta}$ inside the unit circle. Functions in H^2 have nontangential limits almost everywhere on the unit circle and belong to L^2 on the unit circle ([1], p.17 and p.21). The
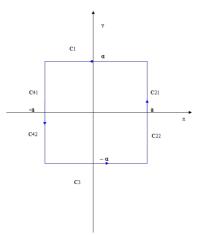
FIGURE 2. Contour for integration over the rectangle A

Hardy space H^2 becomes a Hilbert space on imposing the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{|z|=1} f(z)\overline{g(z)} \, |dz|.$$

Definition 2.2. ([4, Def. 12.6.1]) Let S denote a point set and X a complete inner product space of functions on S. A function of two variables z and w in S, $K(z, w)$ is called a *reproducing kernel* for the space X if

(1) For each fixed $w \in S$, $K(z, w)$ is in X
(2) For every function $f(z) \in X$ and for every point $w \in S$, the *reproducing property*

(2.1) $$f(w) = \langle f(z), K(z, w) \rangle_z.$$

holds.
[The subscript z indicates w is fixed and the inner product is taken on the variable z.]

With respect to the above inner product, H^2 has an orthonormal basis given by $\{1, z, z^2, \cdots, z^n, \cdots\}$ and a reproducing kernel

(2.2) $$K(z, w) := \sum_{n=0}^{\infty} z^n \bar{w}^n = \frac{1}{1 - z\bar{w}}$$

Due to the existence of the reproducing kernel [4, Thm.12.6.1], the *point functionals* P_z defined, for $|z| < 1$, by

$$P_z(f) := f(z)$$

are bounded linear functionals on H^2 satisfying

(2.3) $$|P_z f|^2 \le K(z, z) \, \|f\|^2.$$

2.1. **A residue theorem.**

In the sequel we shall use the following lemma, a version of which is given without proof in [1]. We set

$$(2.4) \qquad \Psi(z) := \begin{cases} -\pi i, & \operatorname{Im} z > 0 \\ \pi i, & \operatorname{Im} z < 0. \end{cases}$$

Lemma 2.3. [1, Lemma 2]) *Suppose that α and h are positive real numbers and f is a function satisfying the following conditions:*

(1) *f is analytic on the closure of the strip $S_\alpha = \{-\alpha < \operatorname{Im} z < \alpha\}$*

(2) *$\lim_{a \to \pm\infty} \int_{a-i\alpha}^{a+i\alpha} |f(z)| dz = 0$*

(3) *$\int_{-\infty}^{\infty} f(x) dx$ and $\sum_{n=-\infty}^{\infty} f(nh)$ exists.*

Then

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z) \Psi(z) \, dz.$$

Moreover,

$$h \sum_{n=-\infty}^{\infty} f(nh) = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z) \, \pi \cot \frac{\pi z}{h} \, dz.$$

Proof. First, we consider integrating f on the boundary of rectangle $A := \{z : -\alpha < \operatorname{Im} z < \alpha$ and $-a < \operatorname{Re} z < a\}$ as shown in Figure 2. Since f is analytic on A, we have

$$\int_{C1} f + \int_{-a}^{a} f = -\int_{C21} f - \int_{C41} f$$

and

$$\int_{C3} f - \int_{-a}^{a} f = -\int_{C22} f - \int_{C42} f.$$

Combining these two equations gives

$$\int_{-a}^{a} f = \frac{1}{2} \left(-\int_{C1} f + \int_{C3} f - \int_{C21} f - \int_{C41} f + \int_{C22} f + \int_{C42} f \right).$$

Letting $a \to \infty$, by the condition 2. of the hypotheses, $\int_{C21} f, \int_{C41} f, \int_{C22} f$ and $\int_{C42} f$ will each converge to zero. Thus, we get

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z) \Psi(z) \, dz.$$

The function $f(z) \, \pi \cot(\pi z/h)$ has poles at nh for integer n with residues $h f(nh)$. According to the residue theorem, we have

$$\frac{1}{2\pi i} \int_{C1+C2+C3+C4} f = h \sum_{n=-N}^{N} f(nh)$$

where N is the largest integer such that $Nh < a$ and $C2 = C21 + C22, C4 = C41 + C42$. Letting $a \to \infty$, we obtain

$$h \sum_{n=-\infty}^{\infty} f(nh) = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z) \pi \cot \frac{\pi z}{h} \, dz,$$

as claimed. $\square$

More generally, similar arguments will establish:

Lemma 2.4. *Suppose, in Lemma 2.3, that f has poles z_i with residues $\mathrm{Res}(z_i)$ inside S_α. Then*

$$\int_{-\infty}^{\infty} f(x)dx \;=\; \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z)\Psi(z)\,dz + \pi i \sum_i \mathrm{Res}(z_i^+) - \pi i \sum_i \mathrm{Res}(z_i^-)$$

$$(2.5) \qquad\qquad\qquad = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z)\Psi(z)\,dz - \sum_i \mathrm{Res}(z_i)\Psi(z_i)$$

and

$$(2.6) \qquad h\sum_{n=-\infty}^{\infty} f(nh) = \frac{1}{2\pi i} \int_{\partial S_\alpha} f(z)\pi \cot\left(\frac{\pi z}{h}\right) dz - \sum_i \mathrm{Res}(z_i)\pi \cot\left(\frac{\pi z_i}{h}\right).$$

Here z_i^+ and z_i^+ represent the poles above and below the real line.

2.2. **Our working notation.** We will work with the following quantities for $h > 0$.

(1) The integral: $If := \int_{-1}^{1} f$.

(2) The *tanh-sinh approximation*:

$$T_{h,N}f := h\sum_{n=-N}^{N} \frac{\pi \cosh(nh)}{2\cosh^2\left(\frac{\pi}{2}\sinh(nh)\right)} f\left(\tanh\left(\frac{\pi}{2}\sinh(nh)\right)\right).$$

(3) The N-th *approximation error*:

$$E_{h,N}f := (I - T_{h,N})\,f.$$

(4) The approximation limit:

$$T_h f = \lim_{N\longrightarrow\infty} T_{h,N}f,$$

which will be shown to exist in lemma 3.1.

(5) The limit error:

$$E_h := (I - T_h)\,f.$$

Then I, $T_{h,N}$, $E_{h,N}$ and are bounded linear functionals on H^2. So are T_h, and E_h once we show they exist.

3. The associated space G^2

Along with H^2, it is helpful to use the same change of variable as in (1.3) to define a corresponding space of functions, [4]. Precisely, we let G^2 be the set of functions of the form

$$\psi'(w)f\left(\psi(w)\right) \qquad f \in H^2 \text{ and } \psi \text{ as in } (1.3).$$

Assumptions. We assume ψ maps region A onto the unit disk, that functions in G^2 are analytic on A, and are defined almost everywhere on ∂A.

Letting $\widehat{f}(w) := \psi'(w)f\left(\psi(w)\right)$, we can induce an inner product in G^2 by

$$\langle \widehat{f}, \widehat{g} \rangle_{G^2} := \langle f, g \rangle_{H^2} \;=\; \frac{1}{2\pi} \int_{|z|=1} f(z)\overline{g(z)}\,|dz|$$

$$(3.1) \qquad\qquad\qquad = \frac{1}{2\pi} \int_{\partial A} f(\psi(w))\overline{g(\psi(w))}|\psi'(w)\,dw|$$

$$\qquad\qquad\qquad = \frac{1}{2\pi} \int_{\partial A} \widehat{f}(w)\overline{\widehat{g}(w)}\,|dw/\psi'(w)|.$$

Then G^2 is a Hilbert space and the mapping $f \mapsto \widehat{f}$ is an isomorphism of H^2 onto G^2. Also, G^2 has an orthonormal basis with elements

$$(3.2) \qquad \phi_n(z) := \psi'(z)\,(\psi(z))^n$$

and

Lemma 3.1. *Assume the reproducing kernel of H^2 is $K(z,w)$, then the reproducing kernel of G^2 is*

$$(3.3) \qquad \widehat{K}(z,w) = K\left(\psi(z),\psi(w)\right) \cdot \psi'(z) \cdot \overline{\psi'(w)}$$

given the transformation $\psi(z)$ from H^2 to G^2.

Proof. Given the reproducing property (2.1) and the definition of the inner product (3.1), if $\widehat{K}(z,w)$ denotes the reproducing kernel of G^2 then

$$
\begin{aligned}
\langle \widehat{f}(z), \widehat{K}(z,w) \rangle_z &= \frac{1}{2\pi} \int_{\partial A} \widehat{f}(z)\overline{\widehat{K}(z,w)}|\,dz/\psi'(z)| \\
&= \widehat{f}(w) \\
&= \psi'(w) \cdot \langle f(z), K(z, \psi(w)) \rangle_z \\
&= \psi'(w) \cdot \frac{1}{2\pi} \int_{\partial A} \widehat{f}(z)\overline{K(\psi(z),\psi(w))\psi'(z)}|\,dz/\psi'(z)| \\
&= \frac{1}{2\pi} \int_{\partial A} \widehat{f}(z)\overline{K(\psi(z),\psi(w))\psi'(z)\overline{\psi'(w)}}|\,dz/\psi'(z)|
\end{aligned}
$$

Comparing the last integral with the one on the first line, we get

$$\widehat{K}(z,w) = K\left(\psi(z),\psi(w)\right) \cdot \psi'(z) \cdot \overline{\psi'(w)}.$$

$\square$

Therefore for the tanh-sinh transformation, by (2.2), G^2 has a reproducing kernel

$$
\begin{aligned}
\widehat{K}(z,w) &= 1/\left(1 - \psi(z)\overline{\psi(w)}\right) \cdot \psi'(z) \cdot \overline{\psi'(w)} \\
&= \frac{\pi^2}{4\cosh\left(\frac{\pi}{2}\sinh(z) - \frac{\pi}{2}\sinh(\overline{w})\right)} \\
(3.4) \qquad &\times \frac{\cosh(z)\cosh(\overline{w})}{\cosh\left(\frac{\pi}{2}\sinh(z)\right)\cosh\left(\frac{\pi}{2}\sinh(\overline{w})\right)}.
\end{aligned}
$$

Consequently, the *point functionals* on G^2, $\widehat{P}_z$, defined for $z \in A$ by

$$\widehat{P}_z\widehat{f} = \widehat{f}(z)$$

are bounded linear functionals (see Thm. 12.6.1 of [4]) and satisfy

$$(3.5) \qquad |\widehat{P}_z f|^2 \le \widehat{K}(z,z)\,\|\widehat{f}\|^2.$$

3.1. **Further working notation.** We use similar notation in G^2 as in H^2:

$$\widehat{I}\widehat{f} := \int_{-1}^{1} f(x)dx = \int_{-\infty}^{\infty} \widehat{f}(u)du,$$

$$\widehat{T}_{h,N}\widehat{f} := T_{h,N}f = h\sum_{n=-N}^{N} \widehat{f}(nh),$$

$$\widehat{E}_{h,N}\widehat{f} := \left(\widehat{I} - \widehat{T}_{h,N}\right)\widehat{f},$$

$$\widehat{T}_h \widehat{f} := \lim_{N \longrightarrow \infty} \widehat{T}_{h,N} \widehat{f} \quad \text{and} \quad \widehat{E}_h \widehat{f} := \left(\widehat{I} - \widehat{T}_h \right) \widehat{f}$$

Lemma 3.2. *For $h > 0$ the operators $\widehat{T}_h$ and $\widehat{E}_h$ are bounded linear functionals on G^2 (as are $\widehat{T}_{h,N}$ and $\widehat{E}_{h,N}$).*

Proof. Define $|\widehat{T}_{h,N}| \widehat{f} = h \sum_{n=-N}^{N} |\widehat{f}(nh)| \geq \widehat{T}_{h,N} \widehat{f}$. The inequality (3.5) shows that

$$
\begin{aligned}
|\widehat{T}_{h,N}| \widehat{f} &\leq h\|\widehat{f}\| \sum_{n=-N}^{N} \widehat{K}(nh, nh)^{1/2} \\
&< h\|\widehat{f}\| \sum_{n=-\infty}^{\infty} \widehat{K}(nh, nh)^{1/2} \\
&= h\|\widehat{f}\| \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{1 - \tanh^2(\pi/2 \sinh(nh))}} \cdot \frac{\pi \cosh(nh)}{2 \cosh^2(\pi/2 \sinh(nh))} \\
&= h\|\widehat{f}\| \sum_{n=-\infty}^{\infty} \frac{\pi \cosh(nh)}{2 \cosh(\pi/2 \sinh(nh))} \\
&\leq h\pi\|\widehat{f}\| \sum_{n=-\infty}^{\infty} e^{nh - \pi/2 |\sinh(nh)|}
\end{aligned}
$$

$$(3.6) \qquad \leq h\pi\|\widehat{f}\| \sum_{n=-\infty}^{\infty} e^{-h|n|/2} = h \cdot \pi \cdot \coth(h/4) \|\widehat{f}\|.$$

where

$$
\begin{aligned}
\|\widehat{f}\|^2 = \|f\|^2 = \langle f, f \rangle &= \frac{1}{2\pi} \int_{|z|=1} f(z) \overline{f(z)} \, |dz| \\
(3.7) \qquad\qquad &= \frac{1}{2\pi} \int_{|z|=1} |f(z)|^2 \, |dz|.
\end{aligned}
$$

Since $f \in H^2$ belongs to L^2 on the unit circle, (3.7) is finite. Equation (3.6) shows that $\widehat{T}_{h,N} \widehat{f}$ is absolutely convergent. Also,

$$\widehat{T}_h \widehat{f} \leq h \cdot \pi \cdot \coth(h/4) \|\widehat{f}\|.$$

Thus, $\widehat{T}_h$ and also $\widehat{E}_h$ are bounded linear operators on G^2. $\qquad \square$

If a space X has a reproducing kernel, every bounded linear functional on X has a very simple expression. In the next result, $L_{(z)}$ denotes a functional L applied to K with respect to variable z while keeping w constant.

Theorem 3.3. [4, Thm 12.6.6 and Cor 12.6.7] *Let X has a reproducing kernel $K(z,w)$, and assume L is a bounded linear functional defined on X, then*

$$\|L\|^2 = L_w \overline{L_z K(z,w)}.$$

When L is any of the previous functionals on G^2,

$$(3.8) \qquad \|L\|^2 = L_{(z)} L_{(w)} \widehat{K}(z,w)$$

since all functionals considered are real.

Our goal is to estimate the approximation error

$$||E_{h,N}|| = ||\widehat{E}_{h,N}||$$

which will be computed by repeated use of Lemma 3.2 and (3.8).

4. EVALUATION OF THE ERROR NORM $||\widehat{E}_{h,N}||$

Let $\widehat{\sigma}_{h,N} := \widehat{T}_h - \widehat{T}_{h,N}$. Then (3.8) implies that

$$
\begin{aligned}
||\widehat{E}_{h,N}||^2 \;&=\; ||\widehat{E}_h + \widehat{\sigma}_{h,N}||^2 \\
(4.1) \qquad &=\; \underbrace{\widehat{E}_{h,(z)}\widehat{E}_{h,(w)}\widehat{K}(z,w)}_{e_1} + \underbrace{\widehat{E}_{h,(z)}\widehat{\sigma}_{h,N,(w)}\widehat{K}(z,w)}_{e_2} \\
&\quad + \underbrace{\widehat{\sigma}_{h,N,(z)}\widehat{E}_{h,(w)}\widehat{K}(z,w)}_{e_3} + \underbrace{\widehat{\sigma}_{h,N,(z)}\widehat{\sigma}_{h,N,(w)}\widehat{K}(z,w)}_{e_4}.
\end{aligned}
$$

In (4.1), the error is divided into four parts denoted by e_1, e_2, e_3, and e_4. In the following three subsections, we will evaluate these quantities term by term.

4.1. Evaluation of e_2 and e_3. The second and third terms of (4.1) are actually equal. Indeed:

$$
\begin{aligned}
e_3 = \widehat{\sigma}_{h,N,(z)}\widehat{E}_{h,(w)}\widehat{K}(z,w) \;&=\; \widehat{E}_{h,(z)}\widehat{\sigma}_{h,N,(w)}\widehat{K}(z,w) \\
&=\; \widehat{E}_{h,(z)}\left(h \sum_{|n|>N} \widehat{K}(z,nh) \right) \\
&=\; \widehat{E}_{h,(z)}\left(h \sum_{|n|>N} F_{nh}(z) \right) \\
(4.2) \qquad &=\; h \sum_{|n|>N} \widehat{E}_h F_{nh}
\end{aligned}
$$

where

$$
\begin{aligned}
F_{nh} \;&:=\; P_{nh,(w)}\widehat{K}(z,w) \\
(4.3) \quad &=\; \frac{\pi^2 \cosh(z)\cosh(nh)}{4\cosh\left(\frac{\pi}{2}\sinh(z) - \frac{\pi}{2}\sinh(nh)\right)\cosh\left(\frac{\pi}{2}\sinh(z)\right)\cosh\left(\frac{\pi}{2}\sinh(nh)\right)}.
\end{aligned}
$$

To calculate (4.2), first we compute $\widehat{I}F_{nh}$

$$
\begin{aligned}
\widehat{I}F_{nh} \;&=\; \int_{-\infty}^{\infty} P_{nh,(w)}\widehat{K}(z,w)\,dz \\
&=\; \int_{-\infty}^{\infty} \frac{\pi^2 \cosh(z)\cosh(nh)}{4\cosh\left(\frac{\pi}{2}\sinh(z) - \frac{\pi}{2}\sinh(nh)\right)\cosh\left(\frac{\pi}{2}\sinh(z)\right)\cosh\left(\frac{\pi}{2}\sinh(nh)\right)}\,dz \\
&=\; \frac{\pi^2 \cosh(nh)}{4\cosh\left(\frac{\pi}{2}\sinh(nh)\right)} \int_{-\infty}^{\infty} \frac{\cosh(z)}{\cosh\left(\frac{\pi}{2}\sinh(z) - \frac{\pi}{2}\sinh(nh)\right)\cosh\left(\frac{\pi}{2}\sinh(z)\right)}\,dz.
\end{aligned}
$$

Using the change of variable $u := \frac{\pi}{2}\sinh(x)$, and letting $a := \frac{\pi}{2}\sinh(nh)$, we get

$$
\begin{aligned}
\widehat{IF}_{nh} &= \frac{\pi\cosh(nh)}{2\cosh(a)}\int_{-\infty}^{\infty}\frac{1}{\cosh(u-a)\cosh(u)}\,du \\
&= \frac{\pi\cosh(nh)}{\cosh(a)}\int_{-\infty}^{\infty}\frac{1}{\cosh(a)+\cosh(2u-a)}\,du.
\end{aligned}
$$

(4.4)

Applying the identity

(4.5)
$$
\frac{\sinh(a)}{\cosh(a)+\cosh(b)} = \frac{1}{1+e^{b-a}} - \frac{1}{1+e^{b+a}},
$$

we see that

$$
\begin{aligned}
(4.4) &= \frac{\pi\cosh(nh)}{\cosh(a)\sinh(a)}\int_{-\infty}^{\infty}\left(\frac{1}{1+e^{2u-2a}} - \frac{1}{1+e^{2u}}\right)du \\
&= \frac{\pi\cosh(nh)}{\cosh(a)\sinh(a)}\cdot a \\
&= \frac{\pi^2\sinh(2nh)}{2\sinh(\pi\sinh(nh))} = \widehat{IF}_{nh}.
\end{aligned}
$$

(4.6)

By a similar manipulation, we get $\widehat{T_nF}_{nh} =$

(4.7)
$$
\frac{\pi^2 h\cosh(nh)}{\sinh(\pi\sinh(nh))}\sum_{r=-\infty}^{\infty}\left(\frac{1}{1+e^{\pi\sinh(rh)-\pi\sinh(nh)}} - \frac{1}{1+e^{\pi\sinh(rh)}}\right)\cosh(rh).
$$

Therefore

(4.8) $\; h\displaystyle\sum_{|n|>N}\widehat{E_hF}_{nh} = \sum_{|n|>N}\frac{\pi^2\cosh(nh)}{\sinh(\pi\sinh(nh))}$

$$
\times\quad\left\{\sum_{r=-\infty}^{\infty}\left(\frac{1}{1+e^{\pi\sinh(rh)-\pi\sinh(nh)}} - \frac{1}{1+e^{\pi\sinh(rh)}}\right)h\cosh(rh) - \sinh(nh)\right\}.
$$

Consider the summation

$$
S_n := \sum_{r=-\infty}^{\infty}\left(\frac{1}{1+e^{\pi\sinh(rh)-\pi\sinh(nh)}} - \frac{1}{1+e^{\pi\sinh(rh)}}\right)h\cosh(rh),
$$

and note that for $h > 0$

$$
\begin{aligned}
S_n &= \left|e^{\pi\sinh(nh)}-1\right|\sum_{r=-\infty}^{\infty}\frac{e^{\pi/2\sinh(rh)}}{e^{\pi\sinh(nh)}+e^{\pi\sinh(rh)}}\frac{h\cosh(rh)}{2\cosh(\pi/2\sinh(rh))} \\
&= \sinh(\pi/2\sinh(nh))\sum_{r=-\infty}^{\infty}\frac{2\,e^{\pi/2\sinh(nh)}\,e^{\pi/2\sinh(rh)}}{e^{\pi\sinh(nh)}+e^{\pi\sinh(rh)}}\frac{h\cosh(rh)}{\cosh(\pi/2\sinh(rh))} \\
&\leq \sinh(\pi/2\sinh(nh))\sum_{r=-\infty}^{\infty}\frac{h\cosh(rh)}{\cosh(\pi/2\sinh(rh))} \\
&= h\sinh(\pi/2\sinh(nh))\left\{1+2\sum_{r=1}^{\infty}\frac{\cosh(rh)}{\cosh(\pi/2\sinh(rh))}\right\}.
\end{aligned}
$$

If we denote

$$(4.9) \qquad C(h) := 1 + 2 \sum_{r=1}^{\infty} \frac{\cosh(rh)}{\cosh(\pi/2 \sinh(rh))}$$

then

$$S_n \le h \, C(h) \sinh(\pi/2 \sinh(nh))$$

and by (4.8)

$$h \sum_{|n|>N} \widehat{E}_h F_{nh} \quad \le \quad \sum_{|n|>N} \frac{\pi^2 \cosh(nh)}{\sinh(\pi \sinh(nh))} \{ hC(h) \sinh(\pi/2 \sinh(nh)) + \sinh(nh) \}$$

$$(4.10) \qquad = \quad \frac{\pi^2}{2} \left(\sum_{|n|>N} \frac{hC(h)\cosh(nh)}{\cosh(\pi \sinh(nh)/2)} + \sum_{|n|>N} \frac{\sinh(2nh)}{\sinh(\pi \sinh(nh))} \right).$$

Since $\frac{\cosh(x)}{\cosh(\pi/2 \sinh(x))}$ is a decreasing function, we have

$$\sum_{n>N} \frac{\cosh(nh)}{\cosh(\pi/2 \sinh(nh))} \quad \le \quad \frac{1}{h} \int_{N-1}^{\infty} \frac{\cosh(x)}{\cosh(\pi/2 \sinh(x))} dx$$

$$= \quad \frac{2}{\pi h} \int_{\pi/2 \sinh(N-1)}^{\infty} \operatorname{sech}(u) du$$

$$= \quad \frac{2}{\pi h} \left(\frac{\pi}{2} - \arctan \left(\sinh \left(\frac{\sinh(N-1)}{\pi} \right) \right) \right)$$

Also, because

$$\lim_{x \to \infty} x \left(\frac{\pi}{2} - \arctan(x) \right) = \frac{1}{2}$$

when x is large enough,

$$O\left(\frac{\pi}{2} - \arctan(x) \right) = O\left(\frac{1}{x} \right)$$

thus

$$(4.11) \qquad \sum_{n>N} \frac{\cosh(nh)}{\cosh(\pi/2 \sinh(nh))} = O\left(\frac{1}{h} e^{-\frac{e^{N-1}}{\pi}} \right)$$

Also, the second summation in (4.10) is less than the first one , which gives

$$(4.12) \qquad e2 = e3 = h \sum_{|n|>N} \widehat{E}_h F_{nh} = O\left(C(h) e^{-\frac{e^{N-1}}{\pi}} \right)$$

4.2. **Evaluation of e_1.** In order to compute $e_1 = \widehat{E}_{h,(z)} \widehat{E}_{h,(w)} \widehat{K}(z,w)$, we have recourse to the results of Section 2.1. In this case, the reproducing kernel $\widehat{K}(z, \overline{w})$ has poles at $w = i \cdot \arcsin(2n+1)$ with residue $\pi \cosh(z)/\sinh(\pi \sinh(z))$ and poles at $w = \operatorname{arcsinh}(\sinh(z) + (2n+1)i)$ with reside $-\pi \cosh(z)/\sinh(\pi \sinh(z))$. Since we are dealing with real functions, $\widehat{K}(z, \overline{w}) = \widehat{K}(z, w)$, as a function of $\overline{w}$, satisfies the conditions of Lemmas 2.3 and 2.4, if we take $\alpha \le i \arcsin(1)$.

Therefore,

$$\widehat{E}_{h,(w)}\widehat{K}(z,w) = \frac{1}{2\pi i}\int_{\partial S_\alpha}\widehat{K}(z,\overline{w})\Phi(w)\,dw$$

$$= \frac{1}{2\pi i}\int_{\partial S_\alpha}\frac{\pi^2\cosh(z)\cosh(w)\Phi(w)}{4\cosh\left(\frac{\pi}{2}\sinh(z)-\frac{\pi}{2}\sinh(w)\right)\cosh\left(\frac{\pi}{2}\sinh(z)\right)\cosh\left(\frac{\pi}{2}\sinh(w)\right)}\,dw,$$

where

$$
(4.13)\qquad
\begin{aligned}
\Phi(z) := \Phi(z;h) \;&=\; \Psi(z) - \pi\cot\frac{\pi z}{h}\\[4pt]
&=\;
\begin{cases}
\frac{-2\pi i}{1-\exp(-2\pi i z/h)}, & \mathrm{Im}(z) > 0\\[6pt]
\frac{2\pi i}{1-\exp(2\pi i z/h)}, & \mathrm{Im}(z) < 0.
\end{cases}
\end{aligned}
$$

Here $\Psi(x)$ is defined as in (2.4). If we let $\alpha \to \infty$ while keeping ∂S_α away from the poles,(2.5) gives

$$\int_{-\infty}^{\infty}f(x)dx$$

$$= \frac{1}{2\pi i}\int_{\partial S_\alpha}f(z)\Psi(z)\,dz - \frac{\pi\cosh(z)}{\sinh(\pi\sinh(z))}\sum_{n=-\infty}^{\infty}\Psi(i\arcsin(2n+1))$$

$$+ \frac{\pi\cosh(z)}{\sinh(\pi\sinh(z))}\sum_{n=-\infty}^{\infty}\Psi\left(\operatorname{arcsinh}\left(\sinh(z)+(2n+1)i\right)\right)$$

and equation (2.6) gives

$$h\sum_{n=-\infty}^{\infty}f(nh)$$

$$= \frac{1}{2\pi i}\int_{\partial S_\alpha}f(z)\pi\cot\frac{\pi z}{h}dz - \frac{\pi\cosh(z)}{\sinh(\pi\sinh(z))}\sum_{n=-\infty}^{\infty}\pi\cot\left(\frac{\pi\cdot i\arcsin(2n+1)}{h}\right)$$

$$+ \frac{\pi\cosh(z)}{\sinh(\pi\sinh(z))}\sum_{n=-\infty}^{\infty}\pi\cot\left(\frac{\pi\cdot\operatorname{arcsinh}(\sinh(z)+(2n+1)i)}{h}\right)$$

We thus have

$$\widehat{E}_{h,(w)}\widehat{K}(z,w)$$

$$= \frac{1}{2\pi i}\int_{\partial S_\alpha}\widehat{K}(z,\overline{w})\Phi(w)\,dw - \frac{\pi\cosh(z)}{\sinh(\pi\sinh(z))}\sum_{n=-\infty}^{\infty}\Phi(i\arcsin(2n+1))$$

$$+ \frac{\pi\cosh(z)}{\sinh(\pi\sinh(z))}\sum_{n=-\infty}^{\infty}\Phi(\operatorname{arcsinh}(\sinh(z)+(2n+1)i))$$

Since $|\Phi(x + iy)| \sim 2\pi \exp(-2\pi|y|/h)$, the integral part will go to zero as $\alpha \to \infty$. Also, since $\Phi(-w) = -\Phi(w)$,

$$\frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \Phi(i \arcsin(2n+1))$$

$$= \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \lim_{N \to \infty} \sum_{n=-N}^{N} \Phi(i \arcsin(2n+1))$$

$$= 0.$$

Therefore

$$(4.14) \quad \widehat{E}_{h,(w)} \widehat{K}(z,w) = \frac{\pi \cosh(z)}{\sinh(\pi \sinh(z))} \sum_{n=-\infty}^{\infty} \Phi(\mathrm{arcsinh}(\sinh(z) + (2n+1)i)).$$

Letting

$$H(z) := \widehat{E}_{h,(w)} \widehat{K}(z,w),$$

it is easy to see that $H(z) \in C^{\infty}$. Also, since we have shown $\widehat{E}_h$ is a bounded linear functional and

$$\widehat{E}_h = \widehat{E}_{h,(z)} H(z) = \int_{-\infty}^{\infty} H(z)dz - h \sum_{n=-\infty}^{\infty} H(nh).$$

Thus, $\sum_{n=-\infty}^{\infty} H(nh)$ exists for all $h > 0$ and $H(z)$ is integrable because the integrand is independent of h. Taking $h \to 0$, the summation part will converge to zero so that the integral is finite. Therefore, $H(z)$ satisfies the conditions of the Poisson summation formula (see the proof of [7, 243–4]). It follows that

$$(4.15) \quad \|\widehat{E}_h\|^2 = \widehat{E}_h \, H = -2 \sum_{r=1}^{\infty} \int_{-\infty}^{\infty} H(x) \cos(2\pi r x/h) \, dx.$$

Set

$$H_n(x) := \Phi\left(\mathrm{arcsinh}\left(\sinh(z) + (2n-1)i\right)\right) + \Phi\left(\mathrm{arcsinh}\left(\sinh(z) - (2n-1)i\right)\right)$$

then

$$(4.16) \quad H(x) = \sum_{n=1}^{\infty} \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} H_n(x).$$

Letting $A(z,n)$ and $B(z,n)$ denote the real part and imaginary part of

$$\mathrm{arcsinh}(\sinh(z) + (2n-1)i),$$

we have

$$A(z,n) = \mathrm{sgn}(z) \cdot \ln\left(\frac{1}{2}\sqrt{\sinh(z)^2 + 4n^2} + \frac{1}{2}\sqrt{\sinh(z)^2 + (2n-2)^2}\right.$$

$$\left. + \sqrt{\left(\frac{1}{2}\sqrt{\sinh(z)^2 + 4n^2} + \frac{1}{2}\sqrt{\sinh(z)^2 + (2n-2)^2}\right)^2 - 1}\right)$$

$$B(z,n) = \arcsin\left(1/2\sqrt{(\sinh(z))^2 + 4n^2} - 1/2\sqrt{(\sinh(z))^2 + (2n-2)^2}\right)$$

where

(4.17)
$$\operatorname{sgn}(z) = \begin{cases} 1, & \operatorname{Re}(z) > 0 \\ -1, & \operatorname{Re}(z) < 0. \end{cases}$$

Therefore

$$\Phi(\operatorname{arcsinh}(\sinh(z) + (2n-1)i)) = \Phi(A(z,n) + B(z,n)i)$$
$$\Phi(\operatorname{arcsinh}(\sinh(z) - (2n-1)i)) = \Phi(A(z,n) - B(z,n)i)$$

and

$$
H_n(z) = 2\pi i \left(\frac{-1}{1 - \exp(\frac{2\pi B(z,n) - i2\pi A(z,n)}{h})} + \frac{1}{1 - \exp(\frac{2\pi B(z,n) + i2\pi A(z,n)}{h})} \right)
$$

(4.18)
$$
= \frac{-4\pi \sin(\frac{2\pi A(z,n)}{h}) \exp(\frac{2\pi B(z,n)}{h})}{1 - 2\exp(\frac{2\pi B(z,n)}{h}) \cos(\frac{2\pi A(z,n)}{h}) + \exp(\frac{4\pi B(z,n)}{h})}
$$

By (4.15) and (4.16),

$$
\|\widehat{E}_h\|^2 = -2 \sum_{r=1}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} H_n(x) \cos(2\pi r x/h)\, dx
$$

(4.19)
$$
= -2 \sum_{r=1}^{\infty} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} H_n(x) \cos(2\pi r x/h)\, dx
$$

Using a more elaborate (and computer algebra assisted) version of the argument given by [1] for tanh, we can estimate e_1 by

(4.20)
$$e_1 = \|\widehat{E}_h\|^2 = O\left(e^{-A/h}\right)$$

where A is a positive constant. This entails integrating the integral component of (4.19) by parts twice to obtain

$$
\int_{-\infty}^{\infty} H_n(x) \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} \cos(2\pi r x/h) dx
$$

(4.21)
$$
= -\left(\frac{h}{2\pi r}\right)^2 \int_{-\infty}^{\infty} \left(H_n(x) \frac{\pi \cosh(x)}{\sinh(\pi \sinh(x))} \right)'' \cos\left(2\pi r x/h\right)\, dx.
$$

which is

$$
O\left(\frac{e^{-A/h}}{r^2 n^2} \right)
$$

Our numerical experiments indicate that A is some constant bounded below by $5/2$ and above by 10.

4.3. Evaluation of e_4. We have:

$$
e_4 = \widehat{\sigma}_{h,N,(z)} \widehat{\sigma}_{h,N,(w)} \widehat{K}(z,w)
$$
$$
= \frac{h^2 \pi^2}{4} \sum_{|r|,|n|>N} \frac{\cosh(nh)\cosh(rh)}{\cosh\left(\frac{\pi}{2}\sinh(nh) - \frac{\pi}{2}\sinh(rh)\right) \cosh\left(\frac{\pi}{2}\sinh(nh)\right) \cosh\left(\frac{\pi}{2}\sinh(rh)\right)}
$$

which can be written as

(4.22)
$$
\frac{h^2 \pi^2}{2} \sum_{r,n>N} \frac{\cosh(nh)\cosh(rh)}{\cosh\left(\frac{\pi}{2}\sinh(nh) - \frac{\pi}{2}\sinh(rh)\right) \cosh\left(\frac{\pi}{2}\sinh(nh)\right) \cosh\left(\frac{\pi}{2}\sinh(rh)\right)}
$$

(4.23)
$$+\frac{h^2\pi^2}{2}\sum_{r,-n>N}\frac{\cosh(nh)\cosh(rh)}{\cosh\left(\frac{\pi}{2}\sinh(nh)-\frac{\pi}{2}\sinh(rh)\right)\cosh\left(\frac{\pi}{2}\sinh(nh)\right)\cosh\left(\frac{\pi}{2}\sinh(rh)\right)}.$$

Then, because of (4.11), equation (4.23) is equal to

$$\frac{h^2\pi^2}{2}\sum_{r,n>N}\frac{\cosh(nh)\cosh(rh)}{\cosh\left(\frac{\pi}{2}\sinh(nh)+\frac{\pi}{2}\sinh(rh)\right)\cosh\left(\frac{\pi}{2}\sinh(nh)\right)\cosh\left(\frac{\pi}{2}\sinh(rh)\right)}$$

(4.24)
$$\leq \frac{h^2\pi^2}{2}\left(\sum_{n>N}\frac{\cosh(nh)}{\cosh(\pi/2\sinh(nh))}\right)^2 = O\left(e^{-\frac{2e^{N-1}}{\pi}}\right)$$

Also, (4.22) can be written as

$$\frac{h^2\pi^2}{2}\sum_{n>N}\frac{\cosh^2(nh)}{\cosh^2(\pi/2\sinh(nh))}$$

$$+\ 2\sum_{n>r>N}\frac{\cosh(nh)\cosh(rh)}{\cosh\left(\frac{\pi}{2}\sinh(nh)-\frac{\pi}{2}\sinh(rh)\right)\cosh\left(\frac{\pi}{2}\sinh(nh)\right)\cosh\left(\frac{\pi}{2}\sinh(rh)\right)}.$$

The first term above is less than

(4.25)
$$O\left(e^{-\frac{2e^{N-1}}{\pi}}\right)$$

and the second term is less than the first. Equation (4.24) and (4.25) together give

(4.26)
$$e_4 = \widehat{\sigma}_{h,N,(z)}\widehat{\sigma}_{h,N,(w)}\widehat{K}(z,w) = O\left(e^{-\frac{2e^{N-1}}{\pi}}\right).$$

5. The main results

Combining (4.12), (4.20) and (4.26) we get

Theorem 5.1. (Tanh-sinh convergence.) *(a) For $f \in H^2$ and $\psi(x) = \tanh(\frac{\pi}{2}\sinh(x))$, the error bound can be evaluated as:*

$$||\widehat{E}_{h,N}||^2 = ||E_{h,N}||^2 = O\left(e^{-A/h}\right) + O\left(C(h)e^{-\frac{e^{N-1}}{\pi}}\right)$$

where the order constant is independent of both N and h. Here as before

(5.1)
$$C(h) := 1 + 2\sum_{r=1}^{\infty}\frac{\cosh(rh)}{\cosh(\pi/2\sinh(rh))} \leq 1 + \frac{4}{\pi h}$$

(b) This method exhibits quadratic convergence as we let $N \to \infty$ and $h \to 0^+$, while keeping Nh a constant.

Proof. (a) We estimate

$$\begin{aligned}||\widehat{E}_{h,N}||^2 = ||E_{h,N}||^2 &= e_1 + e_2 + e_3 + e_4\\ &= O\left(e^{-A/h}\right) + O\left(C(h)e^{-\frac{e^{N-1}}{\pi}}\right) + O\left(e^{-\frac{2e^{N-1}}{\pi}}\right)\\ &= O\left(e^{-A/h}\right) + O\left(C(h)e^{-\frac{e^{N-1}}{\pi}}\right).\end{aligned}$$

To obtain the estimate for $C(h)$ observe that

$$hC(h) \leq h + 2\int_0^{\infty}\frac{\cosh(t)}{\cosh(\pi/2\sinh(t))}\,dt = h + \frac{4}{\pi}\int_0^{\infty}\mathrm{sech}\left(\frac{\pi}{2}x\right)\,dx,$$

since the first integrand is decreasing. The latter integral evaluates to 1.

(b) In this case the order term

$$O\left(e^{-A/h}\right)$$

shows quadratic convergence as $h \to 0$, because when h is halved, $e^{-A/h}$ will be squared. Similarly,

$$O\left(C(h)e^{-\frac{e^{N-1}}{\pi}}\right)$$

exhibits exponential convergence as $N \to \infty$.   □

6. Analysis for Error Function and Other Transformations

As shown in Figure 1, tanh, tanh-sinh and error function have very similar properties: absolutely continuous monotonic increasing functions mapping $(-\infty, \infty)$ onto $(-1, 1)$, infinitely differentiable and every order of their derivatives quickly vanishes at infinity. It's not surprising that a similar approach can be applied to analyze the convergent property of error function and more generally, to any transformation with the property above.

For example, the error quadrature uses the transformation:

$$(6.1) \qquad \psi(x) = \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

with

$$\psi'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

As in section 3, one can introduce a corresponding space $G_*{}^2$, which contains the set of functions of the form

$$\psi'(w) f\left(\psi(w)\right) \qquad f \in H^2 \text{ and } \psi \text{ as in } (6.1).$$

G_*^2 then has a reproducing kernel

$$\begin{aligned}
\widehat{K}(z, w) &= 1 \big/ \left(1 - \psi(z)\overline{\psi(w)}\right) \cdot \psi'(z) \cdot \overline{\psi'(w)} \\
&= \frac{4}{\pi\left(1 - \mathrm{erf}(z)\mathrm{erf}(w)\right)} \cdot e^{-z^2} e^{-w^2}
\end{aligned}$$

The approximation error $\|\widehat{E}_{h,N}\|$ can still be divided into 4 parts as in (4.1) and

$$e_2 = e_3 = h \sum_{|n|>N} \widehat{E}_h F_{nh}$$

$$e_4 = \frac{4}{\pi} \sum_{|r|,|n|>N} \frac{e^{-r^2 h^2} e^{-n^2 h^2}}{1 - \mathrm{erf}(rh)\mathrm{erf}(nh)}$$

where

$$\begin{aligned}
F_{nh} &:= P_{nh,(w)}\widehat{K}(z, w) \\
&= \frac{4}{\pi} \frac{e^{-z^2} e^{-n^2 h^2}}{1 - \mathrm{erf}(z)\mathrm{erf}(nh)}.
\end{aligned}$$

In order to evaluate e_1, one need to find out the poles of

$$\frac{1}{1 - \mathrm{erf}(z)\mathrm{erf}(w)}.$$

Also, the evaluation of e_2 and e_3 involves the careful estimation of

$$\sum_{n=a}^{\infty} \sum_{r=b}^{\infty} \frac{e^{-r^2 h^2}}{1 - \mathrm{erf}(rh)\mathrm{erf}(nh)}$$

Here we will not give further analysis. For any other quadrature rules with similar properties, the similar approach can be taken.

7. Numerical Convergence and Examples

We conclude with a few numerical illustrations and a brief discussion of the optimality of the coefficient of the infinite trapezoidal rule.

7.1. Numerical results. We exhibit results for the following set of test functions, earlier studied in [5]:

  (1) $\int_0^1 t \log(1+t)\,dt \ = \ 1/4$
  (2) $\int_0^1 t^2 \arctan t\,dt \ = \ (\pi - 2 + 2\log 2)/12$
  (3) $\int_0^{\pi/2} e^t \cos t\,dt \ = \ (e^{\pi/2} - 1)/2$
  (4) $\int_0^1 \frac{\arctan(\sqrt{2+t^2})}{(1+t^2)\sqrt{2+t^2}}\,dt \ = \ 5\pi^2/96$
  (5) $\int_0^1 \sqrt{t}\log t\,dt \ = \ -4/9$
  (6) $\int_0^1 \sqrt{1-t^2}\,dt \ = \ \pi/4$
  (7) $\int_0^1 \frac{\sqrt{t}}{\sqrt{1-t^2}}dt \ = \ 2\sqrt{\pi}\Gamma(3/4)/\Gamma(1/4) \ = \ \beta(1/2, 3/4)/2$
  (8) $\int_0^1 \log^2 t\,dt \ = \ 2$
  (9) $\int_0^{\pi/2} \log(\cos t)\,dt \ = \ -\pi \log(2)/2$
  (10) $\int_0^{\pi/2} \sqrt{\tan t}\,dt \ = \ \pi\sqrt{2}/2$
  (11) $\int_0^{\infty} \frac{1}{1+t^2}\,dt \ = \ \pi/2$
  (12) $\int_0^{\infty} \frac{e^{-t}}{\sqrt{t}}\,dt \ = \ \sqrt{\pi}$
  (13) $\int_0^{\infty} e^{-t^2/2}\,dt \ = \ \sqrt{\pi/2}$
  (14) $\int_0^{\infty} e^{-t} \cos t\,dt \ = \ 1/2$

Note that the first four of these integrals involve well-behaved continuous functions on finite intervals. The integrands in problems 5 and 6 have an infinite derivative at an endpoint. In 7 through 10, there is an integrable singularity at an endpoint. In 11 through 13, the interval of integration is infinite, requiring a transformation such as $s = 1/(t+1)$. Problem 14 is an oscillatory function on an infinite interval.

Table 1 gives the actual error resulting from using the summation (1.2) as an approximation to the specified integral, rounded to the nearest power of ten. The precision used in these calculations was 1000 digits, so no results are shown with errors less than 10^{-1000}. The "level" shown in the table is an index of the interval h, according to the rule $h = 2^{-m}$, where m is the level. In other words, at each successive level, the interval h is halved.

It can be seen from the results in the table that the tanh-sinh scheme achieves approximately exponential convergence, in the sense that the number of correct digits approximately doubles with each level. In some cases, such as Problems 3, 5 and 8, the convergence is slightly better than a doubling with each level; in other cases, such as Problems 12, 13 and 14, the rate is slightly less. It is remarkable that Problems 1, 6, 7, 9 and 10 exhibit virtually identical convergence behavior.

Level	Prob. 1	Prob. 2	Prob. 3	Prob. 4	Prob. 5	Prob. 6	Prob. 7
1	10^{-4}	10^{-4}	10^{-4}	10^{-4}	10^{-5}	10^{-5}	10^{-6}
2	10^{-11}	10^{-11}	10^{-9}	10^{-9}	10^{-12}	10^{-12}	10^{-12}
3	10^{-24}	10^{-19}	10^{-21}	10^{-18}	10^{-28}	10^{-25}	10^{-26}
4	10^{-51}	10^{-38}	10^{-49}	10^{-36}	10^{-62}	10^{-50}	10^{-49}
5	10^{-98}	10^{-74}	10^{-106}	10^{-73}	10^{-129}	10^{-99}	10^{-98}
6	10^{-195}	10^{-147}	10^{-225}	10^{-145}	10^{-265}	10^{-196}	10^{-194}
7	10^{-390}	10^{-293}	10^{-471}	10^{-290}	10^{-539}	10^{-391}	10^{-388}
8	10^{-777}	10^{-584}	10^{-974}	10^{-582}		10^{-779}	10^{-777}
9							
10							
11							
Level	Prob. 8	Prob. 9	Prob. 10	Prob. 11	Prob. 12	Prob. 13	Prob. 14
1	10^{-5}	10^{-4}	10^{-6}	10^{-2}	10^{-2}	10^{-1}	10^{-1}
2	10^{-12}	10^{-11}	10^{-12}	10^{-5}	10^{-4}	10^{-3}	10^{-2}
3	10^{-29}	10^{-24}	10^{-25}	10^{-11}	10^{-9}	10^{-6}	10^{-5}
4	10^{-62}	10^{-50}	10^{-48}	10^{-22}	10^{-15}	10^{-9}	10^{-8}
5	10^{-130}	10^{-97}	10^{-98}	10^{-45}	10^{-28}	10^{-19}	10^{-14}
6	10^{-266}	10^{-195}	10^{-194}	10^{-91}	10^{-50}	10^{-37}	10^{-26}
7	10^{-540}	10^{-389}	10^{-388}	10^{-182}	10^{-92}	10^{-66}	10^{-48}
8		10^{-777}	10^{-777}	10^{-365}	10^{-170}	10^{-126}	10^{-88}
9				10^{-731}	10^{-315}	10^{-240}	10^{-164}
10					10^{-584}	10^{-457}	10^{-304}
11						10^{-870}	10^{-564}

TABLE 1. Tanh-sinh errors at level m for the 14 test problems

7.2. Non-optimality of the limiting trapezoidal rule.

Finally, we examine the optimality of the coefficient of the infinite trapezoidal rule $\widehat{T}_h$ with $\psi(x)$ as defined in (1.3). Let $A := \{\ldots, a_{-1}, a_0, a_1 \ldots\}$ be the sequence of coefficients for $\widehat{T}_h$ such that

$$\widehat{T}_{h;A}\widehat{f} = \sum_{n=-\infty}^{\infty} a_n \widehat{f}(nh)$$

for every $\widehat{f} \in G^2$, and

$$\widehat{E}_{h;A} = \widehat{I} - \widehat{T}_{h;A}$$

We may ask whether A_h where $a_n \equiv h$ is *optimal* in the sense that

$$\|\widehat{E}_{h;A_h}\| = \inf \|\widehat{E}_{h;A}\|$$

for all A's for which $\widehat{T}_A$ is a bounded linear functional. As shown in [1], A_h is optimal if and only if $\widehat{T}_h$ integrates F_{nh} exactly for every integer n. But from the analysis in Section 4.1, we know $\widehat{E}_h F_{nh} \neq 0$; consequently A_h is not optimal.

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