Constructions of Uniformly Convex Functions

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Abstract. We give precise conditions under which the composition of a norm with a convex function yields a uniformly convex function on a Banach space. Various applications are given to functions of power type. The results are dualized to study uniform smoothness and several examples are provided.

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1 Introduction and preliminary results

We work in a real Banach space X whose closed unit ball is denoted by B_X, and whose unit sphere is denoted by S_X. By a *proper function* $f : X \to (-\infty, +\infty]$, we mean a function which is somewhere real-valued. A proper function $f : X \to (-\infty, +\infty]$ is *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)y \quad \text{for all } x, y \in \operatorname{dom} f,\ 0 \leq \lambda \leq 1.$$

The *conjugate function* of $f : X \to (-\infty, +\infty]$ is defined for $x^* \in X^*$ by

$$f^*(x^*) = \sup_{x \in X} \langle x^*, x\rangle - f(x).$$

Relevant background material on convex analysis can be found in various fine texts such as [9, 11] and in our own book [3].

Given a proper convex function $f : X \to (-\infty, +\infty]$, its *modulus of convexity* is the function $\delta_f : [0, +\infty) \to [0, +\infty]$ defined by

$$\delta_f(t) := \inf\left\{\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) : \|x-y\| = t,\ x, y \in \operatorname{dom} f\right\},$$

where the infimum over the empty set is $+\infty$. We say that f is *uniformly convex* when $\delta_f(t) > 0$ for all $t > 0$, and f has a *modulus of convexity of power type p* if there exists $C > 0$ so that $\delta_f(t) \geq Ct^p$ for all $t > 0$. In [11], uniformly convex functions are defined using the *gage of uniform convexity*, and it follows from [10, Remark 2.1] that the definition presented here is equivalent to that used in [10, 11].

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Relatedly, the *modulus of convexity of the norm* $\|\cdot\|$, $\delta_{\|\cdot\|}$, is defined for $0 \le \epsilon \le 2$ by

$$\delta_{\|\cdot\|}(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \;:\; \|x\| = \|y\| = 1, \text{ and } \|x-y\| \ge \epsilon\right\}.$$

In the case $\delta_{\|\cdot\|}(\epsilon) > 0$ for each $\epsilon > 0$, $\|\cdot\|$ is said to be *uniformly convex* (as a norm). If there exists $C > 0$ and $p \ge 2$ such that $\delta_{\|\cdot\|}(\epsilon) \ge C\epsilon^p$ for all $0 \le \epsilon \le 2$, then $\|\cdot\|$ is said to have *modulus of convexity of power type p*.

A systematic exposition of uniformly convex norms can be found in [6, Sections IV.4, IV.5], and [11, Section 3.5] presents a thorough account of uniformly convex functions. However, explicit constructions of such functions, especially those derived from a uniformly convex norm, appear to be somewhat sparse. While it is easy to see, for example, that $f := \|\cdot\|^r$ with $r > 1$ is uniformly convex on bounded sets when $\|\cdot\|$ is uniformly convex, it is not necessarily globally uniformly convex. In fact [2] shows when $r \ge 2$, that f is uniformly convex *if and only if* $\|\cdot\|$ has modulus of convexity of power type r. Our goal in this note is provide precise conditions under which $g \circ \|\cdot\|$ is uniformly convex when g is a nondecreasing convex function on $[0, \infty)$.

In many algorithms, uniform convexity on bounded sets and other weaker forms of convexity suffice for their implementation as can be seen, for example, in [4, 5]. Nonetheless, beyond their theoretical interest, uniformly convex functions are dual under conjugation to uniformly smooth convex functions [1]. Also when considered with moduli of power type, there is a tight duality with Hölder continuity conditions on the derivatives (see [11, Theorem 3.5.10, Corollary 3.5.11 and Theorem 3.5.12]. Because uniformly convex norms, and even those with some power type are (abundantly) available on superreflexive spaces as is discussed in the monograph [6], we believe it is important to find explicit conditions under which the composition with a norm yields a uniformly convex function (or even better, one with modulus of power type). Inter alia, we adumbrate the somewhat subtle relationship between notions of uniform convexity for norms—based on behaviour on the sphere—and those for convex functions.

We will use the following simple examples of uniformly convex functions on the real line recorded in [3, Exercise 5.4.2].

Fact 1.1. *Suppose a function f on $\mathbb{R}$ satisfies $f^{(n)} \ge \alpha > 0$ on $[a, \infty)$ where $n \ge 2$ is a fixed integer, and that $f^{(k)} \ge 0$ on $[a, \infty)$ for $k \in \{2, \ldots, n+1\}$. Define the function g by $g(x) := f(x)$ for $x \ge a$ and $g(x) := +\infty$ for $x < a$. Then g is uniformly convex with modulus of convexity of power type n.*

In particular, let $b > 1$ and $g(x) := b^x$ for $x \ge 0$, and $g(x) := +\infty$ otherwise. Then g is uniformly convex with modulus of convexity of power type p for any $p \ge 2$. Similarly, using Taylor series one can show that for $p \ge 2$ and $g(x) := x^p$ for $x \ge 0$, and $g(x) := +\infty$ otherwise, g is uniformly convex with modulus of convexity of power type p.

2 Constructions of Uniformly Convex Functions

Our first objective is to determine precisely when a composition with a norm yields a uniformly convex function.

Theorem 2.1. *Suppose $f : [0, +\infty) \to [0, +\infty)$ is convex and nondecreasing. Then $f \circ \|\cdot\|$ is uniformly convex if and only if f and $\|\cdot\|$ are uniformly convex while*

$$\liminf_{t\to\infty} f'_+(t) \cdot \delta_{\|\cdot\|}\left(\frac{\epsilon}{t}\right) \cdot t > 0 \tag{2.1}$$

for each $\epsilon > 0$.

Proof. $\Rightarrow$: Clearly f is uniformly convex because for fixed $x_0 \in S_X$, we have that $f(t) = f(\|tx_0\|)$ and so f is a uniformly convex function. Similarly, $\|\cdot\|$ is uniformly convex. Indeed, suppose $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| \to 2$. Then

$$\frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - f\left(\left\|\frac{x_n + y_n}{2}\right\|\right) \to 0$$

because f is continuous at 1. The uniform convexity of $f \circ \|\cdot\|$ implies $\|x_n - y_n\| \to 0$; thus $\|\cdot\|$ is uniformly convex.

Thence, suppose for some $\epsilon > 0$ and $t_n \to \infty$ that $\lim_{n\to\infty} f'_+(t_n) \cdot \delta_{\|\cdot\|}\left(\frac{\epsilon}{t_n}\right) \cdot t_n = 0$. Now choose $u_n, v_n \in S_X$ such that $\|u_n - v_n\| \geq \frac{\epsilon}{t_n}$ but

$$\left\|\frac{u_n + v_n}{2}\right\| \geq 1 - 2\delta_{\|\cdot\|}\left(\frac{\epsilon}{t_n}\right).$$

Let $x_n := t_n u_n$ and $y_n := t_n v_n$. Then $\|x_n - y_n\| \geq \epsilon$ for all n, but

$$\begin{aligned} f\left(\left\|\frac{t_n u_n + t_n v_n}{2}\right\|\right) &\geq f(\|t_n u_n\|) - 2t_n\delta_{\|\cdot\|}\left(\frac{\epsilon}{t_n}\right) \cdot f'_+(t_n) \\ &\geq f(\|t_n u_n\|) - 2\epsilon_n \quad \text{where } \epsilon_n = t_n\delta_{\|\cdot\|}\left(\frac{\epsilon}{t_n}\right) \cdot f'_+(t_n) \to 0, \end{aligned}$$

which contradicts the uniform convexity of $f \circ \|\cdot\|$.

$\Leftarrow$: Suppose for each $\epsilon > 0$, $\liminf_{t\to\infty} f'_+(t) \cdot \delta_{\|\cdot\|}\left(\frac{\epsilon}{t}\right) \cdot t > 0$, and f and $\|\cdot\|$ are uniformly convex. Suppose $f \circ \|\cdot\|$ is not uniformly convex. Then there exist $(x_n), (y_n) \subset X$ and $\epsilon > 0$ such that $\|x_n - y_n\| \geq \epsilon$ for all $n \in \mathbb{N}$, but

$$\frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - f\left(\left\|\frac{x_n + y_n}{2}\right\|\right) \to 0. \tag{2.2}$$

We shall consider various cases. First suppose $\limsup_{n\to\infty}\big|\|x_n\| - \|y_n\|\big| > 0$. By switching roles of x_n and y_n as necessary, and passing to a subsequence we may assume $\|x_n\| - \|y_n\| \geq \eta > 0$ for all $n \in \mathbb{N}$. Thus using the fact f is nondecreasing and uniformly convex we have

$$\begin{aligned} \frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - f\left(\left\|\frac{x_n + y_n}{2}\right\|\right) &\geq \frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - f\left(\frac{\|x_n\| + \|y_n\|}{2}\right) \\ &\geq \delta_f(\eta) > 0 \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

This contradicts (2.2). Thus, for the rest of the proof we may and do suppose $(\|x_n\| - \|y_n\|) \to 0$.

(a) Consider the case where (x_n) is a bounded sequence. By passing to a subsequence as necessary we may assume $\|x_n\| \to \alpha$ and $\|y_n\| \to \alpha$ for some $\alpha \geq 0$. Because $\|x_n - y_n\| \geq \epsilon$ it is clear that $\alpha > 0$ and by the uniform convexity of $\|\cdot\|$ we obtain

$$\limsup_{n\to\infty}\left\|\frac{x_n + y_n}{2\alpha}\right\| \leq 1 - \delta\left(\frac{\epsilon}{\alpha}\right).$$

Consequently, $\limsup_{n\to\infty}\left\|\frac{x_n + y_n}{2}\right\| \leq \alpha\left[1 - \delta\left(\frac{\epsilon}{\alpha}\right)\right]$. Using the fact that f is convex and increasing, we obtain

$$\begin{aligned} \liminf_{n\to\infty}\frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - f\left(\left\|\frac{x_n + y_n}{2}\right\|\right) &\geq \liminf_{n\to\infty} f\left(\frac{\|x_n\| + \|y_n\|}{2}\right) - f\left(\left\|\frac{x_n + y_n}{2}\right\|\right) \\ &\geq f(\alpha) - f\left(\alpha - \alpha\delta_{\|\cdot\|}\left(\frac{\epsilon}{\alpha}\right)\right) > 0. \end{aligned}$$

which contradicts (2.2).

(b) It remains to consider the case where (x_n) is unbounded. In fact, any bounded subsequence of (x_n) would yield a contradiction as above, so we let $\alpha_n := \|x_n\|$ and assume $\alpha_n \to \infty$. Further, because we now know that $(\|x_n\| - \|y_n\|) \to 0$, interchanging x_n and y_n as necessary, we write $\|y_n\| = \beta_n$ where $\alpha_n = \beta_n + \eta_n$ and $\eta_n \to 0^+$.

Now let $\tilde{x}_n := \frac{1}{\alpha_n} x_n$ and $\tilde{y}_n := \frac{1}{\beta_n} y_n$. Then $\|\tilde{x}_n - \tilde{y}_n\| \geq \frac{\epsilon - \eta_n}{\alpha_n}$. Fix $N \in \mathbb{N}$ such that $\|\tilde{x}_n - \tilde{y}_n\| \geq \frac{\epsilon}{2\beta_n}$ for $n \geq N$. The uniform convexity of $\|\cdot\|$ ensures that

$$\left\| \frac{\tilde{x}_n + \tilde{y}_n}{2} \right\| \leq 1 - \delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \quad \text{for } n \geq N.$$

Let

$$\tilde{\beta}_n := \frac{\beta_n + \alpha_n}{2} - \delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \cdot \beta_n. \tag{2.3}$$

Note that $\|x_n + y_n\| \leq \beta_n \|\tilde{x}_n + \tilde{y}_n\| + \eta_n$, and that $\tilde{\beta}_n / \beta_n \to 1$ (since $\beta_n \to \infty, \eta_n \to 0$). Then, for $n \geq N$, monotonicity of f ensures that

$$\begin{aligned} f\left(\left\| \frac{x_n + y_n}{2} \right\|\right) &\leq f\left(\beta_n \left\| \frac{\tilde{x}_n + \tilde{y}_n}{2} \right\| + \frac{\eta_n}{2}\right) \\ &\leq f\left(\beta_n - \delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \cdot \beta_n + \frac{\eta_n}{2}\right) \\ &= f(\tilde{\beta}_n). \end{aligned} \tag{2.4}$$

The convexity of f guarantees that

$$\frac{1}{2} f(\alpha_n) + \frac{1}{2} f(\beta_n) \geq f\left(\frac{\alpha_n + \beta_n}{2}\right) \geq f(\tilde{\beta}_n) + \delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n), \quad \text{for } n \geq N. \tag{2.5}$$

Hence

$$\begin{aligned} f(\tilde{\beta}_n) &\leq \frac{1}{2} f(\alpha_n) + \frac{1}{2} f(\beta_n) - \delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n) \\ &= \frac{1}{2} f(\|x_n\|) + \frac{1}{2} f(\|y_n\|) - \delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n). \end{aligned} \tag{2.6}$$

To complete the proof, it remains to verify that

$$\liminf_{n \to \infty} \delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n) > 0 \tag{2.7}$$

and as a consequence it will follow that (2.6) contradicts (2.2). Indeed, since $\tilde{\beta}_n / \beta_n \to 1$, for sufficiently large n, $\tilde{\beta}_n \geq \frac{1}{2}\beta_n$ and because $\delta_{\|\cdot\|}$ is nondecreasing on $[0, 2]$ this additionally ensures $\delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \geq \delta_{\|\cdot\|}\left(\frac{\frac{\epsilon}{4}}{\tilde{\beta}_n}\right)$ for such n. Consequently,

$$\delta_{\|\cdot\|}\left(\frac{\epsilon}{2\beta_n}\right) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n) \geq \frac{1}{2} \delta_{\|\cdot\|}\left(\frac{\frac{\epsilon}{4}}{\tilde{\beta}_n}\right) \cdot \tilde{\beta}_n \cdot f'_+(\tilde{\beta}_n) \ \text{ for sufficiently large } \ n.$$

Applying (2.1) with $\epsilon/4$ replacing ϵ to the right-hand side of the previous inequality, one deduces (2.7) as desired. $\square$

It is perhaps surprising that the previous result means that the composition of a uniformly convex norm with a nondecreasing uniformly convex function on the positive axis is a uniformly convex function if and only if (2.1) holds. Theorem 2.1 also enables us to construct continuous uniformly convex functions using any uniformly convex norm on a superreflexive Banach space.

Example 2.2. Let $\|\cdot\|$ be a uniformly convex norm with modulus $\delta_{\|\cdot\|}$. We define $f(t) := t^2$ for $0 \le t \le 1$ while

$$f(t) := t^2 + \int_1^t \frac{1}{\delta_{\|\cdot\|}(u^{-2})}\, du \quad \text{when } t > 1.$$

We may apply Theorem 2.1 to show $f \circ \|\cdot\|$ is uniformly convex.

Proof. Indeed, f' is nonnegative increasing on $[0, \infty)$ so f is convex and nondecreasing. Moreover, $t \mapsto t^2$ is uniformly convex (hence so is its sum with another convex function) and so f is uniformly convex. Finally, for $t > 1$, $f'(t) = 2t + 1/\delta_{\|\cdot\|}(t^{-2})$. For fixed ϵ when $t^{-1} < \epsilon$ we then have

$$f'_+(t) \cdot \delta_{\|\cdot\|}\left(\frac{\epsilon}{t}\right) \cdot t > \frac{1}{\delta(t^{-2})} \cdot \delta_{\|\cdot\|}\left(\frac{\epsilon}{t}\right) \cdot t > t$$

and so (2.1) holds. □

Further examples will be given after the following more qualitative result concerning moduli of power type.

Theorem 2.3. *Suppose $f : [0, +\infty) \to [0, +\infty)$ is a convex nondecreasing function and $p \ge 2$.*

(a) Suppose f and $\|\cdot\|$ have moduli of convexity of power type p and $f'_+(t) \ge Ct^{p-1}$ for some $C > 0$ and for all $t > 0$. Then $f \circ \|\cdot\|$ also has modulus of convexity of power type p.

(b) Conversely, if $f \circ \|\cdot\|$ has modulus of convexity of power type p, then f and $\|\cdot\|$ have moduli of convexity of power type p. In the case that $\|\cdot\|$ additionally satisfies

$$0 < \liminf_{\epsilon \to 0^+} \frac{\delta_{\|\cdot\|}(\epsilon)}{\epsilon^p} < \infty \tag{2.8}$$

(i.e., the modulus of $\|\cdot\|$ is no better than power type p), then

$$f'_+(t) \ge Kt^{p-1}$$

for some $K > 0$ and for all $t > 0$.

Proof. (a) First we fix positive constants A, B corresponding to the respective moduli, and let $C > 0$ be as given. That is,

$$\delta_f(\epsilon) \ge A\epsilon^p \ \text{ for all } \epsilon > 0, \quad \delta_{\|\cdot\|}(\epsilon) \ge B\epsilon^p \ \text{ for all } 0 \le \epsilon \le 2, \ \text{ and } \ f'_+(t) \ge Ct^{p-1} \ \text{ for all } t > 0.$$

Let $\epsilon > 0$ be fixed, and suppose $x, y \in X$ satisfy $\|x - y\| \ge \epsilon$. We may assume $\|y\| \le \|x\|$.

Suppose first, $\|y\| + \epsilon/2 \le \|x\|$. Using the modulus of convexity of f we obtain

$$\frac{1}{2}f(\|x\|) + \frac{1}{2}f(\|y\|) - f\left(\left\|\frac{x+y}{2}\right\|\right) \ge \frac{1}{2}f(\|x\|) + \frac{1}{2}f(\|y\|) - f\left(\frac{\|x\| + \|y\|}{2}\right) \ge A\left(\frac{\epsilon}{2}\right)^p. \tag{2.9}$$

Thus, for the remainder of the proof we will assume $\|y\| + \epsilon/2 > \|x\|$.

Let $a := \|y\|$ and $\tilde{x} := x/\|x\|$, $\tilde{y} := y/\|y\|$. Then $\|y - a\tilde{x}\| > \epsilon/2$. Consequently, $\|\tilde{y} - \tilde{x}\| > \frac{\epsilon}{2a}$. Thence the modulus of convexity implies

$$\left\|\frac{\tilde{x}+\tilde{y}}{2}\right\| \leq 1 - B\left(\frac{\epsilon}{2a}\right)^p$$

and so

$$\left\|\frac{x+y}{2}\right\| \leq a\left(\left\|\frac{\tilde{x}+\tilde{y}}{2}\right\|\right) + \frac{\|x\|-a}{2} \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\| - Ba\left(\frac{\epsilon}{2a}\right)^p. \tag{2.10}$$

(i) We consider the case, $Ba\left(\frac{\epsilon}{2a}\right)^p \geq a/2$. Recalling that $\|x\| + \|y\| \geq \|x - y\| \geq \epsilon$, we have $\|y\| \geq \epsilon/4$ since $\|y\| \geq \|x\| - \epsilon/2$. Because $a = \|y\|$, it follows that $a/2 \geq \epsilon/8$. Thus, letting $t_0 := (\|x\| + \|y\|)/2 - a/2$, we have $t_0 \geq a/2$ and the nondecreasing property of f ensures

$$f\left(\left\|\frac{x+y}{2}\right\|\right) \leq f(t_0).$$

Now we use this with the convexity of f to compute,

$$\begin{aligned}
\frac{1}{2}f(\|x\|) + \frac{1}{2}f(\|y\|) &\geq f\left(\frac{\|x\|+\|y\|}{2}\right) \geq f(t_0) + f'_+(t_0)\cdot(a/2) \\
&\geq f(t_0) + f'_+(a/2)\cdot(a/2) \geq f(t_0) + f'_+(\epsilon/8)\cdot(\epsilon/8) \\
&\geq f\left(\left\|\frac{x+y}{2}\right\|\right) + C\left(\frac{\epsilon}{8}\right)^p. \qquad (2.11)
\end{aligned}$$

(ii) For our remaining case, we suppose $Ba\left(\frac{\epsilon}{2a}\right)^p \leq a/2$. Then the right hand side of (2.10) is at least $a/2$. Now use the fact $f'(t) \geq C(a/2)^{p-1}$ when $t \geq a/2$ to compute

$$\begin{aligned}
f\left(\left\|\frac{x+y}{2}\right\|\right) &\leq f\left(\frac{1}{2}\|x\| + \frac{1}{2}\|y\|\right) - Ba\left(\frac{\epsilon}{2a}\right)^p \cdot C\left(\frac{a}{2}\right)^{p-1} \\
&\leq \frac{1}{2}f(\|x\|) + \frac{1}{2}f(\|y\|) - BC\left(\frac{\epsilon}{4}\right)^p. \qquad (2.12)
\end{aligned}$$

Putting (2.9), (2.11) and (2.12) together we see that $f \circ \|\cdot\|$ has modulus of convexity of power type p as desired.

(b) Because $f \circ \|\cdot\|$ has modulus of convexity of power type p, one need only fix $x_0 \in S_X$ and consider $f(t) = f(\|tx_0\|)$ for $t \geq 0$ to see that f has modulus of convexity of power type p.

Also, let $\beta := f'_+(1)$ and let $C > 0$ be such that $\delta_{f\circ\|\cdot\|}(\epsilon) \geq C\epsilon^p$ when $\epsilon > 0$. Fix $\epsilon \in (0, 2]$, and choose $x, y \in S_X$ with $\|x - y\| \geq \epsilon$ and $\left\|\frac{x+y}{2}\right\| \geq 1 - 2\delta_{\|\cdot\|}(\epsilon)$. Then

$$f(1) - C\epsilon^p = f\left(\frac{\|x\|+\|y\|}{2}\right) - C\epsilon^p \geq f\left(\left\|\frac{x+y}{2}\right\|\right) \geq f(1) - 2\beta\delta_{\|\cdot\|}(\epsilon)$$

and it follows $\delta_{\|\cdot\|}(\epsilon) \geq \frac{C}{2\beta}\epsilon^p$. Thus $\|\cdot\|$ has modulus of convexity of power type p as desired.

It remains to verify $f'(t) \geq Mt^{p-1}$ for some $M > 0$ and all $t > 0$ when (2.8) is valid. Indeed, in this case, we find $(u_n), (v_n) \subset S_X$ and $K > 0$ such that

$$\epsilon_n := \|u_n - v_n\| \to 0^+ \quad \text{and} \quad \left\|\frac{u_n+v_n}{2}\right\| \geq 1 - K\epsilon_n^p.$$

Now fix $t > 0$, and let $x_n := tu_n$ and $y_n := tv_n$. Then

$$\left\| \frac{x_n + y_n}{2} \right\| \geq t(1 - K\epsilon_n^p). \tag{2.13}$$

Then $\|x_n - y_n\| = t\epsilon_n$ and the modulus of convexity of $f \circ \|\cdot\|$ implies

$$f\left(\left\| \frac{x_n + y_n}{2} \right\|\right) \leq \frac{1}{2} f(\|x_n\|) + \frac{1}{2} f(\|y_n\|) - C(t\epsilon_n)^p = f(t) - Ct^p\epsilon_n^p. \tag{2.14}$$

The convexity of f implies that $f(t - tK\epsilon_n^p) \geq f(t) - f'_+(t)(tK\epsilon_n^p)$. Using this along with (2.13) and the fact f is nondecreasing, we obtain

$$f\left(\left\| \frac{x_n + y_n}{2} \right\|\right) \geq f(t - tK\epsilon_n^p) \geq f(t) - f'_+(t)(tK\epsilon_n^p). \tag{2.15}$$

Combining (2.14) and (2.15) implies $f'_+(t) \geq \frac{C}{K} t^{p-1}$, as desired. □

The following corollary recovers a result from [2] whose original proof proceeded via establishing uniform smoothness and invoking duality results from [1].

Corollary 2.4 (Theorem 2.3, [2])**.** *Let $f := \|\cdot\|^p$ where $p \geq 2$. Then the following are equivalent:*

(a) f is uniformly convex;

(b) $\|\cdot\|$ has modulus of convexity of power type p;

(c) f has modulus of convexity of power type p.

Proof. (a) $\Rightarrow$ (b): Suppose f is uniformly convex, then (2.1) holds with $\epsilon = 1$. Consequently,

$$\liminf_{t\to\infty} pt^p \ \delta_{\|\cdot\|}(t^{-1}) > 0$$

and so there exists $C > 0$ so that $p\, t^p\, \delta_{\|\cdot\|}(t^{-1}) > C$ for $t > t_0$. In particular, for $0 < \epsilon < 1/t_0$, we have $\delta_{\|\cdot\|}(\epsilon) > K\epsilon^p$ where $K := Cp^{-1}$.

(b) $\Rightarrow$ (c): Follows from Theorem 2.3 because the function $t \mapsto |t|^p$ has modulus of convexity of power type p.

(c) $\Rightarrow$ (a): is trivial. □

Example 2.5. Suppose that $b > 1$ and $\|\cdot\|$ has modulus of convexity of power type p where $p \geq 2$. Then $f := b^{\|\cdot\|}$ is uniformly convex with modulus of convexity of power type p. However, even on $\mathbb{R}^2$ there are uniformly convex norms $|\!|\!|\cdot|\!|\!|$ so that $h := b^{|\!|\!|\cdot|\!|\!|}$ is not uniformly convex.

Proof. Let $g(t) := b^t$. Then $g'(t) \geq Ct^p$ for some $C > 0$ and all $t \geq 0$, and g has modulus of convexity of power type p. According to Theorem 2.3, f has modulus of convexity of power type p. For the claim concerning h, we appeal to [7] to obtain a norm $|\!|\!|\cdot|\!|\!|$ on $\mathbb{R}^2$ so that

$$\liminf_{t\to\infty} t\, b^t \ \log(b) \delta_{|\!|\!|\cdot|\!|\!|}(t^{-1}) = 0.$$

Then (2.1) fails, and so Theorem 2.1 ensures h is not uniformly convex. □

One may view the above conditions dually. For this, let us recall the *modulus of smoothness of a norm* $\|\cdot\|$ is defined for $\tau > 0$ by

$$\rho_{\|\cdot\|}(\tau) := \sup\left\{\frac{\|x+\tau h\| + \|x-\tau h\| - 2}{2} \,:\, \|x\| = \|h\| = 1\right\}.$$

Given $1 < q \le 2$, we will say $\|\cdot\|$ has *modulus of smoothness of power type* q if there exists $C > 0$ so that $\rho_{\|\cdot\|}(\tau) \le C\tau^q$ for $\tau > 0$; see [6] for further information. Similarly, *the modulus of smoothness of a convex function* f is defined for $\tau \ge 0$ by

$$\rho_f(\tau) := \sup\left\{\frac{1}{2}f(x+\tau h) + \frac{1}{2}f(x-\tau h) - f(x) \,:\, x \in X,\ \|h\| = 1\right\};$$

as with norms, when $\rho_f(\tau) \le C\tau^q$ for some $C > 0$ and all $\tau > 0$ we will say f has *modulus of smoothness of power type* q. See [11, p. 204ff] or [3, Section 5.4] for further discussion on this and related concepts. We note also that given $h := f \circ \|\cdot\|$, then the conjugate is given by

$$h^*(\phi) = \sup_{x\in X} \phi(x) - f(\|x\|) = \sup_{x\in X} \|\phi\|\|x\| - f(\|x\|) = f^*(\|\phi\|).$$

We may now present the following dual version of Theorem 2.3.

Corollary 2.6. *Suppose $\|\cdot\|$ is uniformly smooth with modulus of smoothness of power type q where $1 < q \le 2$, f is nondecreasing and f has modulus of smoothness of power type q while $f'_+(t) \le Ct^{q-1}$, $t \ge 0$. Then $f \circ \|\cdot\|$ has modulus of smoothness of power type q.*

Conversely, suppose $f \circ \|\cdot\|$ has modulus of smoothness of power type q. Then $\|\cdot\|$ has modulus of smoothness of power type q, f has modulus of smoothness of power type q, and if the modulus of smoothness of $\|\cdot\|$ is not better than power type q, then $f'(t) \le Ct^{q-1}$.

Proof. We may assume $f(0) = 0$ and $f'(0) = 0$ (by subtracting the derivative at 0). Thus, we may further assume $f(t) = 0$ for $t \le 0$. Consequently f^* is nondecreasing, and $f^*(0) = 0$. Let $h := f \circ \|\cdot\|$ as above. According to [6, Proposition IV.1.12], the dual norm $\|\cdot\|$ has modulus of convexity of power type p. Now let $t \in \partial f^*(y)$. Then $t \ge 0$, $y \in \partial f(t)$ and so $y \le Ct^{q-1}$. Thus $t \ge Ky^{1/(q-1)}$, or equivalently $t \ge Ky^{p-1}$.

This implies $f'_+(y) \ge Ky^{p-1}$ for all $y \ge 0$. $t \ge 0$, and f^* has modulus of convexity of power type p. According to Theorem 2.3(a) h^* has modulus of convexity of power type p. By duality, see [11, Corollary 3.5.11], h has modulus of smoothness of power type q.

The details of the converse follow similarly from Theorem 2.3(b); again by invoking duality results of [11, Corollary 3.5.11] and [6, Proposition IV.1.12]. □

In conclusion, we should also mention that [2] provides renormings with moduli of convexity of power type based on growth rates of uniformly convex functions on the space. In fact, [2, Theorem 3.7] can be used as follows to illustrate the restrictiveness of obtaining functions that are simultaneously uniformly convex and uniformly smooth.

Remark 2.7. Suppose X is a Banach space and $f : X \to \mathbb{R}$ is both uniformly convex and uniformly smooth. Then X is isomorphic to a Hilbert space. Moreover, $g := \|\cdot\|^p$ is simultaneously uniformly convex and uniformly smooth if and only if $p = 2$ and $\|\cdot\|$ has modulus of smoothness and modulus of convexity both of power type 2.

Proof. Let f be as given. Then [11, Proposition 3.5.8] implies that

$$\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^2} > 0.$$

Because continuous convex functions are bounded below on bounded sets, we have $f \geq 4a\|\cdot\|^2 + b$ for some $a > 0$ and $b \in \mathbb{R}$. Thus by replacing f with $f - b$, we may assume $f \geq 4a\|\cdot\|^2$. Then $f^* \leq a\|\cdot\|^2$ and so f^* is uniformly convex [11, Theorem 3.5.12].

According to [2, Theorem 3.7], X^* admits a norm with modulus of convexity of power type 2. Arguing similarly with f^*, one can show that $f - B \leq A\|\cdot\|^2$ for some $A > 0$ and constant B. Applying [2, Theorem 3.7] shows that X admits a norm with modulus of convexity of power type 2. It follows from [6, Propositions IV.1.12, IV.5.10, IV.5.12] that X has type and cotype 2 and so X is isomorphic to a Hilbert space by Kwapien's theorem [8].

For the 'moreover' assertion, we note that the 'only if' claim follows from Corollaries 2.4 and 2.6. For the 'if' assertion, as in the previous paragraph, the duality results just cited imply that f and f^* are both uniformly convex and hence [11, Proposition 3.5.8] implies both that $p \geq 2$ and that its conjugate index $q \geq 2$; consequently, $p = 2$ as claimed. $\square$

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