# Seeing Things by Walking on Real Numbers

**Jonathan Borwein** FRSC FAAS FAA FBAS
(Joint work with Francisco Aragón, David Bailey and Peter Borwein)

School of Mathematical & Physical Sciences
The University of Newcastle, Australia

http://carma.newcastle.edu.au/meetings/evims/

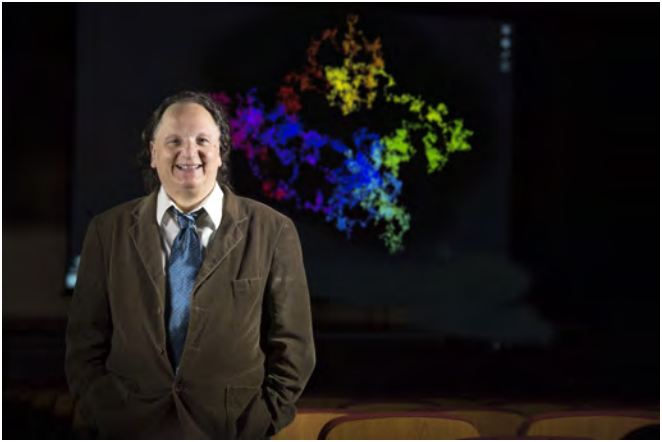
For 2014 Presentations

Revised 10-04-2014

# Contents:

One message is "Try drawing numbers"

# Me and my collaborators



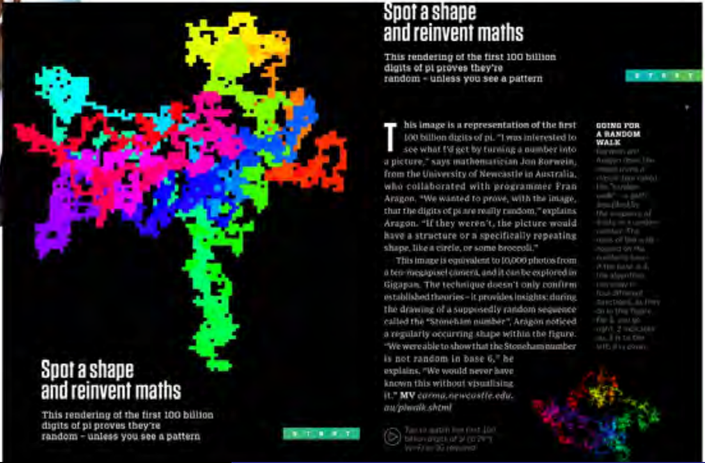
**MAA 3.14**

http://www.carma.newcastle.edu.au/jon/pi-monthly.pdf

# My collaborators

# Outreach: images and animations led to high-level research which went viral


Wired UK August 2013



## Spot a shape and reinvent maths

This rendering of the first 100 billion digits of pi proves they're random – unless you see a pattern

This image is a representation of the first 100 billion digits of pi. "I was interested to see what I'd get by turning a number into a picture," says mathematician Jon Borwein, from the University of Newcastle in Australia, who collaborated with programmer Fran Aragon. "We wanted to prove, with the image, that the digits of pi are really random," explains Aragon. "If they weren't, the picture would have a structure or a specifically repeating shape, like a circle, or some broccoli."

This image is equivalent to 10,000 photos from a ten-megapixel camera, and it can be explored in Gigapan. The technique doesn't only confirm established theories – it provides insights: during the drawing of a supposedly random sequence called the "Stoneham number", Aragon noticed a regularly occurring shape within the figure. "We were able to show that the Stoneham number is not random in base 6," he explains. "We would never have known this without visualising it." **MV** *carma.newcastle.edu.au/piwalk.shtml*

**GOING FOR A RANDOM WALK**

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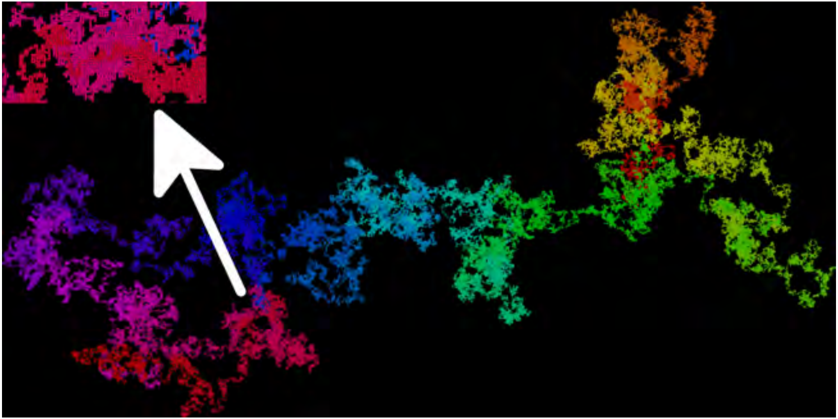

- 100 billion base four digits of $\pi$ on Gigapan
- Really big pictures are often better than movies

# Contents

# Dedication: To my father and colleague David Borwein (1924 –)

The irmacs Centre

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upcoming events

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- SFU Research Chairs Series
- SFU Research Masterclass
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- Colloquiums
- Meetings
- Conferences & Workshops
- Outreach
- 9th Annual IRMACS Day
- Workshop - David Borwein

past events

- All Past Events
- Videos - Scientific Presentations
- Coast to Coast Seminar Series
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- SFU Masterclass Seminar Series
- SFU Research Chairs Seminar Series
- Colloquiums
- Conferences & Workshops
- CRC Seminar Series
- IRMACS: The Interdisciplinary Colloquium

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## Workshop - David Borwein at 90

### Workshop in Honour of David Borwein at 90

### April 16, 2014

### The IRMACS Centre, SFU, Burnaby, BC

Having authored more than 130 publications in a span of 53 years, David Borwein is one of the most significant contributors to the development of Canadian mathematics in the second part of the 20th century. A direct descendant of G.H. Hardy, David is known for his research in the summability theory of series and integrals, measure theory and probability theory, and in number theory. He has also published on generalized subgradients and coderivatives, and on the remarkable properties of single- and many-variable sinc integrals. Even at the age of 90, David Borwein remains actively involved in research.

David Borwein served as president of the Canadian Mathematical Society (CMS) from 1985 - 1987. To the wider Canadian mathematical community David Borwein is known as the eponym of the CMS Distinguished Career Award.

**Schedule:**

| | |
|---|---|
| 1:30 - 1:35 | Opening Remarks and Welcome by Peter Borwein |
| 1:35 - 2:20 | *"Seeing things in mathematics by walking on real numbers"*, Jon Borwein, University of Newcastle |
| 2:30 - 3:45 | *"Legendre Polynomials and Legendre-Stirling Numbers"*, Lance Littlejohn, Baylor University |
| 3:15 - 3:30 | Coffee Break |
| 3:30 - 4:15 | *"From Hilbert Matrices to Fourier Bootstrapping"*, Gordon Simmons, University of Western Ontario |
| 4:15 - 5:00 | *"Nonlinear recurrences related to Chebyshev polynomials"*, Karl Dilcher, Dalhousie University |
| 5:00 - 5:15 | *"The j-invariant of an elliptic curve associated to an imaginary quadratic field"*, Joshua Nevin, University of British Columbia |
| 5:15 – 5:25 | Launching the online archive of David's papers |

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  - Chief scientist for *NeXT*
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- Developer of the *Pixar* compression format
  - and the *iPod shuffle*

http://en.wikipedia.org/wiki/Richard_Crandall

# Some early conclusions:

So I am sure they get made

**Key ideas**: randomness, normality of numbers, planar walks, and fractals


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DHB and JMB, **Exploratory Experimentation in Mathematics** (2011), www.ams.org/notices/201110/rtx111001410p.pdf

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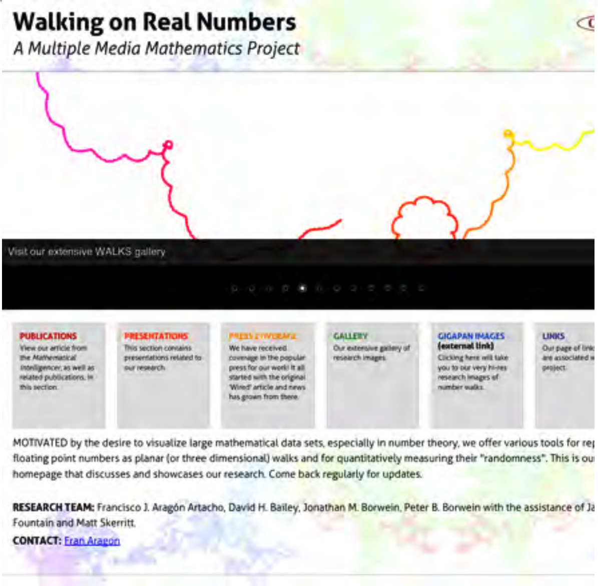
- In an **1808** letter to his friend Farkas (father of Janos Bolyai)
- Archimedes, Euler, Gauss are the big three





Almost all I mention is at `http://carma.newcastle.edu.au/walks/`

# A surprising fan?

26-07-2013

*He [David Attenborough] described current pop music as "hugely sexual and even lets slip that if he were not one of the world's most famous broadcasters, he would like to try his hand at academia. "I wish I was a mathematician, he said. "I know a mathematician would talk about the beauty of an equation. And you can sense that when you hear a five-part fugue by Bach, which also has a mathematical beauty.*

www.independent.co.uk/arts-entertainment/tv/features/

when-bjrk-met-attenborough-the-icelandic-punk-the-national-treasure-and-a-display-of-rather-remarkable-human-

html

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# We shall explore things like: How random is Pi?

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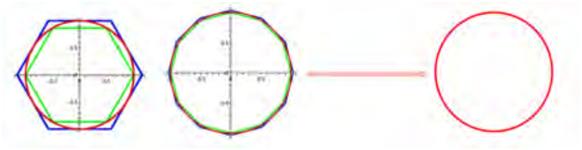

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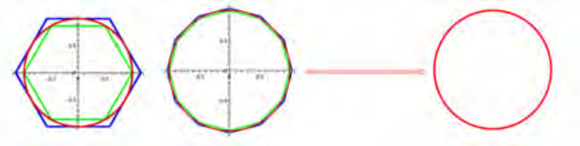

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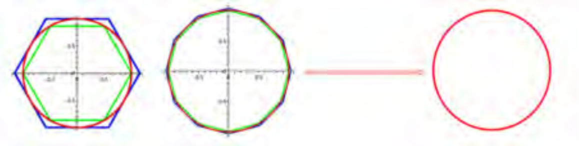
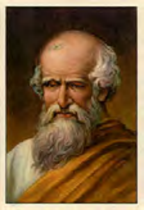


$\mathbf{6} \mapsto \mathbf{12} \mapsto 24 \mapsto 48 \mapsto \mathbf{96}$ to obtain the estimate

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

# Where Greece was: Magna Graecia



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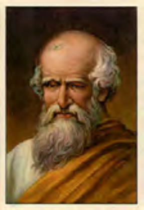

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2. Troy
3. Byzantium
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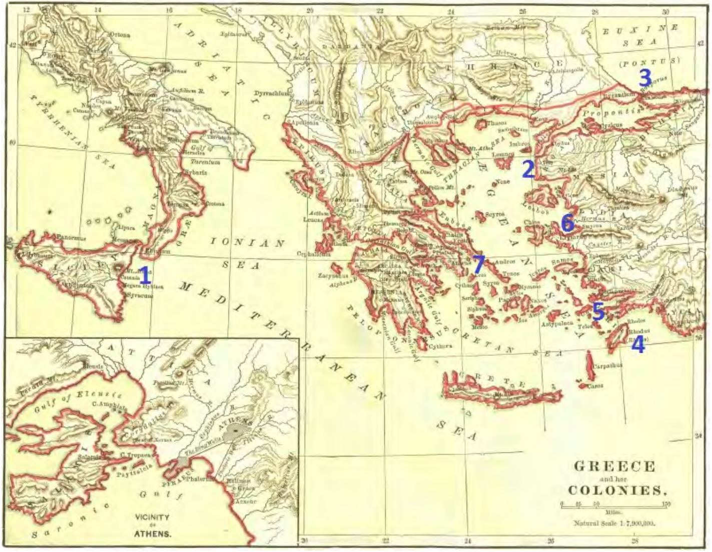
4. Rhodes
   (Helios)
5. Hallicarnassus
   (Mausolus)
6. Ephesus
   (Artemis)
7. Athens (Zeus)



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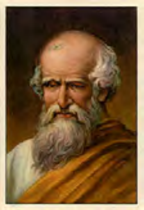

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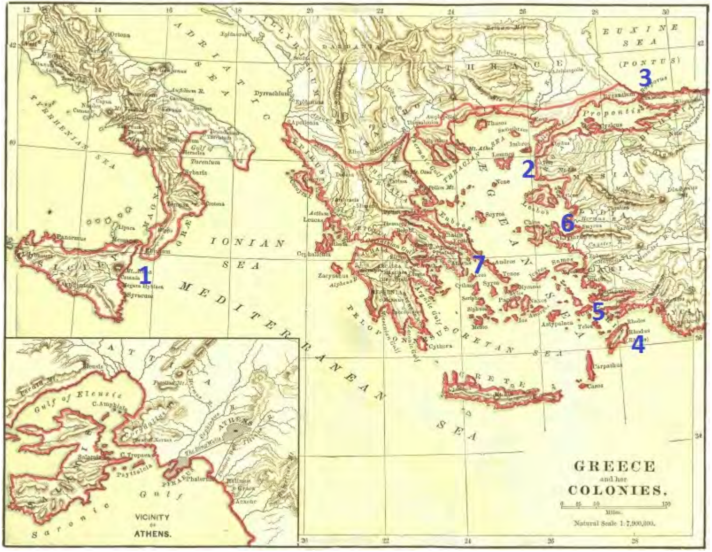


The others of the Seven Wonders of the Ancient World: Lighthouse of Alexandria, Pyramids of Giza, Gardens of Babylon

# Randomness

- The digits expansions of $\pi, e, \sqrt{2}$ appear to be “random”:

$$\pi = 3.14159265358979323846264338327950288419716939937 5\ldots$$

$$e = 2.7182818284590452353602874713526624977572470936 99\ldots$$

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**Are they really?**

- **1949 ENIAC** (*Electronic Numerical Integrator and Calculator*) computed of $\pi$ to **2,037** decimals (in **70** hours)—proposed by polymath John von Neumann (**1903-1957**) to shed light on distribution of $\pi$ (and of $e$).

# Two continued fractions

Change representations often

**Gauss map.** Remove the integer, invert the fraction and repeat: for 3.1415926 and 2.7182818 to get the fractions below.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}$$

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$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

**Leonhard Euler** (**1707-1783**) named $e$ and $\pi$.

"Lisez Euler, lisez Euler, c'est notre maître à tous." Simon Laplace (**1749-1827**)

# Are the digits of $\pi$ random?

| Digit | Ocurrences |
|---|---|
| 0 | 99,993,942 |
| 1 | 99,997,334 |
| 2 | 100,002,410 |
| 3 | 99,986,911 |
| 4 | **100,011**,958 |
| 5 | **99,998**,885 |
| 6 | 100,010,387 |
| 7 | 99,996,061 |
| 8 | 100,001,839 |
| 9 | 100,000,273 |
| Total | **1,000,000,000** |

Table : Counts of first billion digits of $\pi$. Second half is 'right' for law of large numbers.

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- There are equally many zeroes and ones in the binary expansion of $\pi$
- Or pretty much anything else...

# What is "random"?

## A hard question

# What is “random”? A hard question


It might be:

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- Without structure (noise)?
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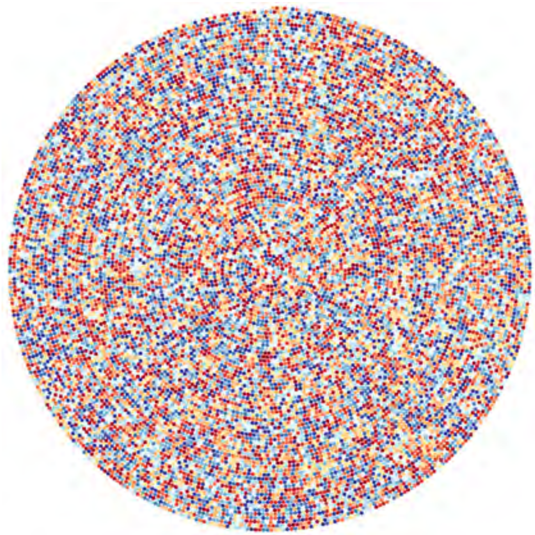
**Conjecture** (Borel) All irrational algebraic numbers are $b$-normal

**Best Theorem** [BBCP, 04] (Feeble but hard) Asymptotically all degree $d$ algebraics have at least $n^{1/d}$ ones in binary (should be $n/2$)

# Randomness in Pi?

http://mkweb.bcgsc.ca/pi/art/



# Normality

A property random numbers must possess

## Definition

A real constant $\alpha$ is $b$-normal if, given the positive integer $b \geq 2$ (the base), every $m$-long string of base-$b$ digits appears in the base-$b$ expansion of $\alpha$ with precisely the expected limiting frequency $1/b^m$.

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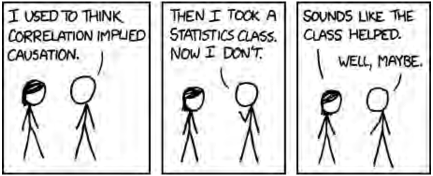
- Given an integer $b \geq 2$, almost all real numbers, with probability one, are $b$-normal (Borel).



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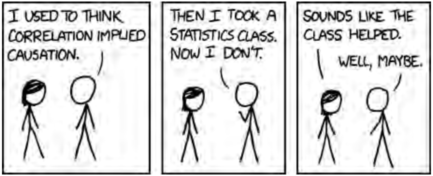

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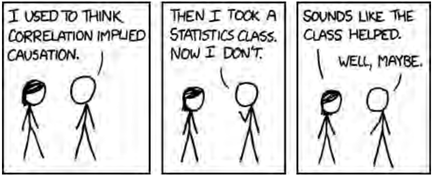
- Indeed, almost all real numbers are $b$-normal simultaneously for all positive integer bases ("absolute normality").
- Unfortunately, it has been very difficult to prove normality for any number in a given base $b$, much less all bases simultaneously.



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- Normality proofs are not known for $\pi, e, \log 2, \sqrt{2}$ etc.

# Contents

# Is $\pi$ 10-normal?

| String | Occurrences | String | Occurrences | String | Occurrences |
|---|---|---|---|---|---|
| 0 | 99,993,942 | 00 | 10,004,524 | 000 | 1,000,897 |
| 1 | 99,997,334 | 01 | 9,998,250 | 001 | 1,000,758 |
| 2 | 100,002,410 | 02 | 9,999,222 | 002 | 1,000,447 |
| 3 | 99,986,911 | 03 | 10,000,290 | 003 | 1,001,566 |
| 4 | 100,011,958 | 04 | 10,000,613 | 004 | 1,000,741 |
| 5 | **99,998,885** | 05 | **10,002,048** | 005 | **1,002,881** |
| 6 | 100,010,387 | 06 | 9,995,451 | 006 | 999,294 |
| 7 | 99,996,061 | 07 | 9,993,703 | 007 | 998,919 |
| 8 | 100,001,839 | 08 | 10,000,565 | 008 | 999,962 |
| 9 | 100,000,273 | 09 | 9,999,276 | 009 | 999,059 |
| | | 10 | 9,997,289 | 010 | 998,884 |
| | | 11 | 9,997,964 | 011 | 1,001,188 |
| | | ⋮ | ⋮ | ⋮ | ⋮ |
| | | 99 | 10,003,709 | 099 | 999,201 |
| | | | | ⋮ | ⋮ |
| | | | | 999 | 1,000,905 |
| **TOTAL** | 1,000,000,000 | **TOTAL** | 1,000,000,000 | **TOTAL** | 1,000,000,000 |

Table : Counts for the first billion digits of $\pi$.

# Is $\pi$ 16-normal

That is, in Hex?

$\hookleftarrow$ Counts of first trillion hex digits

| | |
|---|---|
| 0 | 62499881108 |
| 1 | 62500212206 |
| 2 | 62499924780 |
| 3 | 62500188844 |
| 4 | 62499807368 |
| 5 | 62500007205 |
| 6 | 62499925426 |
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- They are $353CB3F7F0C9ACCFA9AA215F2$

See www.karrels.org/pi/index.html

# Modern $\pi$ Calculation Records: and IBM Blue Gene/L at LBL

| Name | Year | Correct Digits |
|---|---|---|
| Miyoshi and Kanada | 1981 | 2,000,036 |
| Kanada-Yoshino-Tamura | 1982 | 16,777,206 |
| Gosper | 1985 | 17,526,200 |
| Bailey | Jan. 1986 | 29,360,111 |
| Kanada and Tamura | Sep. 1986 | 33,554,414 |
| Kanada and Tamura | Oct. 1986 | 67,108,839 |
| Kanada et. al | Jan. 1987 | 134,217,700 |
| Kanada and Tamura | Jan. 1988 | 201,326,551 |
| Chudnovskys | May 1989 | 480,000,000 |
| Kanada and Tamura | Jul. 1989 | 536,870,898 |
| Kanada and Tamura | Nov. 1989 | 1,073,741,799 |
| Chudnovskys | Aug. 1991 | 2,260,000,000 |
| Chudnovskys | May 1994 | 4,044,000,000 |
| Kanada and Takahashi | Oct. 1995 | 6,442,450,938 |
| Kanada and Takahashi | Jul. 1997 | 51,539,600,000 |
| Kanada and Takahashi | Sep. 1999 | 206,158,430,000 |
| Kanada-Ushiro-Kuroda | Dec. 2002 | 1,241,100,000,000 |
| Takahashi | Jan. 2009 | 1,649,000,000,000 |
| Takahashi | April 2009 | 2,576,980,377,524 |
| Bellard | Dec. 2009 | 2,699,999,990,000 |
| Kondo and Yee | Aug. 2010 | **5,000,000,000,000** |
| Kondo and Yee | Oct. 2011 | **10,000,000,000,000** |
| Kondo and Yee | Dec. 2013 | **12,100,000,000,000** |

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This is based on the following then new formula for $\pi$:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{1}$$

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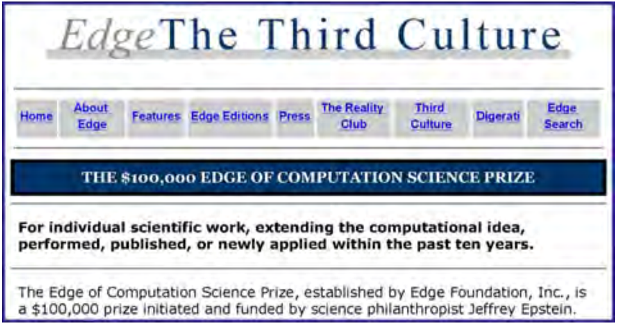
$$\pi = 4\,{}_2F_1\left(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}\right) + 2\tan^{-1}\left(\frac{1}{2}\right) - \log 5$$

where ${}_2F_1(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gaussian hypergeometric function.

# Edge of Computation Prize Finalist



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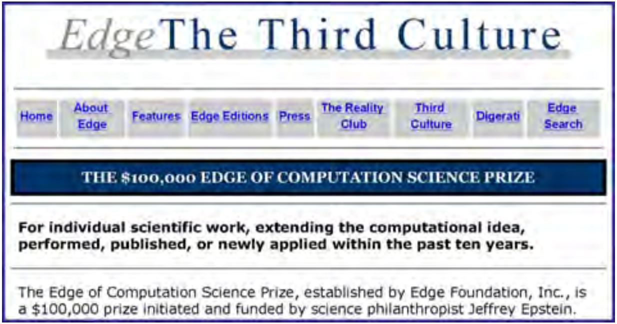


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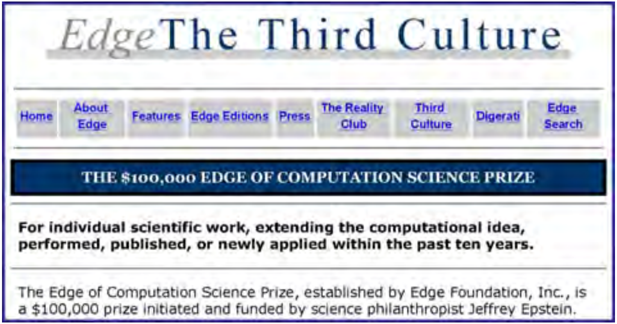


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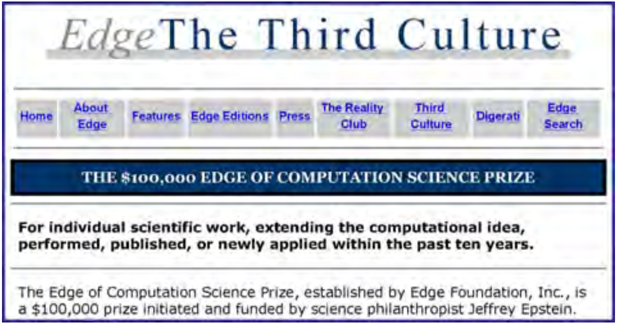


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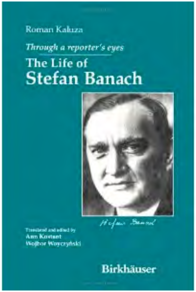
- Won by David Deutsch — discoverer of Quantum Computing.

# Stefan Banach (1892-1945) Another Nazi casuality

*A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories.* [1]



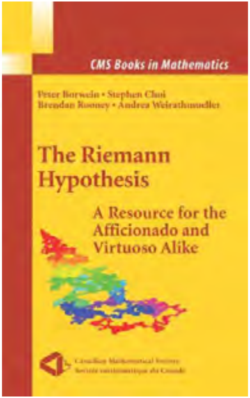


[1] Only the best get stamps. Quoted in www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html .

# Contents

# One 1500-step ramble: a familiar picture

Liouville function



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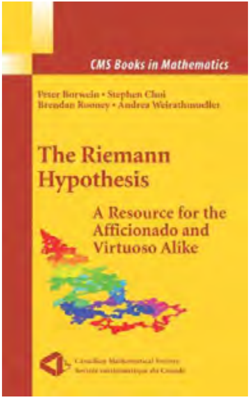


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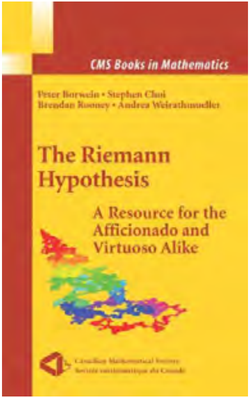
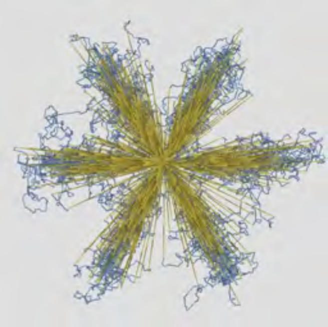



- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).
- 1D or 2D *lattice*: probability one of returning to the origin.

# 1000 three-step rambles: a less familiar picture?

# Art meets science

AAAS & Bridges conference



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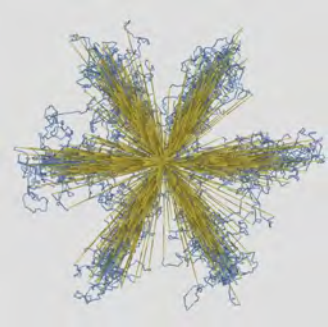

*A visualization of six routes that 1000 ants took after leaving their nest in search of food. The jagged blue lines represent the breaking off of random ants in search of seeds.*

(Nadia Whitehead 2014-03-25 16:15)

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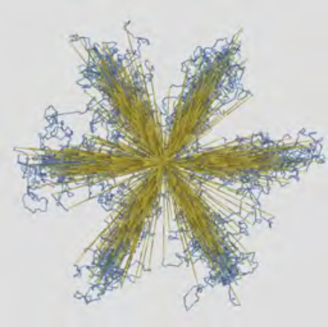

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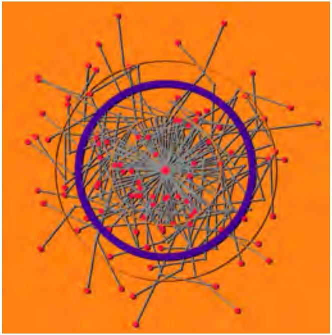
(Nadia Whitehead 2014-03-25 16:15)

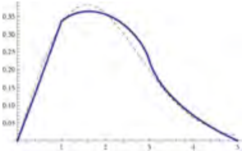
(JonFest 2011 Logo) *Three-step random walks. The (purple) expected distance travelled is 1.57459 ...*

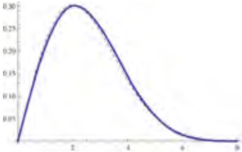
*The closed form $W_3$ is given below.*



$$W_3 = \frac{16\sqrt[3]{4}\,\pi^2}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4}\,\pi^4}$$

# A Little History: From a vast literature





L: Pearson posed question about a 'rambler' taking unit random steps (*Nature*, '05).

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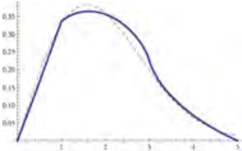

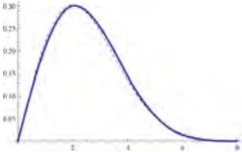

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**John William Strutt** (Lord Rayleigh) **(1842-1919)**: discoverer of Argon, explained why sky is blue.

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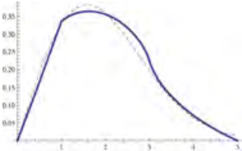

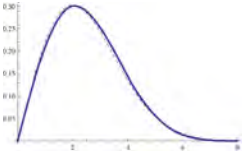

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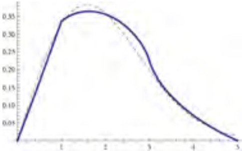
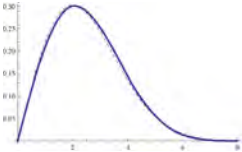


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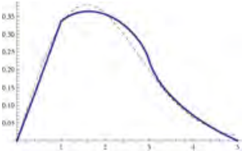
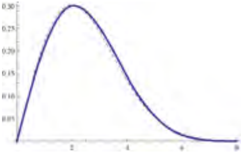


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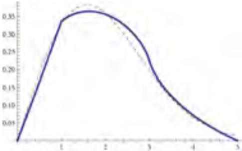
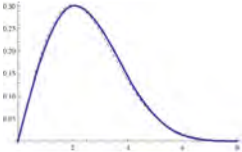


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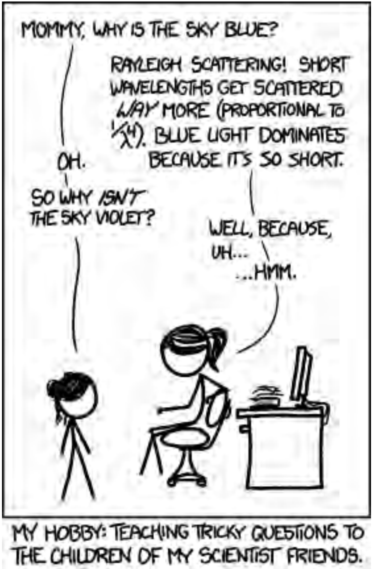
- appear in graph theory, quantum chemistry, in quantum physics as hexagonal and diamond lattice integers, etc ...

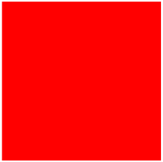
# Why is the sky blue?



# Contents

# What is a (base four) random walk ?

Pick a random number in $\{0,1,2,3\}$ and move according to $0 = \rightarrow$, $1 = \uparrow$, $2 = \leftarrow$, $3 = \downarrow$



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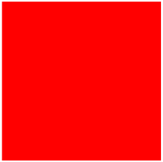

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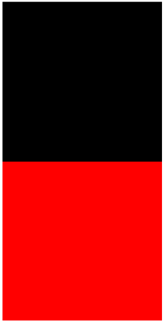

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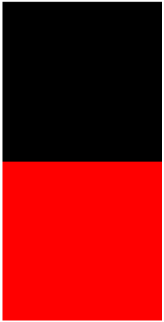

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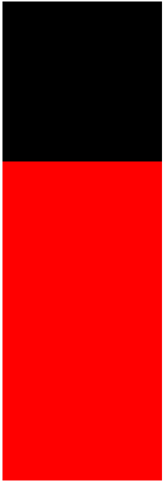

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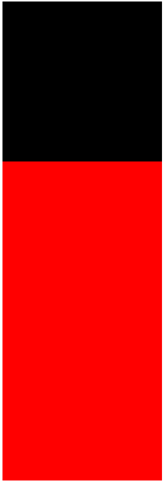

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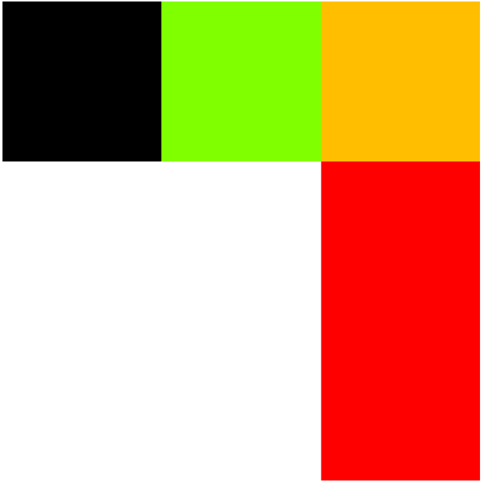

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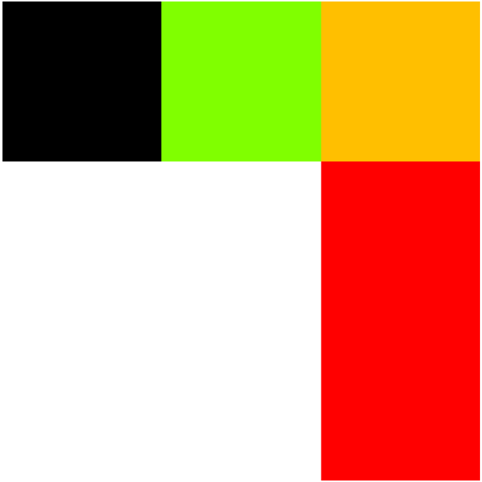

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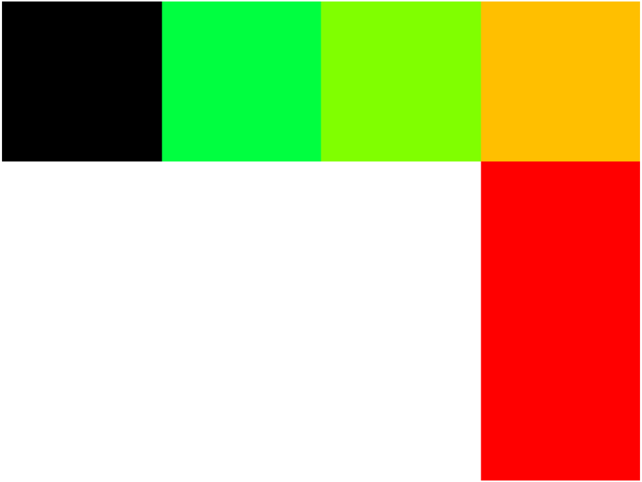

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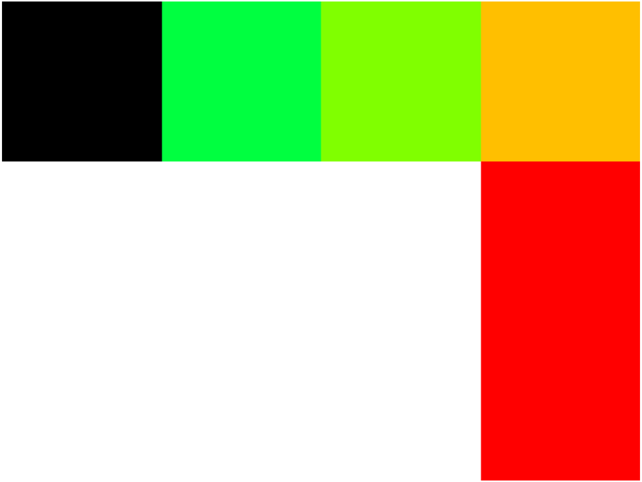

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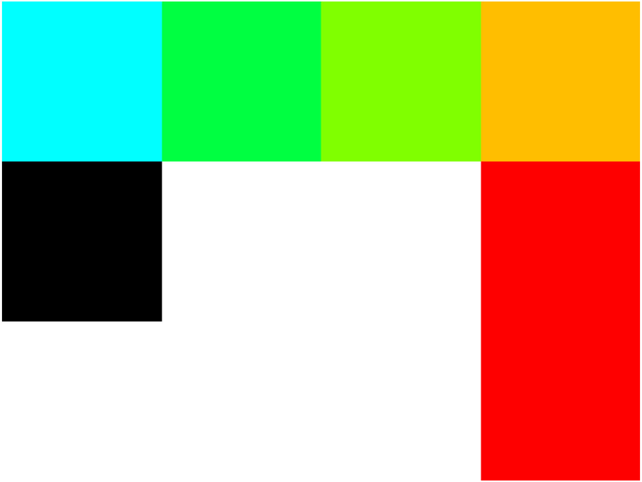

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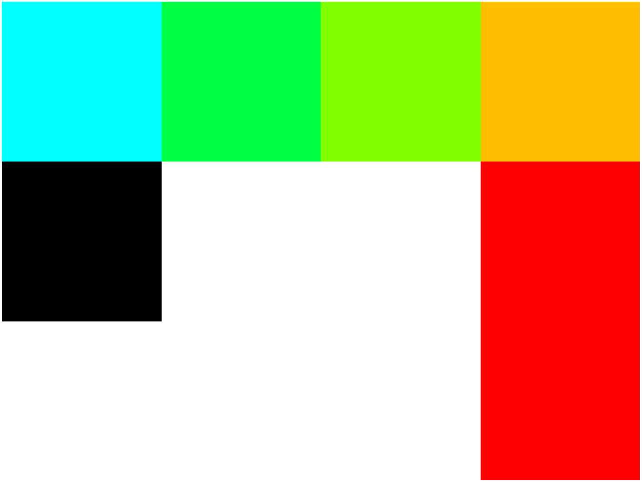

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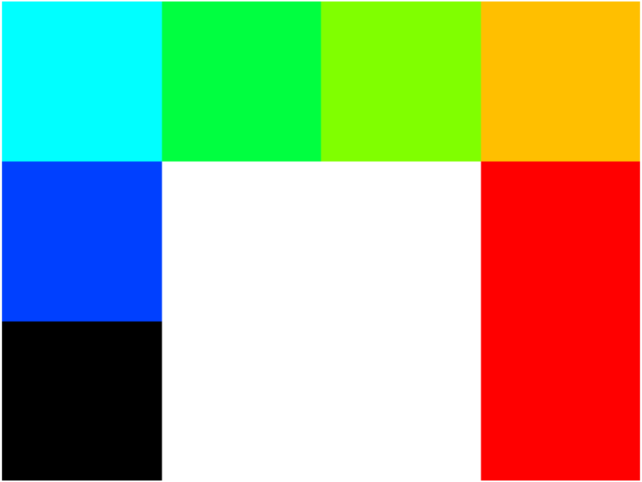

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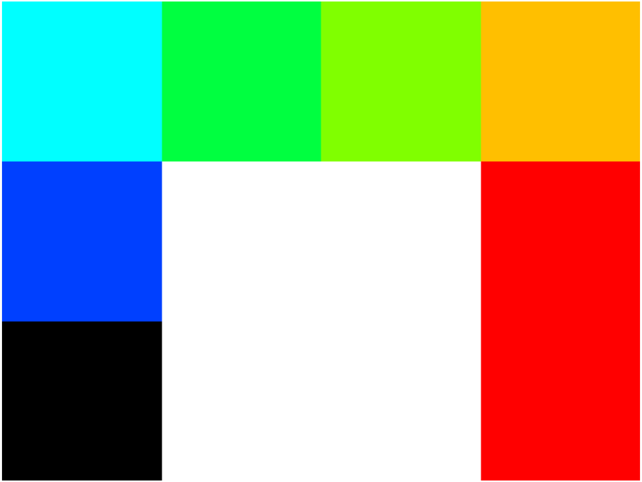

$$0 = \rightarrow$$

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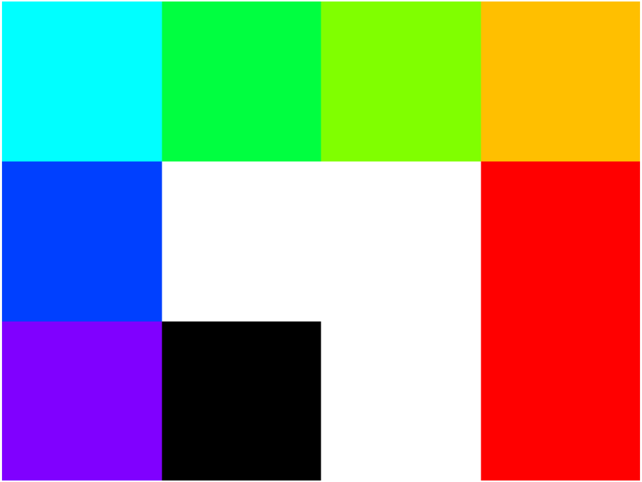

11222330

# What is a random walk (base 4)?

Pick a random number in $\{0,1,2,3\}$ and move $0 = \rightarrow$, $1 = \uparrow$, $2 = \leftarrow$, $3 = \downarrow$ ANIMATION

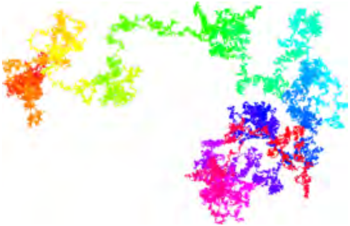

Figure : A million step base-4 pseudorandom walk. We use the spectrum to show when we visited each point (ROYGBIV and R).

# Random walks look similarish

Chaos theory (order in disorder)

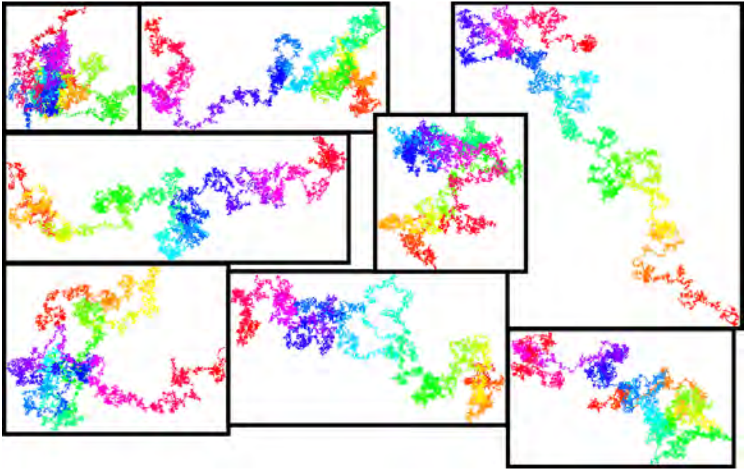

Figure : Eight different base-4 (pseudo)random[2] walks of one million steps.

[2] Python uses the *Mersenne Twister* as the core generator. It has a period of $2^{19937} - 1 \approx 10^{6002}$.

# Base-$b$ random walks: Our direction choice

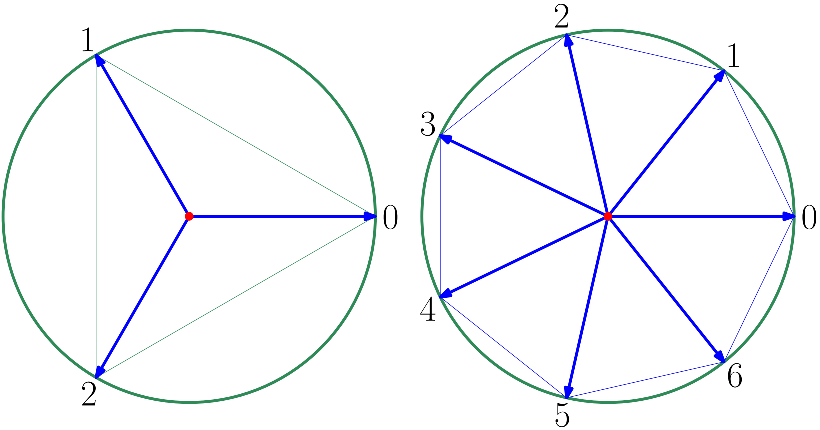


Figure : Directions for base-3 and base-7 random walks.

We are all base-four numbers (**AGCT**/U)

# Contents

# Two rational numbers

ANIMATION

The base-4 digit expansion of $Q1$ and $Q2$:

```
Q1=
0.221221012232121200122101223121001222100011232123121000122210001222
1000122210001222100001222100012220110301012201001201031103333333333
3333333333333333011111111111111111111111111100100000000300300320032
0032003022300032220300032223000302222030003222300032223000322230 0032
2232000002322230003222300322213300233212330232132321121121212223 23233
3330300000100032300323003220303203011033301110330110310111101133 2333
3232322321221211211121122322222122...
```

```
Q2=
0.221221012232121200122101223121001222100011232123121000122210001222
1000122210001222100001222100012220110301012201001201031103333333333
3333333333333333011111111111111111111111111100100000000300300320032
0032003022300032220300032223000302222030003222300032223000322230 0032
2232000002322230003222300322213300233212330232132321121121212223 23233
3330300000100032300323003220303203011033301110330110310111101100 0000
000000...
```

# Two rational numbers

ANIMATION

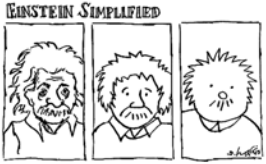


Figure : Self-referent walks on the rational numbers $Q1$ (top) and $Q2$ (bottom).

# Two more rationals

Hard to tell from their decimal expansions

The following relatively small rational numbers [G. Marsaglia, 2010]

$$Q3 = \frac{3624360069}{7000000001} \quad \text{and} \quad Q4 = \frac{123456789012}{10000000000061},$$

have base-10 periods with huge length of **1,750,000,000** digits and **1,000,000,000,060** digits, respectively.

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(a) $Q3$ (b) $Q4$

Figure : Walks on the first million base-10 digits of the rationals $Q3$ and $Q4$.

# Walks on the digits of numbers

ANIMATION

Figure : A walk on the first 10 million base-4 digits of $\pi$.

# Walks on the digits of numbers

Coloured by hits (more pink is more hits)

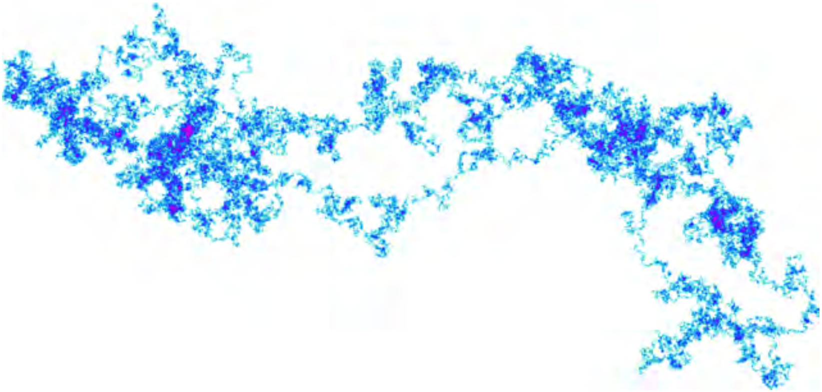

Figure : 100 million base-4 digits of $\pi$ coloured by number of returns to points.

# Contents

# The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

**1973** Richard Stoneham proved some of the following (nearly 'natural') constants are $b$-normal for relatively prime integers $b, c$:

$$\alpha_{b,c} := \frac{1}{cb^c} + \frac{1}{c^2 b^{c^2}} + \frac{1}{c^3 b^{c^3}} + \ldots.$$

Such super-geometric sums are Stoneham constants. To 10 places

$$\alpha_{2,3} = \frac{1}{24} + \frac{1}{3608} + \frac{1}{3623878656} + \ldots$$

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- Since $3 < 2^{3-1} = 4$, $\alpha_{2,3}$ is 2-normal and 6-nonnormal !

# The Stoneham numbers

$\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

Figure : $\alpha_{2,3}$ is 2-normal (top) but 6-nonnormal (bottom). Is seeing believing?

# The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

Figure : Is $\alpha_{2,3}$ 3-normal or not? Is it strongly 2-normal?

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# The expected distance to the origin

$\frac{\sqrt{\pi N}}{2\,d_N} \to 1$

**Theorem**

The expected distance $d_N$ to the origin of a base-$b$ *random walk* of $N$ steps behaves like to $\sqrt{\pi N}/2$.

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## Theorem

The expected distance $d_N$ to the origin of a base-$b$ *random walk* of $N$ steps behaves like to $\sqrt{\pi N}/2$.

| Number | Base | Steps | Average normalized dist. to the origin: $\frac{1}{\text{Steps}}\sum_{N=2}^{\text{Steps}} \frac{\text{dist}_N}{\frac{\sqrt{\pi N}}{2}}$ | Normal |
|---|---|---|---|---|
| Mean of 10,000 random walks | 4 | 1,000,000 | **1.00315** | Yes |
| Mean of 10,000 walks on the digits of $\pi$ | 4 | 1,000,000 | **1.00083** | ? |
| $\alpha_{2,3}$ | 3 | 1,000,000 | 0.89275 | ? |
| $\alpha_{2,3}$ | 4 | 1,000,000 | 0.25901 | Yes |
| $\alpha_{2,3}$ | 6 | 1,000,000 | 108.02218 | No |
| $\pi$ | 4 | 1,000,000 | 0.84366 | ? |
| $\pi$ | 6 | 1,000,000 | 0.96458 | ? |
| $\pi$ | 10 | 1,000,000 | 0.82167 | ? |
| $\pi$ | 10 | 1,000,000,000 | 0.59824 | ? |
| $\sqrt{2}$ | 4 | 1,000,000 | 0.72260 | ? |
| Champernowne $C_{10}$ | 10 | 1,000,000 | 59.91143 | Yes |

# Contents

# Number of points visited

For a 2D lattice

- The expected number of distinct points visited by an $N$-step random walk on a two-dimensional lattice behaves for large $N$ like $\pi N/\log(N)$ (Dvoretzky–Erdős, **1951**).

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- **1988** D. Downham and S. Fotopoulos gave better bounds on the expectation. It lies in:

$$\left(\frac{\pi(N+0.84)}{1.16\pi - 1 - \log 2 + \log(N+2)}, \frac{\pi(N+1)}{1.066\pi - 1 - \log 2 + \log(N+1)}\right).$$

- For example, for $N = 10^6$ these bounds are $(199256.1, 203059.5)$, while $\pi N/\log(N) = 227396$, which overestimates the expectation.

# Catalan's constant

$G = 1 + 1/4 + 1/9 + 1/16 + \cdots$

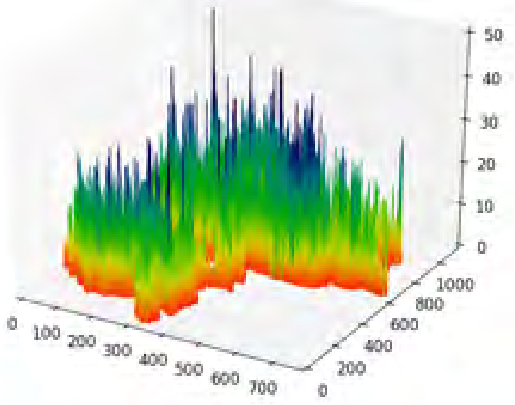
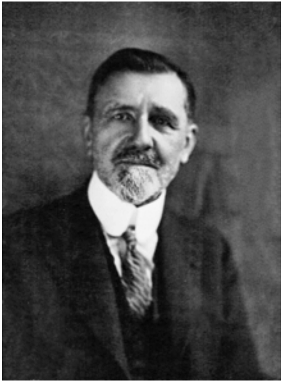



Figure : A walk on one million quad-bits of $G$ with height showing frequency

# Paul Erdős (1913-1996) "My brain is open"

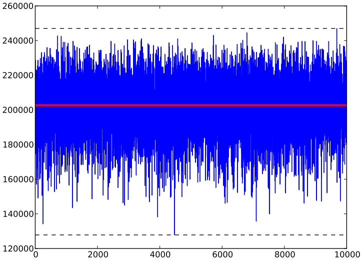
(a) Paul Erdős (Banff 1981. I was there)



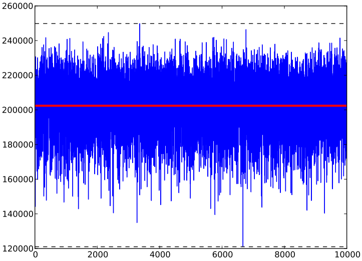
(b) Émile Borel (1871–1956)

Figure : Two of my favourites. Consult MacTutor.

# Number of points visited: Again $\pi$ looks random




(a) (Pseudo)random walks.




(b) Walks built by chopping up 10 billion digits of $\pi$.

Figure : Number of points visited by $10,000$ million-steps base-4 walks.

# Points visited by various base-4 walks

| Number | Steps | Sites visited | Bounds on the expectation of sites visited by a random walk | |
|---|---|---|---|---|
| | | | Lower bound | Upper bound |
| Mean of 10,000 random walks | 1,000,000 | **202,684** | 199,256 | 203,060 |
| Mean of 10,000 walks on the digits of $\pi$ | 1,000,000 | **202,385** | 199,256 | 203,060 |
| $\alpha_{2,3}$ | 1,000,000 | 95,817 | 199,256 | 203,060 |
| $\alpha_{3,2}$ | 1,000,000 | 195,585 | 199,256 | 203,060 |
| $\pi$ | 1,000,000 | **204,148** | 199,256 | 203,060 |
| $\pi$ | 10,000,000 | 1,933,903 | 1,738,645 | 1,767,533 |
| $\pi$ | 100,000,000 | 16,109,429 | 15,421,296 | 15,648,132 |
| $\pi$ | 1,000,000,000 | 138,107,050 | 138,552,612 | 140,380,926 |
| $e$ | 1,000,000 | 176,350 | 199,256 | 203,060 |
| $\sqrt{2}$ | 1,000,000 | **200,733** | 199,256 | 203,060 |
| $\log 2$ | 1,000,000 | **214,508** | 199,256 | 203,060 |
| Champernowne $C_4$ | 1,000,000 | 548,746 | 199,256 | 203,060 |
| Rational number $Q_1$ | 1,000,000 | 378 | 199,256 | 203,060 |
| Rational number $Q_2$ | 1,000,000 | 939,322 | 199,256 | 203,060 |

# Normal numbers need not be so “random” ...

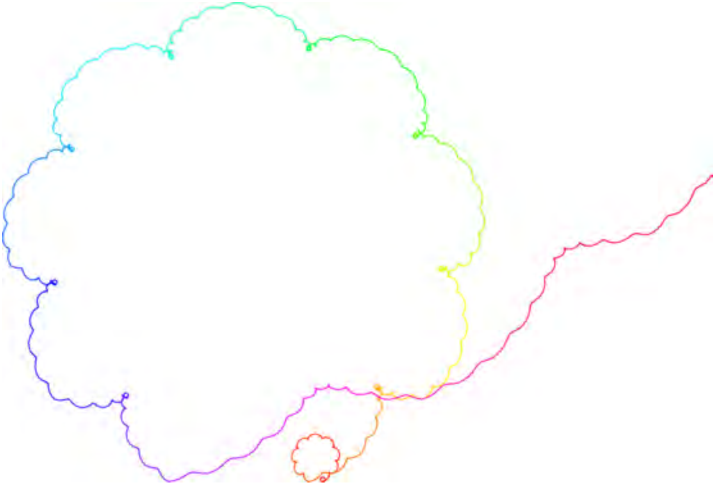


Figure : Champernowne $C_{10} = 0.123456789101112\ldots$ (normal).
Normalized distance to the origin: 15.9 (50,000 steps).

# Normal numbers need not be so "random" ...

Figure : Champernowne $C_4 = 0.123101112132021\ldots$ (normal).
Normalized distance to the origin: 18.1 (100,000 steps).
Points visited: 52760. Expectation: (23333, 23857).

# Normal numbers need not be so "random" ...

Figure : Stoneham $\alpha_{2,3} = 0.0022232032\ldots_4$ (normal base 4).
Normalized distance to the origin: 0.26 (1,000,000 steps).
Points visited: 95817. Expectation: (199256, 203060).

# Normal numbers need not be so "random" ...

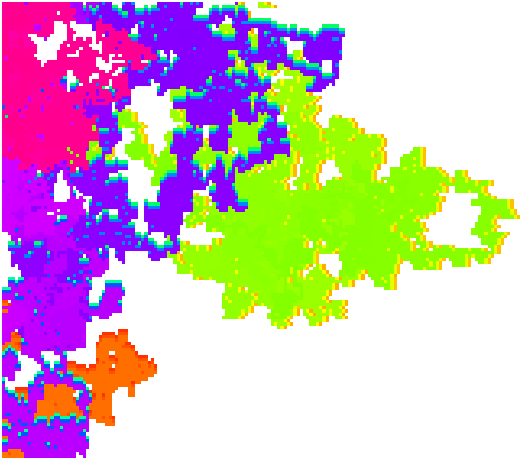


Figure : Stoneham $\alpha_{2,3} = 0.0022232032\ldots_4$ (normal base 4).
Normalized distance to the origin: 0.26 (1,000,000 steps).
Points visited: 95817. Expectation: (199256, 203060).

# $\alpha_{2,3}$ is 4-normal but not so “random” ANIMATION

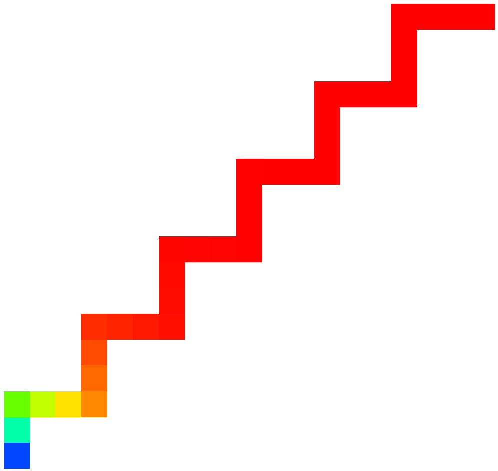


Figure : A pattern in the digits of $\alpha_{2,3}$ base 4. We show only positions of the walk after $\frac{3}{2}(3^n+1), \frac{3}{2}(3^n+1)+3^n$ and $\frac{3}{2}(3^n+1)+2\cdot 3^n$ steps, $n=0,1,\ldots,11$.

# *Experimental* conjecture

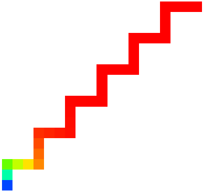
Proven 12-12-12 by Coons

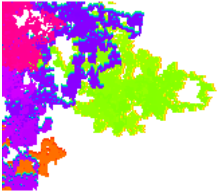
**Theorem (Base-4 structure of Stoneham $\alpha_{2,3}$)**

Denote by $a_k$ the $k^{\text{th}}$ digit of $\alpha_{2,3}$ in its base 4 expansion: $\alpha_{2,3} = \sum_{k=1}^{\infty} a_k/4^k$, with $a_k \in \{0,1,2,3\}$ for all $k$. Then, for all $n = 0,1,2,\ldots$ one has:

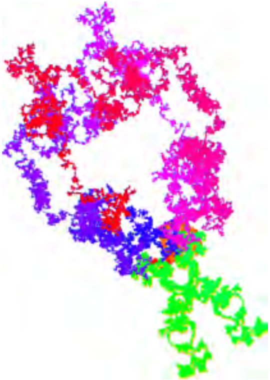
(i) $$\sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n} e^{a_k \pi i/2} = \begin{cases} -i, & \text{n odd} \\ -1, & \text{n even} \end{cases};$$

(ii) $a_k = a_{k+3^n} = a_{k+2\cdot 3^n}$ if $k = \frac{3(3^n+1)}{2}, \frac{3(3^n+1)}{2}+1, \ldots, \frac{3(3^n+1)}{2}+3^n-1$.





# Likewise, $\alpha_{3,5}$ is 3-normal ... but not very "random" ANIMATION



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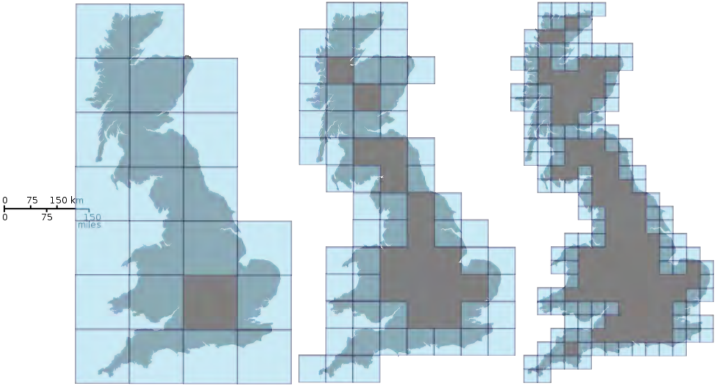
# Box-dimension:

Tends to '2' for a planar random walk

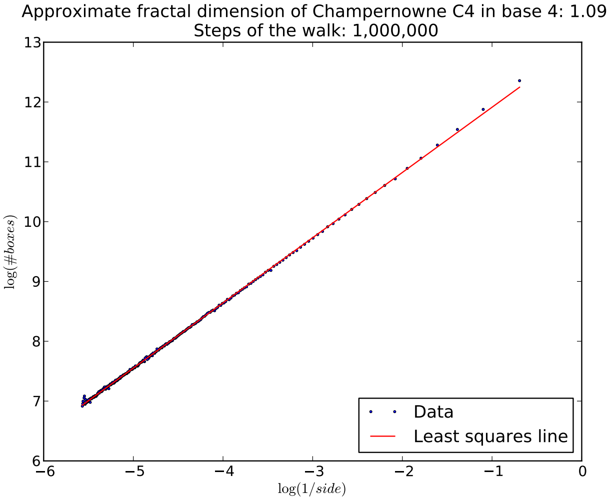



$$\text{Box-dimension} = \lim_{\text{side}\to 0} \frac{\log(\#\ \text{boxes})}{\log(1/\text{side})}$$

Norway is "frillier" — *Hitchhiker's Guide to the Galaxy*

# Box-dimension:

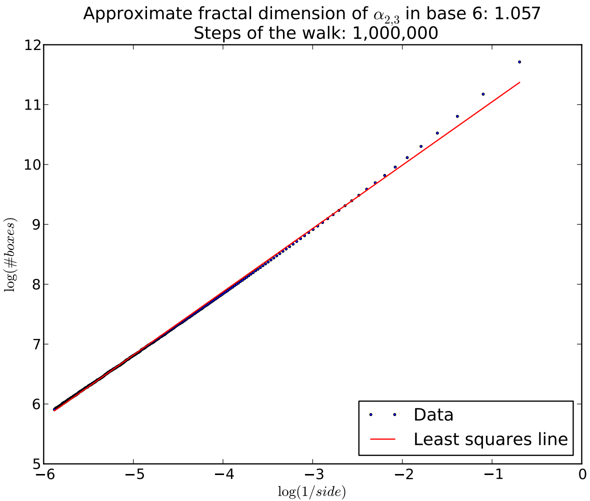
Tends to ‘2’ for a planar random walk




Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

# Box-dimension:

Tends to ‘2’ for a planar random walk




Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

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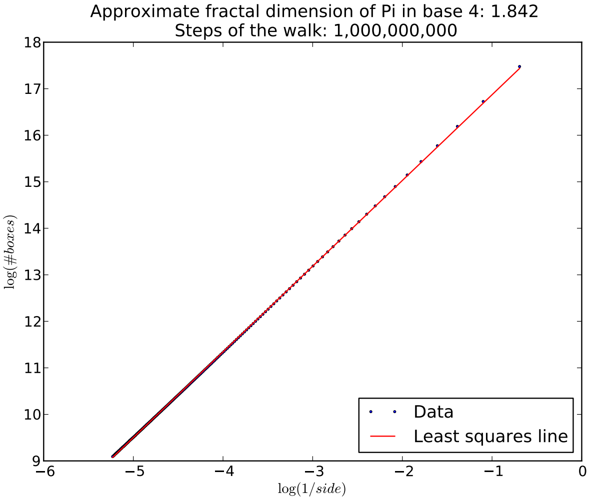
Tends to ‘2’ for a planar random walk




Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

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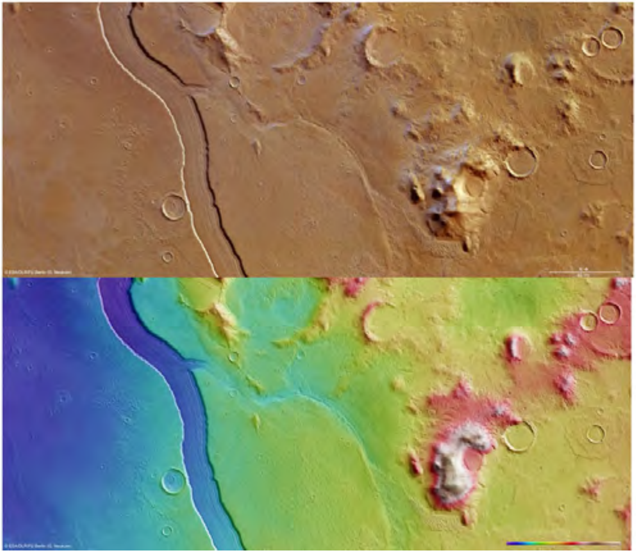
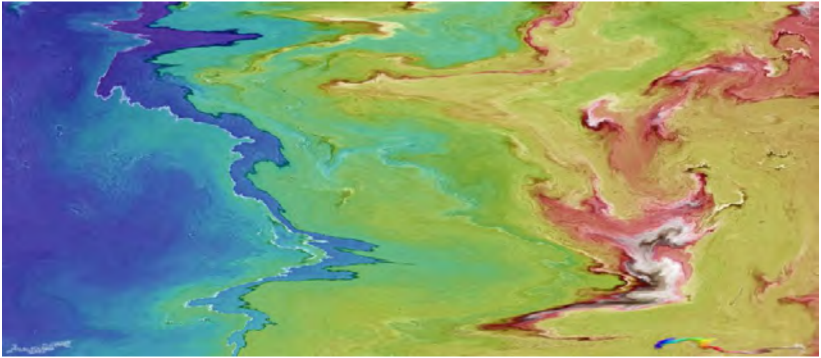
# Fractals everywhere From Mars



# Fractals everywhere

## From Mars



The picture fractalized by the Barnsley's

`http://frangostudio.com/frangocamera.html`

# Fractals everywhere

## From Space



# Fractals everywhere

$1 \mapsto 3$ or $1 \mapsto 8$ or ...

# Fractals everywhere

$1 \mapsto 3$ or $1 \mapsto 8$ or ...

# Fractals everywhere

$1 \mapsto 3$ or $1 \mapsto 8$ or ...

# Fractals everywhere

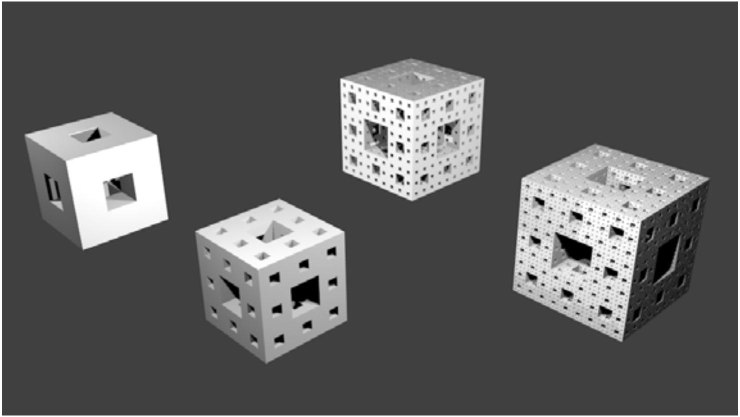
$1 \mapsto 3$ or $1 \mapsto 8$ or ...

Pascal triangle modulo two

[1] [1,1] [1,2,1] [1,3,3,1,] [1,4,6,4,1] [1,510,10,5,1] ...

# Fractals everywhere

$1 \mapsto 3$ or $1 \mapsto 8$ or ...



Steps to construction of a Sierpinski cube

# Fractals everywhere — The Sierpinski Triangle

$1 \mapsto 3 \mapsto 9$

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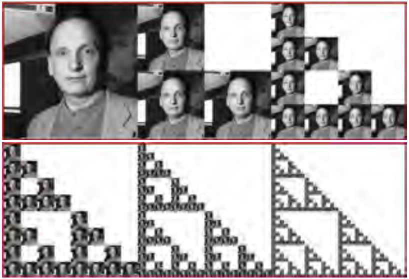
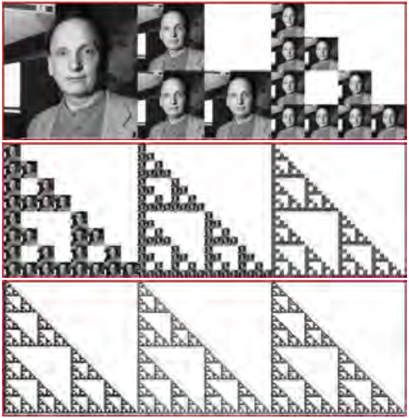
$1 \mapsto 3 \mapsto 9$



# Fractals everywhere — The Sierpinski Triangle

$1 \mapsto 3 \mapsto 9$



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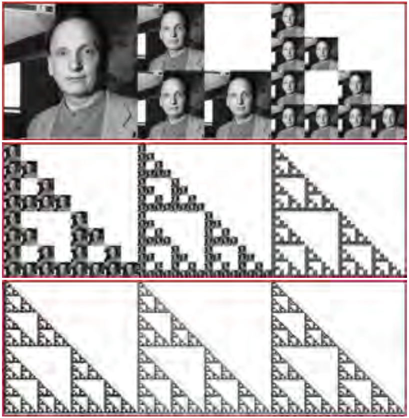

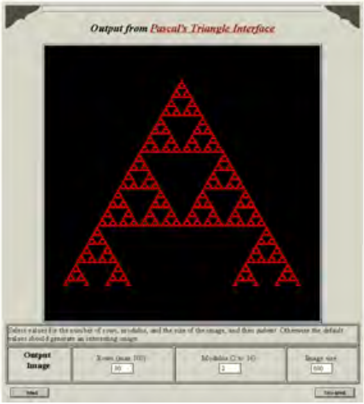


http://oldweb.cecm.sfu.ca/cgi-bin/organics/pascalform

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# Three dimensional walks: Using base six — soon on 3D screen

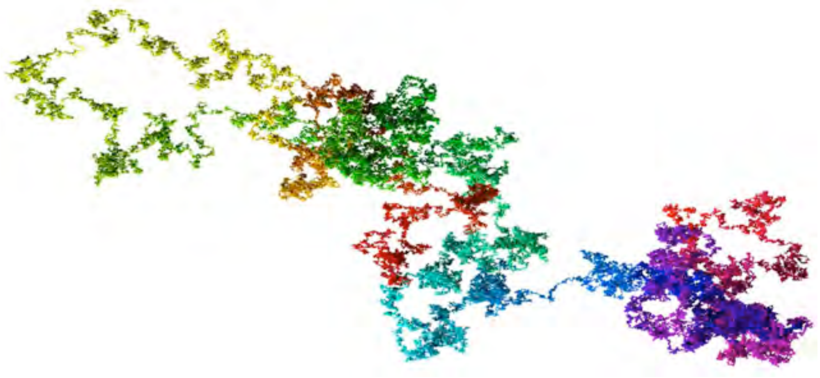

Figure : Matt Skerritt's 3D walk on $\pi$ (base 6), showing one million steps. But 3D random walks are not recurrent.

# Three dimensional walks: Using base six — soon on 3D screen

Figure : Matt Skerritt's 3D walk on $\pi$ (base 6), showing one million steps. But 3D random walks are not recurrent.

"A drunken man will find his way home, a drunken bird will get lost forever." (Kakutani)

# Three dimensional printing:

3D everywhere


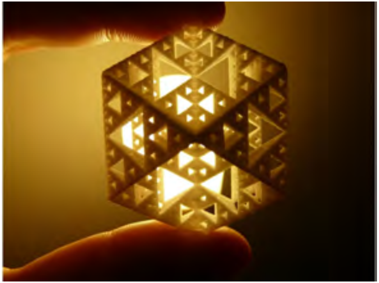

Figure : The future is here ...

www.digitaltrends.com/cool-tech/the-worlds-first-plane-created-entirely-by-3d-printing-takes-flight/

www.shapeways.com/shops/3Dfractals

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# Chaos games: Move half-way to a (random) corner

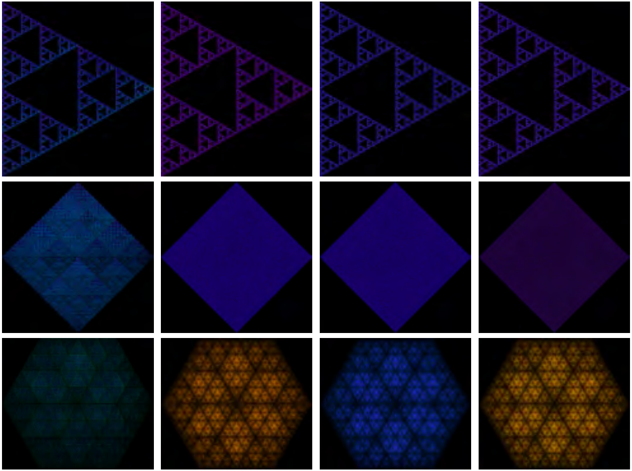

Figure : Coloured by frequency — leads to random fractals.
Row 1: Champernowne $C_3$, $\alpha_{3,5}$, random, $\alpha_{2,3}$. Row 2: Champernowne $C_4$, $\pi$, random, $\alpha_{2,3}$. Row 3: Champernowne $C_6$, $\alpha_{3,2}$, random, $\alpha_{2,3}$.

# Contents

# Automatic numbers: Thue–Morse and Paper-folding

Automatic numbers are never normal. They are given by simple but fascinating rules...giving structured/boring walks:

Figure : **Paper folding**. The sequence of left and right folds along a strip of paper that is folded repeatedly in half in the same direction. Unfold and read 'right' as '1' and 'left' as '0': 1 0 1 1 0 0 1 1 1 0 0 1 0 0

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**Thue–Morse constant** (transcendental; 2-automatic, hence nonnormal):

$$TM_2 = \sum_{n=1}^{\infty} \frac{1}{2^{t(n)}} \text{ where } t(0)=0, \text{ while } t(2n)=t(n) \text{ and } t(2n+1)=1-t(n)$$

$$0.01101001100101101001011001101001\ldots$$

# Automatic numbers: Thue–Morse and Paper-folding

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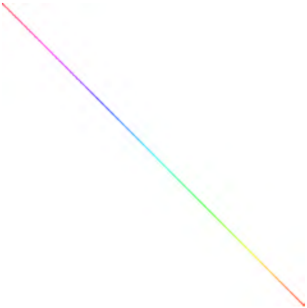
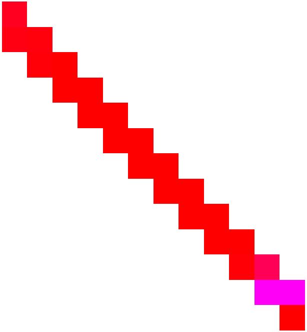


(a) 1,000 bits of Thue–Morse sequence.



(b) 10 million bits of paper-folding sequence.

Figure : Walks on two automatic and so nonnormal numbers.

# Automatic numbers: Turtle plots look great!

(a) Ten million digits of the paper-folding sequence, rotating $60^\circ$.

(b) One million digits of the paper-folding sequence, rotating $120^\circ$ (a dragon curve).

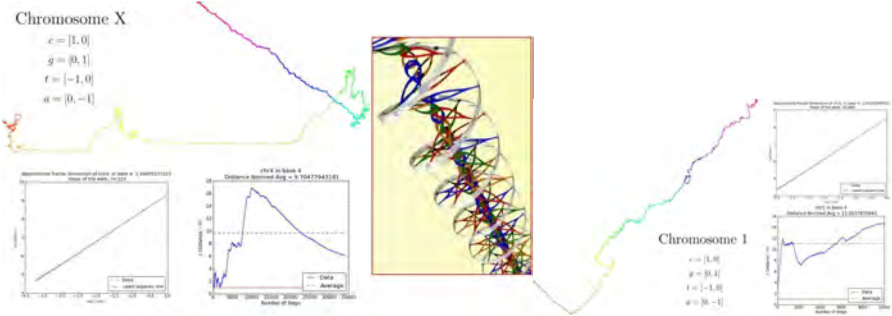
(c) 100,000 digits of the Thue–Morse sequence, rotating $60^\circ$ (a Koch snowflake).

(d) One million digits of $\pi$, rotating $60^\circ$.

Figure : Turtle plots on various constants with different rotating angles in base 2—where ‘**0**’ yields forward motion and ‘**1**’ rotation by a fixed angle.

# Genomes as walks:

## we are all base 4 numbers (ACGT/U)



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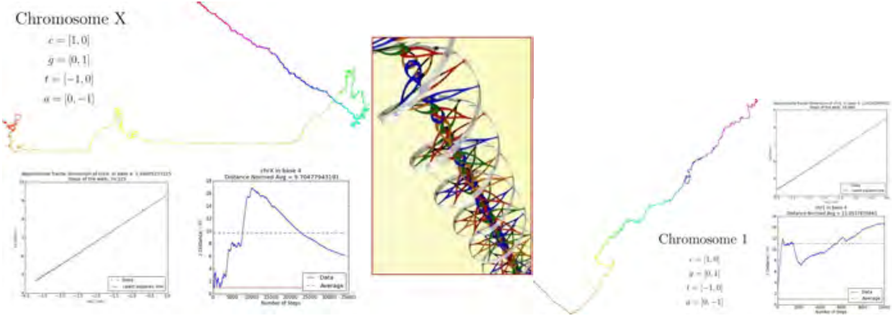


The X Chromosome (34K) and Chromosome One (10K).

# Genomes as walks:

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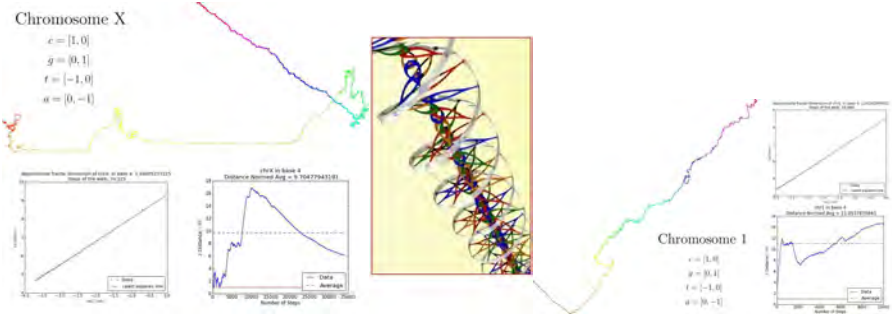


The X Chromosome (34K) and Chromosome One (10K).

® Chromosomes look less like $\pi$ and more like concatenation numbers?

# DNA for Storage: we are all base 4 numbers (ACGT/U)

News › Science › Biochemistry and molecular biology

## Shakespeare and Martin Luther King demonstrate potential of DNA storage

All 154 Shakespeare sonnets have been spelled out in DNA to demonstrate the vast potential of genetic data storage

**Ian Sample**, science correspondent
The Guardian, Thursday 24 January 2013
Jump to comments (...)

When written in DNA, one of Shakespeare's sonnets weighs 0.3 millionths of a millionth of a gram. Photograph: Oli Scarff/Getty

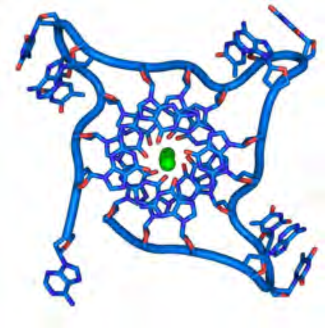

Figure : The potential for DNA storage (L) and the quadruple helix (R)

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# 2012 walk on $\pi$ (went *viral*)

Biggest mathematics picture ever?

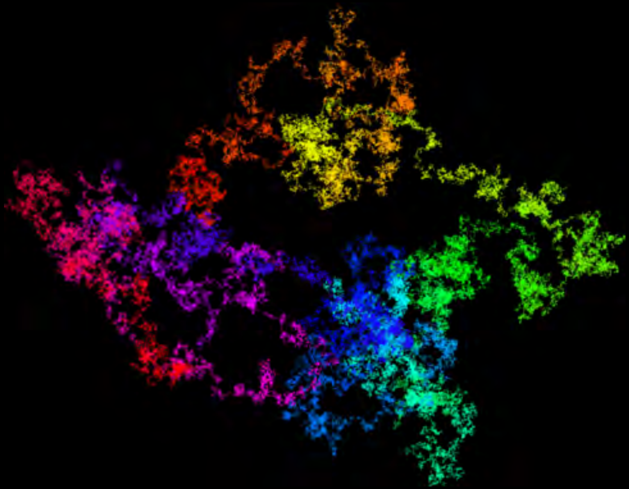

Figure : Walk on first 100 billion base-4 digits of $\pi$ (normal?).

# 2012 walk on $\pi$ (went *viral*)

Biggest mathematics picture ever?

Resolution: 372,224×290,218 pixels (108 gigapixels)

Computation: took roughly a month where several parts of the algorithm were run in parallel with 20 threads on CARMA's MacPro cluster.

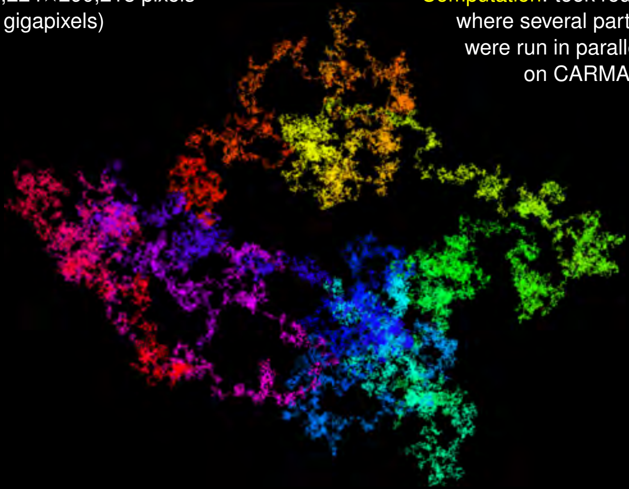

Figure : Walk on first 100 billion base-4 digits of $\pi$ (normal?).

http://gigapan.org/gigapans/106803

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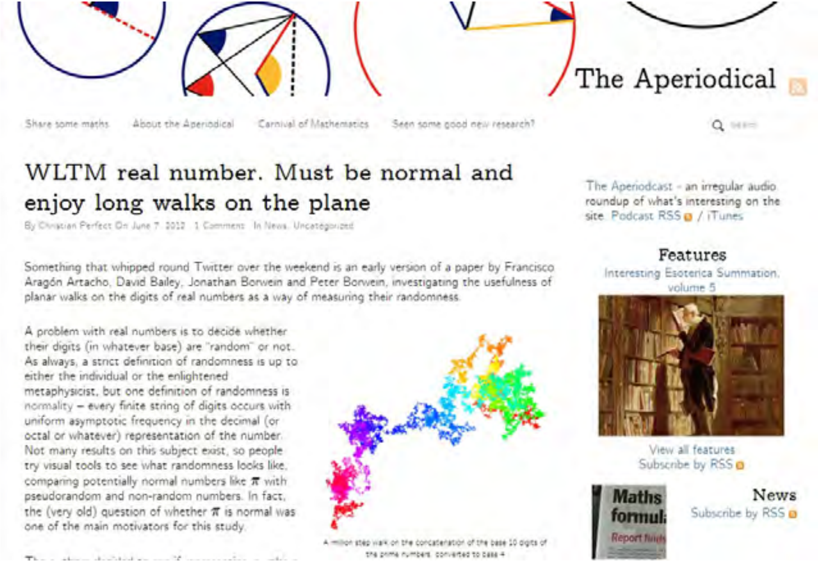

The Aperiodical

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# WLTM real number. Must be normal and enjoy long walks on the plane

By Christian Perfect On June 7, 2012 · 1 Comments · In News, Uncategorized

Something that whipped round Twitter over the weekend is an early version of a paper by Francisco Aragón Artacho, David Bailey, Jonathan Borwein and Peter Borwein, investigating the usefulness of planar walks on the digits of real numbers as a way of measuring their randomness.

A problem with real numbers is to decide whether their digits (in whatever base) are "random" or not. As always, a strict definition of randomness is up to either the individual or the enlightened metaphysicist, but one definition of randomness is normality – every finite string of digits occurs with uniform asymptotic frequency in the decimal (or octal or whatever) representation of the number. Not many results on this subject exist, so people try visual tools to see what randomness looks like, comparing potentially normal numbers like $\pi$ with pseudorandom and non-random numbers. In fact, the (very old) question of whether $\pi$ is normal was one of the main motivators for this study.

A million step walk on the concatenation of the base 10 digits of the prime numbers, converted to base 4

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http://aperiodical.com/2012/06/wltm-real-number-must-be-normal-and-enjoy-long-walks-on-the-plane/


Figure : Is Grandma's letter normal?

SOCIAL DIMENSION

A Random Walk with Pi

By Samuel Arbesman | June 12, 2012 | 12:12 pm | Categories: Science Blogs, Social Dimension

Photo: Mykl Roventine/Flickr

We've gotten really good at generating big datasets. From what we search for on Google to all the stuff we do on Facebook, we generate a lot of data. And there have in turn been a proliferation of methods of visualizing big data, slicing and dicing it and providing meaning to it.

But what's the mother of all big datasets? The digits of π. Of course, the number of digits in π is infinite, but over the years, as computers have become more powerful, we know many more of them than ever before.

But it's one thing to say that I can download trillions of digits of π and another to make some sort of sense out of all of these digits. Into this stage a team of researchers from Australia, Lawrence Berkeley National Laboratory, and Simon Fraser University with some elegant visualizations of π. In a recent paper, they used a classic method of visualizing large strings of numbers: the random walk.

A true random walk is the path described by a sequence of randomly generated numbers. For example, if we have a string of -1's and 1's, then one way to describe this is by using time as the x-axis, and the y-axis as the cumulative sum of numbers so far. This yields a nice little sawtooth shape.

Alternatively, you can do a random walk on a plane, with each digit describing the direction and distance being moved. In the case of π and the method used by these researchers, they used color to designate additional information. For example, here color is used to designate time, with the blue being earlier in the sequence and the further along the spectrum is later in the sequence.

Champions by Design

About Samuel Arbesman

BOOKS BY Samuel Arbesman

The Half-Life of Facts

http://www.wired.com/wiredscience/2012/06/a-random-walk-with-pi/

# Especially in Japan


Figure : Decisions, decisions


http://wired.jp/2012/06/15/a-random-walk-with-pi/


wired.jp/category/Science/
TECHNOLOGY
CULTURE
SCIENCE
BUSINESS
MAGAZINE
ARCHIVES
NEWS
LONG STORY
MIND HACK
HOTTEST TOPIC
1 203 RT
2 130 RT
3 92 RT
4 73 RT
5 67 RT
RANKING
LATEST NEWS
2012SS
NEW YORK
COLLECTION
BRANDS for FRIENDS
WIRED.jp
読者アンケート＆
プレゼント実施中!
WIREDの
未来生物学講義

**HOTTEST TOPIC**

1. 276 RT
円周率をランダムウォークで視覚化
2. 186 RT
新MacBook Proは「ほとんど修理不可能」
3. 90 RT
アップルとグーグルが、GPS専用端末を殺すとき
4. 85 RT
グーグルマップと別れ、アップルは成功への道を走れるか
5. 85 RT
赤ん坊のように言葉を学ぶロボット：動画

**RANKING**

1. 円周率をランダムウォークで視覚化
2. 新MacBook Proは「ほとんど修理不可能」
3. アップルとグーグルが、GPS専用端末を殺すとき
4. グーグルマップと別れ、アップルは成功への道を走れるか
5. 「新MacBook Proは種の進化だ」

### HOTTEST TOPIC

1. 276 RT 円周率をランダムウォークで視覚化
2. 186 RT 新MacBook Proは「ほとんど修理不可能」
3. 90 RT アップルとグーグルが、GPS専用端末を殺すとき
4. 85 RT グーグルマップと別れ、アップルは成功への道を走れるか
5. 85 RT 赤ん坊のように言葉を学ぶロボット：動画

### RANKING

1. 円周率をランダムウォークで視覚化
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4. グーグルマップと別れ、アップルは成功への道を走れるか
5. 「新MacBook Proは種の進化だ」

### HOTTEST TOPIC

1. 276 RT Random walk in the Pi visualization
2. 186 RT The new MacBook Pro is "almost impossible to repair"
3. 90 RT Apple and Google are, kill a dedicated terminal at GPS
4. 85 RT Google Maps and separation, or Apple can run on the road to success
5. 85 RT Video: Robots learn the language like a baby

### RANKING

1. Random walk in the Pi visualization
2. The new MacBook Pro is "almost impossible to repair"
3. Apple and Google are, kill a dedicated terminal at GPS
4. Google Maps and separation, or Apple can run on the road to success
5. "The evolution of species' s new MacBook Pro"

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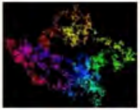
Illustration

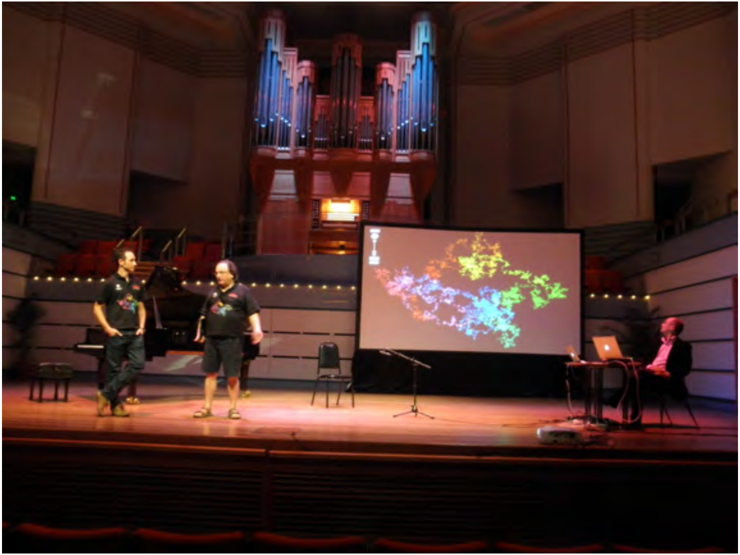


Walking on pi
By Francisco Javier Aragón Artacho Sep 21, 2012

6 Comments

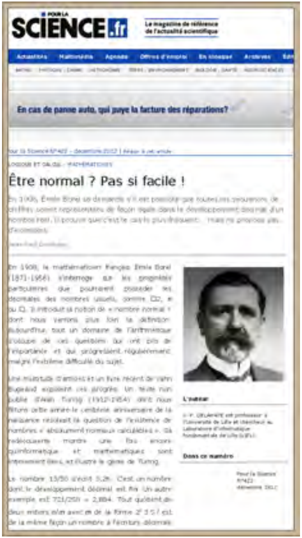
180 votes

Learn about Pi at http://www.carma.newcastle.edu.au/jon/pi-2012.pdf



October 25 2012: Music and Maths Concert http://carma.newcastle.edu.au/pdf/music_maths.pdf

Hear Pi at http://carma.newcastle.edu.au/walks/


POUR LA SCIENCE.fr
La magazine de référence de l'actualité scientifique

En cas de panne auto, qui paye la facture des réparations?

## Être normal ? Pas si facile !



December 2012: Normality of Pi and Stoneham numbers

Our analysis of 5 trillion hex-digits suggests $\pi$ is *very* probably normal!

http://www.pourlascience.fr/ewb_pages/f/fiche-article-etre-normal-pas-si-facile-30713.php

GIZMODO

TOP STORIES

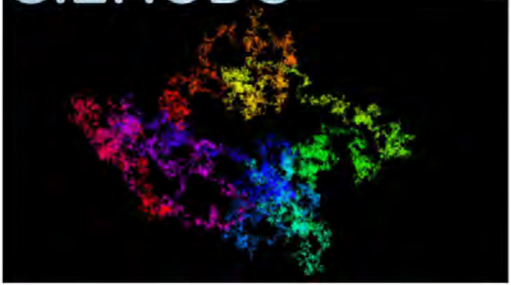


IMAGE CACHE

## What Is This?

Jamie Condliffe

Share +1 Like 152

This ragged cloud of color looks messy and unstructured—but in fact it's a rare and unusual view of one of the most fundamental things in science. Can you work out what it is?

Sadly for you, we're going to let you puzzle over the answer for a little while. To stop you all going round in circles, though, here are a couple of clues: it was generated by a computer and the thing it depicts is used in every branch of science, from mathematics to engineering.

We'll post the solution here in an hour or so. Until then, try and work out exactly what it is amongst yourselves in the comments—without cheating and resorting to Google Images.

**Update:** You can find the answer here.

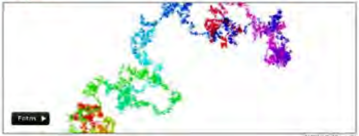
January 10, 2013 http://gizmodo.com/5974779/what-is-this

- Spiegel. The mysterious circular number: Pi contains Goethe (not Shakespeare)

SPIEGEL ONLINE WISSENSCHAFT

**Rätselhafte Kreiszahl: In Pi könnte Goethes "Faust" stecken**

Von Holger Dambeck




CARMA/ UC of Newcastle

**Die Kreiszahl Pi fasziniert Mathematiker seit Jahrtausenden. Mittlerweile ist sie auf zehn Billionen Stellen genau berechnet, doch eines ihrer größten Geheimnisse hat noch niemand gelüftet: Ist in ihr jeder jemals geschriebene Text kodiert?**

Montag, 29.04.2013 – 08:26 Uhr

Berlin - Stellen Sie sich vor, es gibt ein Buch, in dem alle je von Menschen geschriebenen Texte vereint sind. Shakespeare, Goethe, der erste Schulaufsatz von Albert Einstein - es gibt keinen Gedanken, der darin fehlt. Mehr noch: Das Buch enthält obendrein auch alles, was in Zukunft geschrieben werden wird.

So ein dickes Buch kann es gar nicht geben, werden Sie sagen und haben damit im Grunde recht. Doch trotzdem existiert dieses Buch wahrscheinlich - virtuell in der unendlich langen Zahl Pi. Sie kennen Pi als das Verhältnis von Kreisumfang zu Durchmesser - die Zahl beginnt mit 3,1415926535...

Falls die unendlich vielen Ziffern von Pi zufällig verteilt sind, wovon viele Mathematiker ausgehen, steckt in Pi jede beliebige Ziffernfolge, die wir uns ausdenken können. Nehmen wir zum Beispiel 00000000 - also acht Nullen hintereinander. An Position 172.330.850 nach dem Komma tauchen die acht Nullen tatsächlich auf. Und an Position 184.688.988 gleich noch mal. Dort aufgespürt hat sie übrigens die Webseite Pi-Search, die auf Knopfdruck die ersten 200 Millionen Stellen der Kreiszahl durchsucht.

Wir können Buchstaben problemlos mit Zahlen kodieren, ein Computer macht nichts anderes. Wenn in Pi aber jede beliebige Ziffernfolge steckt, ist in Pi auch jeder beliebige Text kodiert. Die Frage ist nur, wie viele Millionen, Milliarden oder Billiarden Stellen wir benötigen, bis wir beispielsweise auf den Text von Goethes "Faust" stoßen.

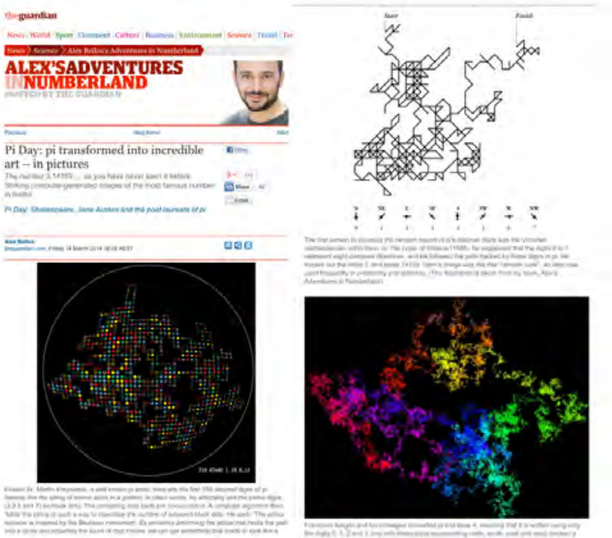
Der Schauspieler George Takei ("Raumschiff Enterprise") hat dieses verblüffende Phänomen kürzlich auf seiner Facebook-Seite beschrieben -

THEMA Mathematik

VIDEO

April 29, 2013 www.spiegel.de/wissenschaft/mensch/mathematik-ist-die-kreiszahl-pi-normal-a-895876.html

- **Guardian**. 3.14.14




March 14, 2014 www.theguardian.com/science/alexs-adventures-in-numberland/gallery/2014/mar/14/pi-day-pi-transformed-into-incredible-art-in-pictures

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http://carma.newcastle.edu.au/walks/


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