ERGODIC BEHAVIOUR OF A DOUGLAS-RACHFORD OPERATOR AWAY FROM THE ORIGIN

JONATHAM M. BORWEIN AND OHAD GILADI

Abstract. Considering the Douglas-Rachford iteration scheme with respect to a line and a sphere in $\mathbb{R}^d$, $d \geq 2$, it is shown that by smoothing the operator in a neighbourhood of the origin, it can be approximated by another operator that satisfies a weak ergodic theorem. Other iteration schemes are also discussed.

1. Introduction

1.1. **Background and statement of the main result.** In this note we consider the Douglas-Rachford operator in $\mathbb{R}^d$, which is defined as follows. Given two sets $A, B \subseteq \mathbb{R}^d$, define

$$T_{A,B} = \frac{I + R_B R_A}{2}. \tag{1.1}$$

Here I is the identity operator in $\mathbb{R}^d$ and R_A, R_B are the reflection operators with respect to A and B, that is, $R_A = 2P_A - I$, $R_B = 2P_B - I$, where for a given set $A \subseteq \mathbb{R}^d$, P_A denotes the projection operator on A,

$$P_A x = \left\{ y \in A \;\middle|\; \|x - y\| = \inf_{z \in A} \|x - z\| \right\}.$$

Here and in what follows, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. In the general case, P_A, P_B, as well as R_A, R_B, and $T_{A,B}$ can be multi-valued operators. One of the questions regarding the Douglas-Rachford operator is the following. Given $x \in \mathbb{R}^d$, study the asymptotic behaviour of a sequence $\{x_n\}_{n=1}^{\infty}$ which is generated by the iterations of $T_{A,B}$,

$$x_{n+1} \in T_{A,B} x_n = (T_{A,B})^n x, \quad x_1 = x. \tag{1.2}$$

In the case where both A and B are convex, it is known that $T_{A,B}$ is single-valued and strictly non-expansive, and the sequence $\{x_n\}_{n=1}^{\infty}$ is norm convergent. See for example [GK90, Opi67]. In the non-convex case, several attempts have also been made. One of the simplest non-convex cases is when one of the sets is a line in $\mathbb{R}^d$ and the other is the Euclidean unit sphere. Indeed, let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^d$. Fix $\alpha \in [0, 1]$, and let $a, b \in \mathbb{R}^d$ be two independent vectors. Define the following sets,

$$S = \left\{ x \in \mathbb{R}^d \;\middle|\; \|x\| = 1 \right\}, \quad L_\alpha = \left\{ x \in \mathbb{R}^d \;\middle|\; x = \lambda a + \alpha b, \lambda \in \mathbb{R} \right\}. \tag{1.3}$$

It can be assumed without loss of generality that $a \perp b$ and that $\|a\| = \|b\| = 1$. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$, be an orthonormal basis of $\mathbb{R}^d$ such that $\mathbf{e}_1 = a$ and $\mathbf{e}_2 = b$. Write $x = \sum_{j=1}^d x_j \mathbf{e}_j \in \mathbb{R}^d$, where $x_j = \langle x, \mathbf{e}_j \rangle$, $\langle \cdot, \cdot \rangle$ being the standard inner product in $\mathbb{R}^d$. In such case, R_{L_α}, R_S are given explicitly by

$$R_{L_\alpha} x = x_1 \mathbf{e}_1 + (2\alpha - x_2)\mathbf{e}_2 + \sum_{j=3}^d x_j \mathbf{e}_j, \quad R_S x = \left(\frac{2}{\|x\|} - 1 \right) x. \tag{1.4}$$

Date: May 25, 2016.
2010 *Mathematics Subject Classification.* 49J99, 47H25, 37J25.

As a result, the Douglas-Rachford operator is given by the following explicit formula,

$$T_{S,L_\alpha} x = \frac{x_1}{\|x\|}\mathbf{e}_1 + \left(\alpha + \left(1 - \frac{1}{\|x\|}\right) x_2\right) \mathbf{e}_2 + \sum_{j=3}^{d} \left(1 - \frac{1}{\|x\|}\right) x_j \mathbf{e}_j. \tag{1.5}$$

Note that T_{S,L_α} is single-valued for every $x \neq 0$. It will always be assumed that $d \geq 2$. In the case $d = 2$, the sum in (1.4) and (1.5) is empty.

Remark 1.1. The sets defined in (1.3) can also be defined in an infinite dimensional Hilbert space. In such case, (1.4) and (1.5) still hold true, but with an infinite sum in (1.5). See Section 5.2 for a discussion of another iteration scheme in an infinite dimensional setting. $\diamond$

Regarding the behaviour of the sequence defined in (1.2), it was shown in [BS11] that if $\alpha \in [0, 1)$, then (1.2) is locally convergent around the two intersection points $\pm\sqrt{1 - \alpha^2}a + \alpha b$. Later, in [AAB13] a more detailed study of the asymptotic behaviour of the sequence (1.2) was studied in the case $d = 2$. Recently, it was shown in [Ben15] that the sequence (1.2) converges for all $x \in \mathbb{R}^d$ with $\langle x, a\rangle \neq 0$ (the global convergence for the case $\alpha = 0$ was already shown in [BS11]). The result in [Ben15] holds true in an infinite dimensional setting as well. In particular, it follows that for every $x, y \in \mathbb{R}^d$ such that $\langle x, a\rangle$, $\langle x, a\rangle$, are either both positive or both negative, and for every $\varepsilon > 0$, there exists $N = N(x, y, \varepsilon)$ such that for all $n \geq N$, $\|T^n_{S,L_\alpha} x - T^n_{S,L_\alpha} y\| < \varepsilon$. However, given $\varepsilon > 0$ and a set $K \subseteq \mathbb{R}^d$, the results in [Ben15] do not yield an estimate on $N = N(K, \varepsilon)$ such that for all $n \geq N$ we have

$$\sup_{x,y\in K} \|T^n_{S,L_\alpha} x - T^n_{S,L_\alpha} y\| < \varepsilon. \tag{1.6}$$

The type of convergence that appears in (1.6) is the main focus of this note. Recall first the following notion.

Definition 1.1 (Weak ergodic theorem)**.** Given a set $K \subseteq \mathbb{R}^d$, an operator $G : K \to K$ is said to satisfy a weak ergodic theorem if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\sup_{x,y\in K} \|G^n x - G^n y\| < \varepsilon.$$

The main result below, Theorem 1.1, says that by somehow 'smoothing out' the problematic behaviour of the Douglas-Rachford operator close to the origin, we can approximate it with another operator that satisfies a weak ergodic theorem.

To fix some notation, given a normed space $(X, \|\cdot\|)$ let $B[x, r]$ be the *closed* ball around x with radius r, that is, $B[x, r] = \{y \in X \mid \|y - x\| \leq r\}$. $B(x, r)$ will denote an *open* ball, that is, $B(x, r) = \{y \in X \mid \|y - x\| < r\}$.

We are now in a position to state the main result.

Theorem 1.1. *Assume that $\alpha, \beta \in [0, 1)$, and $\gamma, \varepsilon \in (0, 1)$. Assume that $R, r \in [0, \infty)$ are such that $R \geq \frac{4}{1-\alpha}$, $r \leq \frac{1-\alpha}{4}$. Then there exists a map $G : \mathbb{R}^d \to \mathbb{R}^d$ satisfying*

$$\sup_{x\in B[0,R/2]\setminus B(0,1-\beta)} \|T_{S,L_\alpha} x - Gx\| \leq \left(1 - \frac{(1-\gamma)^2}{\frac{1}{1-\beta} + \frac{d\beta}{2r}}\right) 2R + \frac{r}{1-\beta} + \beta, \tag{1.7}$$

and for every $n \in \mathbb{N}$ satisfying

$$n \geq \frac{\log\left(\frac{\varepsilon}{64R}\right)}{\log\left(1 - \frac{\gamma}{2}\right)} + 2, \tag{1.8}$$

we have,

$$\sup_{x,y\in B[0,R/2]} \|G^n x - G^n y\| < \varepsilon. \tag{1.9}$$

Moreover, if $H : B\left[0, \sqrt{R^2 - \frac{(1-\alpha)}{2}R}\right] \to B\left[0, \sqrt{R^2 - \frac{(1-\alpha)}{2}R}\right]$ *satisfies*

$$\sup_{x\in B\left[0,\sqrt{R^2-\frac{(1-\alpha)}{2}R}\right]} \|Gx - Hx\| \leq \frac{\varepsilon}{200},$$

then for every $n \in \mathbb{N}$ *satisfying* (1.8)*, we have*

$$\sup_{x,y\in B[0,R/2]} \|H^n x - H^n y\| < \varepsilon. \tag{1.10}$$

Theorem 1.1 as stated above is by no means optimal. The version above is presented for the sake of simplicity. Also, note that when $\beta = 0$, we can choose $r = \gamma = 0$. In such case $T_{S,L_\alpha} = G$ on $\mathbb{R}^d \setminus B(0,1)$. Next, we consider a simple example to illustrate the use of Theorem 1.1.

Example 1.1. To consider a concrete example, let $d = 2$ and choose $r = \sqrt{\beta}$. In order to make such a choice, we would have to assume that $\beta \leq \frac{(1-\alpha)^2}{16}$. Also, in such case we have the following trivial bounds,

$$\frac{r}{1-\beta} + \beta \leq \frac{2\sqrt{\beta}}{1-\beta}, \qquad \frac{1}{1-\beta} + \frac{\beta}{r} = \frac{1}{1-\beta} + \sqrt{\beta} \leq \frac{1+\sqrt{\beta}}{1-\beta} = \frac{1}{1-\sqrt{\beta}}.$$

Hence, we have

$$\left(1 - \frac{(1-\gamma)^2}{\frac{1}{1-\beta} + \frac{\beta}{r}}\right) 2R + \frac{r}{1-\beta} + \beta \leq \left(1 - (1-\gamma)^2\left(1-\sqrt{\beta}\right)\right) 2R + \frac{2\sqrt{\beta}}{1-\beta}$$

If we choose $\gamma = \sqrt{\beta}$, then we have

$$\left(1 - (1-\gamma)^2\left(1-\sqrt{\beta}\right)\right) R + \frac{2\sqrt{\beta}}{1-\beta} = \left(1 - \left(1-\sqrt{\beta}\right)^3\right) 2R + \frac{2\sqrt{\beta}}{1-\beta} \leq 8\sqrt{\beta}R + \frac{2\sqrt{\beta}}{1-\beta}.$$

If we further assume that $\beta \leq \frac{1}{(8R)^4}$, we have

$$8\sqrt{\beta}R + \frac{2\sqrt{\beta}}{1-\beta} \leq \beta^{1/4} + \frac{2\sqrt{\beta}}{1-\beta} \leq \frac{3\sqrt{\beta}}{1-\beta}.$$

Note that in order to use Theorem 1.1, we need to assume $R \geq \frac{4}{1-\alpha}$, in which case $\frac{1}{(8R)^4} \leq \left(\frac{1-\alpha}{4}\right)^2$. Hence, it is enough to assume $R \geq \frac{4}{1-\alpha}$ and $\beta \leq \frac{1}{(8R)^4}$. Altogether, applying Theorem 1.1, we have that there exists $G : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\sup_{x\in B[0,R/2]\setminus B(0,1-\beta)} \|T_{S,L_\alpha}x - Gx\| \leq \frac{3\sqrt{\beta}}{1-\beta},$$

and for every $n \in \mathbb{N}$ satisfying

$$n \geq \frac{\log\left(\frac{\varepsilon}{64R}\right)}{\log\left(1 - \frac{\sqrt{\beta}}{2}\right)} + 2,$$

we have

$$\sup_{x,y\in B[0,R/2]} \|G^n x - G^n y\| < \varepsilon.$$

Note that this example is not optimal. It is merely an example of how Theorem 1.1 can be used.$\diamond$

1.2. **Comparing new and old results.** By (1.5), the sets $\{x \in \mathbb{R}^d \mid \langle x, a\rangle \geq 0\}$, $\{x \in \mathbb{R}^d \mid \langle x, a\rangle \leq 0\}$, are both invariant under T_{S,L_α}. Indeed, the global convergence result in [Ben15] holds in each half space separately. However, during the process of 'smoothing' T_{S,L_α} in the proof of Theorem 1.1, this invariance property is lost. See in particular Theorem 2.3 and the proof of Theorem 2.1 below. This smoothing process is robust enough to yield a map on $B[0,R/2]$ which satisfies an ergodic theorem and which approximates T_{S,L_α} on $B[0,R/2]\setminus B(0,1-\beta)$. However, we do not how to preserve such invariance during the smoothing process, and it would be interesting to find a way to do so.

Another difference between the results of [Ben15] and Theorem 1.1 is the following. While Theorem 1.1 does not imply norm convergence as in [Ben15], it does show that as long as we avoid approaching the origin, the Douglas-Rachford operator has Lipschitz behaviour that in turn can be used to well approximate it with an operator that satisfies a weak ergodic theorem. Theorem 1.1 and particularly Theorem 2.1 below show that the Lipschitz behaviour of T_{S,L_α} depends on the distance from the origin, not on the particular location in the space. In particular, Theorem 1.1 would imply the following. Assume that $K \subseteq \{x \in \mathbb{R}^d \mid \langle x, a\rangle \neq 0\}$. Let $\varepsilon > 0$, and choose R, r, β, γ in Theorem 1.1 such that the bound in (1.7) is smaller than ε. By [Ben15] it is known that the sequence $\{T^m_{S,L_\alpha}x\}_{m=1}^\infty$ satisfies $\|T^m_{S,L_\alpha}x\| \overset{m\to\infty}{\longrightarrow} 1$ for all $x \in K$. Assume that it were known further that there exists $N' = N'(K,\varepsilon)$ such that $T^m_{S,L_\alpha}(K) \subseteq B[0,R/2] \setminus B(0,1-\beta)$ for all $m \geq N'$. Then, Theorem 1.1 would yield a map $G : \mathbb{R}^d \to \mathbb{R}^d$ and $N = N(\varepsilon)$ such that for all $n \geq N$, we have

$$\sup_{x\in B[0,R/2]\setminus B(0,1-\beta)} \|T_{S,L_\alpha}x - Gx\| < \varepsilon, \qquad \sup_{x,y\in B[0,R/2]} \|G^n x - G^n y\| < \varepsilon.$$

Since $\{T^m_{S,L_\alpha}x\}_{m=N'}^\infty, \{T^m_{S,L_\alpha}x\}_{m=N'}^\infty \subseteq B[0,R/2]\setminus B(0,1-\beta)$, we have in particular for all $m \geq N'$, $n \geq N$,

$$\sup_{x,y\in K} \|G^n T^m_{S,L_\alpha}x - G^n T^m_{S,L_\alpha}y\| < \varepsilon. \tag{1.11}$$

In other words, if we knew that the norm of the Douglas-Rachford iteration scheme converges to 1 uniformly on some sets, then that would imply a weak ergodic theorem, not for the operator T_{S,L_α} itself, but involving another operator which well approximates it. It would thus be interesting to see whether $N' = N'(K,\varepsilon)$ can be evaluated, at least for certain sets.

Note that even if (1.11) were true, it would not automatically imply a result involving T_{S,L_α} only. Indeed, in order to make (1.7) smaller than ε, then since $\frac{1}{1-\beta} + \frac{d\beta}{2r} \geq 1$, we must have in particular $(1-(1-\gamma)^2)R < \varepsilon$, which means that

$$\gamma \leq 1 - \sqrt{1 - \frac{\varepsilon}{R}} \leq \frac{\varepsilon}{R}. \tag{1.12}$$

Next, we have for $x \in K$, $n \geq N$, $m \geq N'$,

$$\begin{aligned}
\|G^n T^m_{S,L_\alpha}x - T^{n+m}_{S,L_\alpha}x\| &\leq \|G^n T^m_{S,L_\alpha}x - GT^{n+m-1}_{S,L_\alpha}x\| + \|GT^{n+m-1}_{S,L_\alpha}x - T^{n+m}_{S,L_\alpha}x\| \\
&\overset{(*)}{\leq} \|G\|_{\text{lip}}\|G^{n-1}T^m_{S,L_\alpha}x - T^{n+m-1}_{S,L_\alpha}x\| + \varepsilon \\
&\overset{(**)}{\leq} \left(1 - \frac{\gamma}{2}\right)\|G^{n-1}T^m_{S,L_\alpha}x - T^{n+m-1}_{S,L_\alpha}x\| + \varepsilon,
\end{aligned} \tag{1.13}$$

where in $(*)$ we used fact that G is Lipschitz, as well as the fact that $T^{n+m-1}_{S,L_\alpha}x \in B[0,R/2]\setminus B(0,1-\beta)$ for all $n \geq N$, $m \geq N'$, and in $(**)$ we used Remark 4.1 below. Altogether, by iterating (1.13), we have for $x \in K$, $n \geq N$, $m \geq N'$,

$$\|G^n T_{S,L_\alpha}^m x - T_{S,L_\alpha}^{n+m} x\| \leq \varepsilon \sum_{k=0}^{n-1} \left(1 - \frac{\gamma}{2}\right)^k = \frac{2\varepsilon}{\gamma} \left(1 - \left(1 - \frac{\gamma}{2}\right)^n\right).$$

By the choice of N in (1.8), we have

$$\left(1 - \frac{\gamma}{2}\right)^n \leq \left(1 - \frac{\gamma}{2}\right)^{\frac{\log\left(\frac{\varepsilon}{64R}\right)}{\log\left(1-\frac{\gamma}{2}\right)}} = \frac{\varepsilon}{64R}.$$

Combining this with (1.12), we have

$$\frac{2\varepsilon}{\gamma} \left(1 - \left(1 - \frac{\gamma}{2}\right)^n\right) \geq 2R \left(1 - \frac{\varepsilon}{64R}\right),$$

which is not a good bound as $\varepsilon \to 0$.

Organisation of the note. The proof of Theorem 1.1 is done in two steps. Firstly, it is shown that by removing a ball around the origin, we can approximate the Douglas-Rachford operator with another map which satisfies a Lipschitz condition. This is done in Section 2. Then it is shown that Lipschitz maps can be approximated by operators that satisfy a weak ergodic theorem. This is very similar to the main result in [RZ03]. See Section 3. The proof of Theorem 1.1 is presented in Section 4. Finally, in Section 5, we briefly discuss how the tools in this note can be applied to other iteration schemes.

2. Lipschitz approximation of the Douglas-Rachford operator

The first step in the proof of Theorem 1.1 is to study of some aspects of the Lipschitz behaviour of the Douglas-Rachford operator. Given two normed space $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, and a map $f : X \to Y$, define

$$\|f\|_{\text{lip}} = \sup_{x \neq y} \frac{\|f(x) - f(y)\|_Y}{\|x - y\|_X}. \tag{2.1}$$

A function $f : X \to Y$ is said to be Lipschitz if $\|f\|_{\text{lip}} < \infty$. The first auxiliary result in the proof of Theorem 1.1 reads as follows.

Theorem 2.1. *Assume that $\alpha, \beta \in [0, 1)$, and $R, \rho \in [0, \infty)$ are such that*

$$R \geq \sqrt{\left(\frac{\rho^2 + 4}{2(1 - \alpha)}\right)^2 + \rho^2 + 5}.$$

Assume that r is such that $0 < r \leq R - \sqrt{R^2 - \rho^2}$. Then there exists a map $F : \mathbb{R}^d \to \mathbb{R}^d$ which satisfies the following properties.

(1) *Bounded Lipschitz constant:*

$$\|F\|_{\text{lip}} \leq \frac{1}{1 - \beta} + \frac{d\beta}{2r}. \tag{2.2}$$

(2) *Approximation of T_{S,L_α} on $\mathbb{R}^d \setminus B(0, 1 - \beta)$:*

$$\sup_{x \in \mathbb{R}^d \setminus B(0,1-\beta)} \|Fx - T_{S,L_\alpha} x\| \leq \frac{r}{1 - \beta} + \beta. \tag{2.3}$$

(3) *Invariance:*

$$F\left(B\left[0, \sqrt{R^2 - \rho^2}\,\right]\right) \subseteq B\left[0, \sqrt{R^2 - \rho^2}\,\right]. \tag{2.4}$$

We begin by showing that if we remove a neighbourhood of the origin, then the Douglas-Rachford operator satisfies a Lipschitz condition, but with some error term. First we show that the Douglas-Rachford operator satisfies a Lipschitz conditions on spheres and rays passing at the origin.

Proposition 2.1. *If $\|x\| = \|y\|$ and $x \neq 0$, then*

$$\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| \le \max\left\{\frac{1}{\|x\|}, 1 - \frac{1}{\|x\|}\right\}\|x-y\|. \tag{2.5}$$

If $y = tx$, $t \in \mathbb{R} \setminus \{0\}$, then

$$\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| \le \|x-y\|. \tag{2.6}$$

Proof. First, note that by (1.4), we have for all $x, y \in \mathbb{R}^d$,

$$\|R_{L_\alpha}x - R_{L_\alpha}y\| = \|x-y\|. \tag{2.7}$$

Assume first that $\|x\| = \|y\|$. Then we have

$$\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| \overset{(1.1)}{\le} \frac{1}{2}\|x-y\| + \frac{1}{2}\|R_{L_\alpha}R_Sx - R_{L_\alpha}R_Sy\| \overset{(2.7)}{=} \frac{1}{2}\|x-y\| + \frac{1}{2}\|R_Sx - R_Sy\|$$
$$\overset{(1.4)}{=} \frac{1}{2}\|x-y\| + \frac{1}{2}\left|\frac{2}{\|x\|} - 1\right|\|x-y\| \le \max\left\{\frac{1}{\|x\|}, 1 - \frac{1}{\|x\|}\right\}\|x-y\|,$$

which proves (2.5). If $y = tx$, $t \in \mathbb{R} \setminus \{0\}$, we have

$$\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| \le \frac{1}{2}\|x-y\| + \frac{1}{2}\|R_Sx - R_Sy\|$$
$$= \frac{|t-1|\|x\|}{2} + \frac{1}{2}\left\|\left(\frac{2}{\|x\|} - 1\right)x - \left(\frac{2}{t\|x\|} - 1\right)tx\right\| = |t-1|\|x\| = \big|\|x\| - \|y\|\big| \le \|x-y\|.$$

This proves (2.6) and the proof is complete. $\square$

Next, we show the 'almost Lipschitz' property of the Douglas-Rachford operator on the domain $\mathbb{R}^d \setminus B(0, 1-\beta)$.

Proposition 2.2. *Assume that $\beta \in [0,1)$ and $x, y \in \mathbb{R}^d \setminus B(0, 1-\beta)$. Then*

$$\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| \le \frac{\|x-y\|}{1-\beta} + \beta.$$

Proof. Assume first that $\|x\| \ge 1$, $\|y\| \ge 1$. In such case, we have that $R_Sx = R_{B[0,1]}x$, $R_Sy = R_{B[0,1]}y$. Since $B[0,1]$ is a *convex* set, it is known that in such case, the reflection map is non-expansive. See for example Theorem 12.2 in [GK90]. Thus,we have that $\|R_Sx - R_Sy\| \le \|x-y\|$. As before, we have $\|R_{L_\alpha}x - R_{L_\alpha}y\| = \|x-y\|$ for all $x, y \in \mathbb{R}^d$. Hence, we have

$$\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| \le \frac{1}{2}\|x-y\| + \frac{1}{2}\|R_{L_\alpha}R_Sx - R_{L_\alpha}R_Sy\|$$
$$= \frac{1}{2}\|x-y\| + \frac{1}{2}\|R_Sx - R_Sy\| \le \|x-y\|. \tag{2.8}$$

Next, assume that $\|x\| \le 1$, $\|y\| \le 1$. Assume without loss of generality that $\|x\| \le \|y\|$. Note that since $\|x\| \in [1-\beta, 1]$, we have

$$\max\left\{\frac{1}{\|x\|}, 1 - \frac{1}{\|x\|}\right\} \le \frac{1}{1-\beta}.$$

Also, since $\|x\|, \|y\| \in [1-\beta, 1]$, we have $\big|\|x\| - \|y\|\big| \le \beta$. Hence, we have

$$
\begin{aligned}
\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| &\le \left\|T_{S,L_\alpha}x - T_{S,L_\alpha}\left(\frac{\|x\|}{\|y\|}y\right)\right\| + \left\|T_{S,L_\alpha}\left(\frac{\|x\|}{\|y\|}y\right) - T_{S,L_\alpha}y\right\| \\
&\overset{(2.5)\wedge(2.6)}{\le} \max\left\{\frac{1}{\|x\|}, 1 - \frac{1}{\|x\|}\right\}\|x-y\| + \left\|y - \frac{\|y\|}{\|x\|}y\right\| \le \frac{\|x-y\|}{1-\beta} + \big|\|x\| - \|y\|\big| \\
&\le \frac{\|x-y\|}{1-\beta} + \beta. \qquad (2.9)
\end{aligned}
$$

Finally, consider the case where $\|x\| \le 1 \le \|y\|$. In such case, there exists $s \in [0,1]$ such that $\|sx + (1-s)y\| = 1$. Therefore, we have

$$
\begin{aligned}
\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| &\le \|T_{S,L_\alpha}x - T_{S,L_\alpha}(sx+(1-s)y)\| + \|T_{S,L_\alpha}(sx+(1-s)y) - T_{S,L_\alpha}y\| \\
&\overset{(2.8)\wedge(2.9)}{\le} \frac{\|x - (sx+(1-s)y)\|}{1-\beta} + \beta + \|(sx+(1-s)y) - y\| \\
&= \frac{(1-s)\|x-y\|}{1-\beta} + \beta + s\|x-y\| \le \frac{\|x-y\|}{1-\beta} + \beta. \qquad (2.10)
\end{aligned}
$$

Combining (2.8), (2.9), and (2.10), the proof is complete. ☐

Next, we study some of the invariance properties of T_{S,L_α}. The following is a simple generalisation of Proposition 6.7 in [BS11].

Proposition 2.3. *Assume that $\alpha \in [0,1)$ and $\rho \ge 0$. Assume that $\|x\| \ge 1 + \frac{\alpha^2+\rho^2}{2(1-\alpha)}$. Then*

$$\|T_{S,L_\alpha}x\| \le \sqrt{\|x\|^2 - \rho^2}.$$

Proof. First, note that if we assume that $\|x\| \ge 1 + \frac{\alpha^2+\rho^2}{2(1-\alpha)}$ and since $\alpha \in [0,1)$, we have

$$\|x\| \ge 1 + \frac{\rho^2}{2} = \left(\frac{\rho}{\sqrt{2}} - 1\right)^2 + \sqrt{2}\rho \ge \rho.$$

Hence, $\|x\|^2 - \rho^2 \ge 0$. Next, by (1.5), we have

$$
\begin{aligned}
\|T_{S,L_\alpha}x\|^2 &= \frac{x_1^2}{\|x\|^2} + \alpha^2 + 2\alpha x_2\left(1 - \frac{1}{\|x\|}\right) + \left(1 - \frac{1}{\|x\|}\right)^2 \sum_{j=2}^d x_j^2 \\
&= \alpha^2 + 2\alpha x_2\left(1 - \frac{1}{\|x\|}\right) + 1 + \left(1 - \frac{2}{\|x\|}\right)\sum_{j=2}^d x_j^2 \\
&\le \alpha^2 + 2\alpha x_2\left(1 - \frac{1}{\|x\|}\right) + 1 + \left(1 - \frac{1}{\|x\|}\right)^2 \sum_{j=2}^d x_j^2 \\
&\le \alpha^2 + 2\alpha x_2\left(1 - \frac{1}{\|x\|}\right) + 1 + \left(1 - \frac{1}{\|x\|}\right)^2 \|x\|^2 \\
&= \alpha^2 + 2\alpha x_2\left(1 - \frac{1}{\|x\|}\right) + \|x\|^2 + 2(1 - \|x\|) \\
&= \alpha^2 + \|x\|^2 + 2(1 - \|x\|)\left(1 - \frac{\alpha x_2}{\|x\|}\right).
\end{aligned}
$$

In particular, if $\|x\| \geq 1 + \frac{\alpha^2+\rho^2}{2(1-\alpha)}$, we have

$$\|T_{S,L_\alpha}x\|^2 \leq \alpha^2 + \|x\|^2 + 2(1-\|x\|)\left(1 - \frac{\alpha x_2}{\|x\|}\right) \leq \alpha^2 + \|x\|^2 + 2(1-\|x\|)(1-\alpha) \leq \|x\|^2 - \rho^2,$$

and the proof is complete. $\square$

Proposition 2.4. *Assume that* $\alpha \in [0,1)$. *If* $\|x\| \leq 1 + \frac{\alpha^2+\rho^2}{2(1-\alpha)}$, *then*

$$\|T_{S,L_\alpha}x\| \leq \frac{\rho^2+4}{2(1-\alpha)}.$$

Proof. As in the proof of Proposition 2.3, we have

$$\|T_{S,L_\alpha}x\|^2 \leq \|x\|^2 + \alpha^2 + 2(1-\|x\|)\left(1 - \frac{\alpha x_2}{\|x\|}\right).$$

If $\|x\| \geq 1$, then since it was assumed that $\|x\| \leq 1 + \frac{\alpha^2+\rho^2}{2(1-\alpha)}$, we have

$$\|T_{S,L_\alpha}x\| \leq \sqrt{\|x\|^2+\alpha^2} \leq \|x\| + \alpha \leq 1 + \frac{\alpha^2+\rho^2}{2(1-\alpha)} + \alpha = \frac{\rho^2-\alpha^2+2}{2(1-\alpha)} \leq \frac{\rho^2+4}{2(1-\alpha)}. \tag{2.11}$$

If $\|x\| \leq 1$, we have

$$\|T_{S,L_\alpha}x\| \leq \sqrt{\|x\|^2+\alpha^2+2} \leq \sqrt{3+\alpha^2} \leq 2 \leq \frac{\rho^2+4}{2(1-\alpha)}. \tag{2.12}$$

Combining (2.11) and (2.12), the result follows. $\square$

Using Proposition 2.3 and Proposition 2.4, we can prove the following invariance property of the operator T_{S,L_α}.

Corollary 2.1. *Assume that* $\alpha \in [0,1)$. *If* $R, \rho \geq 0$ *are such that*

$$R \geq \sqrt{\left(\frac{\rho^2+4}{2(1-\alpha)}\right)^2 + \rho^2},$$

then

$$T_{S,L_\alpha}\big(B[0,R]\big) \subseteq B\big[0, \sqrt{R^2-\rho^2}\,\big].$$

Proof. If $\|x\| \leq R$ and $\|x\| \geq 1 + \frac{\alpha^2+\rho^2}{2(1-\alpha)}$, then by Proposition 2.3 we have

$$\|T_{S,L_\alpha}x\| \leq \sqrt{\|x\|^2-\rho^2} \leq \sqrt{R^2-\rho^2}.$$

Otherwise, if $\|x\| \leq 1 + \frac{\alpha^2+\rho^2}{2(1-\alpha)}$, then by Proposition 2.4, we have

$$\|T_{S,L_\alpha}x\| \leq \frac{\rho^2+4}{2(1-\alpha)} \leq \sqrt{R^2-\rho^2},$$

and this completes the proof. $\square$

Next, we show that if we consider the Douglas-Rachford operator on $\mathbb{R}^d \setminus B(0, 1-\beta)$, we can extend it to all of $\mathbb{R}^d$ while preserving some of its invariance properties. First, we would like to extend the function so that its domain includes $B[0, 1-\beta]$. In order to do so, we use Kirszbraun's Theorem. See for example [BL00].

Theorem 2.2. *Assume that* $A \subseteq \mathbb{R}^d$ *is a subset of* $\mathbb{R}^d$. *Assume that* $f : A \to \mathbb{R}^d$ *is Lipschitz. Then there exists* $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$ *such that* $\tilde{f}\big|_A = f$ *and* $\|\tilde{f}\|_{\mathrm{lip}} = \|f\|_{\mathrm{lip}}$.

Using Theorem 2.2, we deduce the following simple proposition.

Proposition 2.5. *Assume that $r > 0$ and $f : \partial B[0, r] \to \mathbb{R}^d$ is a Lipschitz map. Then there exists a function $\tilde{f} : B[0, r] \to \mathbb{R}^d$ such that $\tilde{f}\big|_{\partial B[0,r]} = f$, $\|\tilde{f}\|_{\mathrm{lip}} = \|f\|_{\mathrm{lip}}$, and*

$$\sup_{x \in B[0,r]} \|\tilde{f}(x)\| \le \sup_{x \in \partial B[0,r]} \|f(x)\| + r\|f\|_{\mathrm{lip}}.$$

Proof. Let $\tilde{f} : B[0, r] \to \mathbb{R}^d$ be the extension from Theorem 2.2. For every $x \in B[0, r]$ there exists $y \in \partial B[0, r]$ such that $\|x - y\| \le r$. Hence, we have

$$\begin{aligned}\|\tilde{f}(x)\| \le \|\tilde{f}(y)\| + \|\tilde{f}(x) - \tilde{f}(y)\| &= \|f(y)\| + \|\tilde{f}(x) - \tilde{f}(y)\| \\ &\le \|f(y)\| + r\|\tilde{f}\|_{\mathrm{lip}} = \|f(y)\| + r\|f\|_{\mathrm{lip}} \le \sup_{y \in \partial B[0,r]} \|f(y)\| + r\|f\|_{\mathrm{lip}}.\end{aligned}$$

□

Using Proposition 2.5, we can now extend T_{S,L_α} from $\mathbb{R}^d \setminus B(0, 1-\beta)$ to all of $\mathbb{R}^d$ while preserving its Lipschitz and its invariance properties.

Proposition 2.6. *Assume that $\alpha, \beta \in [0, 1)$. Then there exists a map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $T\big|_{\mathbb{R}^d \setminus B(0,1-\beta)} = T_{S,L_\alpha}$, and for all $x, y \in \mathbb{R}^d$,*

$$\|Tx - Ty\| \le \frac{\|x - y\|}{1 - \beta} + \beta. \tag{2.13}$$

Moreover, if we assume that $R, \rho \in [0, \infty)$ are such that $R \ge \sqrt{\left(\frac{\rho^2+4}{2(1-\alpha)}\right)^2 + \rho^2 + 5}$, then we have

$$T\big(B[0, R]\big) \subseteq B\big[0, \sqrt{R^2 - \rho^2}\,\big]. \tag{2.14}$$

Proof. Define T so that its domain includes $B[0, 1-\beta]$. To do that, use Kirszbraun's Theorem. There exists $\tilde{f} : B[0, 1-\beta] \to \mathbb{R}^d$ such that $\tilde{f}\big|_{\partial B[0,1-\beta]} = T_{S,L_\alpha}$ and $\|\tilde{f}\|_{\mathrm{lip}} = \big\|T_{S,L_\alpha}\big|_{\partial B[0,1-\beta]}\big\|_{\mathrm{lip}}$. Then, define for $x \in \mathbb{R}^d$,

$$Tx = \begin{cases} \tilde{f}(x) & x \in B[0, 1-\beta] \\ T_{S,L_\alpha} & x \notin B[0, 1-\beta]. \end{cases}$$

Clearly we have $T\big|_{\mathbb{R}^d \setminus B(0,1-\beta)} = T_{S,L_\alpha}$. Note also that for every $x, y \in \partial B[0, 1-\beta]$ we have by (2.5)

$$\|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| \le \max\left\{\frac{1}{1-\beta}, 1 - \frac{1}{1-\beta}\right\} \|x - y\| = \frac{1}{1-\beta}\|x - y\|.$$

Assume that $x, y \in \mathbb{R}^d$. We consider several cases. Assume first that $x, y \in B[0, 1-\beta]$. Then we have

$$\|Tx - Ty\| = \|\tilde{f}(x) - \tilde{f}(y)\| \le \frac{1}{1-\beta}\|x - y\| \le \frac{\|y - x\|}{1-\beta} + \beta. \tag{2.15}$$

Next, assume that $\|x\| \ge 1-\beta$, $\|y\| \ge 1-\beta$. Then $x, y \in \mathbb{R}^d \setminus B(0, 1-\beta)$ and so by Proposition 2.2, we have

$$\|Tx - Ty\| = \|T_{S,L_\alpha}x - T_{S,L_\alpha}y\| \le \frac{\|y - x\|}{1-\beta} + \beta, \tag{2.16}$$

Finally, we need to consider the case $\|x\| \le 1-\beta \le \|y\|$. In such case, there exists $s \in [0,1]$ such that $\|sx+(1-s)y\| = 1-\beta$. Hence, we have

$$\begin{aligned}\|Tx-Ty\| &\le \|Tx - T(sx+(1-s)y)\| + \|T(sx+(1-s)y) - Ty\| \\ &= \|\tilde{f}(x) - \tilde{f}(sx+(1-s)y)\| + \|T_{S,L_\alpha}(sx+(1-s)y) - T_{S,L_\alpha}y\| \\ &\le \frac{1}{1-\beta}\|x-(sx+(1-s)y)\| + \frac{\|sx+(1-s)y-y\|}{1-\beta} + \beta \\ &= \frac{(s-1)\|x-y\|}{1-\beta} + \frac{s\|x-y\|}{1-\beta} + \beta = \frac{\|x-y\|}{1-\beta} + \beta. \end{aligned} \tag{2.17}$$

Combining (2.15), (2.16), (2.17), inequality (2.13) follows. To prove the boundedness property, note that if $x \in B[0, 1-\beta]$, then $Tx = \tilde{f}(x)$ and so by Proposition 2.5, we have

$$\|Tx\| \le \sup_{\|x\|=1-\beta} \|T_{S,L_\alpha}x\| + \frac{1-\beta}{1-\beta} \overset{(2.12)}{\le} \sqrt{3+\alpha^2}+1 \overset{(*)}{\le} 3 \overset{(**)}{\le} \sqrt{R^2-\rho^2}, \tag{2.18}$$

where in $(*)$ we used the fact that $\alpha \le 1$ and in $(**)$ we used the fact that from the choice of R, we have

$$\sqrt{R^2-\rho^2} \ge \sqrt{\left(\frac{\rho^2+4}{2(1-\alpha)}\right)^2 + 5} \ge \sqrt{4+5} = 3.$$

Finally, if $1-\beta \le \|x\| \le R$, then we have by Corollary 2.1,

$$\|Tx\| = \|T_{S,L_\alpha}x\| \le \sqrt{R^2-\rho^2}. \tag{2.19}$$

Combining (2.18) and (2.19), (2.14) follows, and this completes the proof. □

Remark 2.1. Note that so far we have not used the fact the dimension is finite (Kirszbraun's Theorem holds in infinite dimensional Hilbert spaces). ◇

Proposition 2.6 shows how to construct a map on $\mathbb{R}^d$ which is 'almost Lipschitz'. However, in order to use Theorem 3.1 below, we need to have Lipschitz functions. In order to achieve this, we use the following Theorem, proved in [Beg99], which is a simplification of a result that appeared in [Bou87]. See also Lemma 3.1 in [GNS12]. The following theorem holds only in finite dimensional spaces.

Theorem 2.3. *Assume that $M, \delta \in [0,\infty)$. Assume that $T : \mathbb{R}^d \to \mathbb{R}^d$ satisfies for all $x, y \in \mathbb{R}^d$*

$$\|Ty - Tx\| \le M\big(\|y-x\| + \delta\big).$$

Then the map $F : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$Fx = \frac{1}{\mathrm{Vol}\big(B[0,r]\big)} \int_{B[0,r]} T(x+y)dy,$$

is a Lipschitz map with

$$\|F\|_{\mathrm{lip}} \le M\left(1 + \frac{d\delta}{2r}\right),$$

and for all $x \in \mathbb{R}^d$,

$$\|Fx - Tx\| \le M(r+\delta).$$

Using Theorem 2.3, we can prove Theorem 2.1.

Proof of Theorem 2.1. Apply Theorem 2.3 to the map T from Proposition 2.6, with $M = \frac{1}{1-\beta}$ and $\delta = \beta(1-\beta)$. Then (2.2) follows immediately. Also, (2.3) follows from the fact that $T = T_{S,L_\alpha}$ on $\mathbb{R}^d \setminus B(0, 1-\beta)$ combined with Theorem 2.3. To prove (2.4), note that if $x \in B\big[0, \sqrt{R^2-\rho^2}\,\big]$ and $r \le R - \sqrt{R^2-\rho^2}$, then $B[x,r] \subseteq B[0,R]$. Hence, by (2.14) in Proposition 2.6, we have that $\|Ty\| \le \sqrt{R^2-\rho^2}$ for all $y \in B[x,r]$. Therefore, we have

$$\|Fx\| \le \frac{1}{\mathrm{Vol}\big(B[0,r]\big)} \int_{B[x,r]} \|Ty\| dy \le \sqrt{R^2-\rho^2},$$

and this completes the proof. $\square$

Remark 2.2. In [Bou87], F is defined using the Poisson semigroup

$$Fx = \int_{\mathbb{R}^d} P_r(y) T(x+y) dy, \quad P_r(y) = \frac{c_d r}{(r^2 + \|y\|^2)^{\frac{d+1}{2}}},$$

c_d being a normalisation constant. Note, however, that since P_r is not compactly supported, using this version of F would not yield the desired invariance property (2.4). $\diamond$

3. A weak ergodic theorem for sequences of Lipschitz maps

The second auxiliary result is a general weak ergodic theorem for sequences of Lipschitz maps on a normed space. It is a straightforward modification of the main result in [RZ03]. Assume that $(X, \|\cdot\|)$ is a Banach space and $K \subseteq X$ is a closed, bounded and convex set. Let $M \ge 1$. For a sequence of Lipschitz maps $\{F_j\}_{j=1}^\infty$, $F_j : K \to K$, define the following set

$$\mathcal{B}_M = \left\{ \{F_j\}_{j=1}^\infty \;\middle|\; \limsup_{j\to\infty} \|F_j\|_{\mathrm{lip}} \le M \right\}.$$

Let $\theta \in K$ be a fixed vector, and let $\gamma \in [0,1]$. Given a Lipschitz map $F : K \to K$, define the following operator:

$$F_\gamma x = (1-\gamma)Fx + \gamma\theta.$$

Note that since K is convex, we have $F_\gamma : K \to K$. Inductively, for $k \ge 2$, define

$$F_{\gamma^{(k)}} x = \left(F_{\gamma^{(k-1)}}\right)_\gamma x = (1-\gamma)F_{\gamma^{(k-1)}} x + \gamma\theta = (1-\gamma)^k Fx + \left(1 - (1-\gamma)^k\right)\theta. \tag{3.1}$$

The operation $F \to F_\gamma$ is a smoothing operator, with some loss of precision. Next, define the weak distance on $\mathcal{B}_M$. Given $\{F_j\}_{j=1}^\infty, \{G_j\}_{j=1}^\infty \in \mathcal{B}_M$, define

$$d_w(\{F_j\}_{j=1}^\infty, \{G_j\}_{j=1}^\infty) = \sup_{j\in\mathbb{N}} \sup_{x\in K} \|F_j x - G_j x\|.$$

In [RZ03], the notion of strong distance is also defined. However, for the purposes of this note, we will consider only weak distances between sequences of maps. For $\{F_j\}_{j=1}^\infty \in \mathcal{B}_M$ and $r > 0$, let $B_{d_w}(\{F_j\}_{j=1}^\infty, r)$ be the *open* ball around $\{F_j\}_{j=1}^\infty$ with radius r (recall that the notation $B[x,r]$ was used to denote the closed ball). A sequence $\{F_j\}_{j=1}^\infty$ is said to be constant if $F_j = F_1$ for all $j \in \mathbb{N}$.

We are now in a position to state the second main auxiliary result.

Theorem 3.1. *Let $M \ge 1$. Let $(X, \|\cdot\|)$ be a Banach space and $K \subseteq X$ be a closed, bounded, convex set. Then there exists a set $\mathcal{F} \subseteq \mathcal{B}_M$ which is G_δ in the weak-distance topology, which has the following properties.*

(1) *For every $\gamma \in (0,1)$, there exists $k = k(\gamma, M)$, such that for every $\{F_j\}_{j=1}^\infty$, the sequence $\{(F_j)_{\gamma^{(k)}}\}_{j=1}^\infty$ is in $\mathcal{F}$, and we have*

$$d_w(\{F_j\}_{j=1}^\infty, \{(F_j)_{\gamma^{(k)}}\}_{j=1}^\infty) \le \left(1 - \frac{(1-\gamma)^2}{M}\right) \operatorname{diam}(K), \tag{3.2}$$

(2) *For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if*

$$\{H_j\}_{j=1}^\infty \in B_{d_w}\left(\{(F_j)_{\gamma^{(k)}}\}_{j=1}^\infty, \frac{\varepsilon}{100\left(\max\left\{(1-\gamma)^k \sup_{j\in\mathbb{N}} \|F_j\|_{\text{lip}}, 1\right\}\right)^n}\right),$$

then for every injective map $\tau : \{1, \dots, n\} \to \mathbb{N}$, we have

$$\sup_{x,y\in K} \|H_{\tau(n)} \cdots H_{\tau(1)}x - H_{\tau(n)} \cdots H_{\tau(1)}y\| < \varepsilon. \tag{3.3}$$

If $\{F_j\}_{j=1}^\infty$ is a constant sequence, then we can choose

$$\{H_j\}_{j=1}^\infty \in B_{d_w}\left(\{(F_j)_{\gamma^{(k)}}\}_{j=1}^\infty, \frac{\varepsilon}{200}\right),$$

and (3.3) holds for every n satisfying

$$n \ge \frac{\log\left(\frac{\varepsilon}{64\operatorname{diam}(K)}\right)}{\log\left(1 - \frac{\gamma}{2}\right)} + 2. \tag{3.4}$$

Remark 3.1. The case $M = 1$ was studied in [RZ03] with a better bound in (3.2), in addition to other properties which will not be studied here. $\diamond$

Remark 3.2. If $K' \subseteq K$ and $T : K \to K'$, then we can replace the term $\operatorname{diam}(K)$ by $\operatorname{diam}(K')$ in (3.2). $\diamond$

We begin with the following simple proposition, which follows trivially from (3.1).

Proposition 3.1. *Assume that $F : K \to K$ be a Lipschitz map, $\gamma \in (0,1)$ and $k \in \mathbb{N}$. Then*

$$\begin{gathered}
\sup_{x\in K} \left\|F_{\gamma^{(k)}}x - Fx\right\| \le (1 - (1-\gamma)^k)\operatorname{diam}(K), \\
\|F_{\gamma^{(k)}}\|_{\text{lip}} = (1-\gamma)^k \|F\|_{\text{lip}}, \\
\|F - F_{\gamma^{(k)}}\|_{\text{lip}} \le (1 - (1-\gamma)^k)\|F\|_{\text{lip}}.
\end{gathered} \tag{3.5}$$

In particular,

$$d_w(\{F_j\}_{j=1}^\infty, \{(F_j)_{\gamma^{(k)}}\}_{j=1}^\infty) \le (1 - (1-\gamma)^k) \operatorname{diam}(K).$$

The following is a simple modification of Lemma 2.1 in [RZ03].

Lemma 3.1. *Assume that $\{F_j\}_{j=1}^\infty \in \mathcal{B}_M$, $M \ge 1$, $\gamma \in (0,1)$, and $\varepsilon > 0$. Assume that $k \in \mathbb{N}$ is given by*

$$k = \left\lceil \left| \frac{\log M}{\log(1-\gamma)} \right| \right\rceil + 1. \tag{3.6}$$

Then there exists $N \in \mathbb{N}$ such that for every $n \ge N$, every injective $\tau : \{1, \dots, n\} \to \mathbb{N}$, we have

$$\sup_{x,y\in K} \|(F_{\tau(n)})_{\gamma^{(k)}} \cdots (F_{\tau(1)})_{\gamma^{(k)}}x - (F_{\tau(n)})_{\gamma^{(k)}} \cdots (F_{\tau(1)})_{\gamma^{(k)}}y\| < \varepsilon. \tag{3.7}$$

In the case where $\{F_j\}_{n=0}^{\infty}$ is a constant sequence, we can choose

$$N = \frac{\log\left(\frac{\varepsilon}{\operatorname{diam}(K)}\right)}{\log\left(1-\frac{\gamma}{2}\right)} + 2. \tag{3.8}$$

Proof. First, by Lemma 3.1, $\|(F_j)_{\gamma^{(k)}}\|_{\mathrm{lip}} \leq (1-\gamma)^k \|F_j\|_{\mathrm{lip}}$. Since $\{F_j\}_{j=1}^{\infty} \in \mathcal{B}_M$, there exists $j_0 \in \mathbb{N}$ such that if $j \geq j_0$, then

$$\|(F_j)_{\gamma^{(k)}}\|_{\mathrm{lip}} \leq (1-\gamma)^{k-1}\left(1-\frac{\gamma}{2}\right) M.$$

By the choice of k, it follows that for all $j \geq j_0$,

$$\|(F_j)_{\gamma^{(k)}}\|_{\mathrm{lip}} \leq 1 - \frac{\gamma}{2}. \tag{3.9}$$

Next, continue as in Lemma 2.1 in [RZ03]. Choose $j_1 \geq 2$ sufficiently large such that

$$\left(\sup_{j\in\mathbb{N}} \|F_j\|_{\mathrm{lip}} + 1\right)^{j_0} \left(1-\frac{\gamma}{2}\right)^{j_1} \operatorname{diam}(K) < \varepsilon, \tag{3.10}$$

and let $N = j_0 + j_1 + 1$. Let $n \geq N$, $\tau : \{1, \dots, n\} \to \mathbb{N}$ an injective map, and define

$$E_1 = \big\{j \in \{1, \dots, n\} \;\big|\; \tau(j) < j_0\big\}, \quad E_2 = \{1, \dots, n\} \setminus E_1.$$

Since τ is injective, it follows that $|E_1| < j_0$, $|E_2| > j_1$. Therefore, we have

$$\begin{aligned} \Big\|\Big(\prod_{j=1}^{m} (F_{\tau(j)})_{\gamma^{(k)}}\Big)x - \Big(\prod_{j=1}^{m} (F_{\tau(j)})_{\gamma^{(k)}}\Big)y\Big\| &\leq \prod_{j\in E_1} \|(F_{\tau(j)})_{\gamma^{(k)}}\|_{\mathrm{lip}} \prod_{j\in E_2} \|(F_{\tau(j)})_{\gamma^{(k)}}\|_{\mathrm{lip}} \operatorname{diam}(K) \\ \leq \left(\sup_{j\in\mathbb{N}} \|F_j\|_{\mathrm{lip}} + 1\right)^{|E_1|} \left(1-\frac{\gamma}{2}\right)^{|E_2|} \operatorname{diam}(K) &\leq \left(\sup_{j\in\mathbb{N}} \|F_j\|_{\mathrm{lip}} + 1\right)^{j_0} \left(1-\frac{\gamma}{2}\right)^{j_1} \operatorname{diam}(K) < \varepsilon, \end{aligned}$$

which completes the proof of (3.7). In case $\{F_j\}_{j=1}^{\infty}$ is a constant sequence, we can choose $j_0 = 0$, making E_1 an empty set. In order that j_1 satisfy (3.10), we can choose

$$j_1 = \frac{\log\left(\frac{\varepsilon}{\operatorname{diam}(K)}\right)}{\log\left(1-\frac{\gamma}{2}\right)} + 1.$$

Then, choosing $N = j_1 + 1$ completes the proof of (3.8). $\square$

Proposition 3.2. *Assume that $M \geq 1$. If $k \in \mathbb{N}$ is chosen as in Lemma 3.1, then*

$$1 - (1-\gamma)^k \leq 1 - \frac{(1-\gamma)^2}{M}.$$

In particular, for every $\{F_j\}_{j=1}^{\infty} \in \mathcal{B}_M$,

$$d_w(\{F_j\}_{j=1}^{\infty}, \{(F_j)_{\gamma^{(k)}}\}_{j=1}^{\infty}) \leq \left(1 - \frac{(1-\gamma)^2}{M}\right) \operatorname{diam}(K).$$

Proof. Recall that k was chosen to be the smallest integer such that $(1-\gamma)^{k-1} \leq \frac{1}{M}$. Therefore, we have $\frac{1}{M} \leq (1-\gamma)^{k-2}$, which implies that $1 - (1-\gamma)^k \leq 1 - \frac{(1-\gamma)^2}{M}$. $\square$

Lemma 3.2 (Lemma 2.2 in [RZ03]). *Let $\varepsilon > 0$. Assume that $\{F_j\}_{j=1}^{\infty}$ is a sequence of Lipschitz maps. Assume also that there exists $n \in \mathbb{N}$ such that for every injective $\tau : \{1, \dots, n\} \to \mathbb{N}$ we have*

$$\sup_{x,y\in K} \|F_{\tau(n)} \cdots F_{\tau(1)}x - F_{\tau(n)} \cdots F_{\tau(1)}y\| < \frac{\varepsilon}{8}.$$

Assume that

$$\{H_j\}_{j=1}^{\infty} \in B_{d_w}\left(\{F_j\}_{j=1}^{\infty}, \frac{\varepsilon}{16\left(\max\left\{\sup_{j\in\mathbb{N}}\|F_j\|_{\text{lip}},\, 1\right\}\right)^n}\right).$$

Then for every injective $\tau : \{1, \dots, n\} \to \mathbb{N}$*, we have*

$$\sup_{x,y\in K} \|H_{\tau(n)} \cdots H_{\tau(1)}x - H_{\tau(n)} \cdots H_{\tau(1)}y\| < \varepsilon.$$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\{F_j\}_{j=1}^{\infty} \in \mathcal{B}_M$, $\gamma \in (0,1)$ and $l \in \mathbb{N}$. By Lemma 3.1, there exist $k = k(\gamma, M)$ and $N = N(\{F_j\}_{j=1}^{\infty}, \gamma, l)$ such that for every $n \geq N$ and every injective $\tau : \{1, \dots, n\} \to \mathbb{N}$,

$$\sup_{x,y\in K} \|(F_{\tau(n)})_{\gamma^{(k)}} \cdots (F_{\tau(1)})_{\gamma^{(k)}}x - (F_{\tau(n)})_{\gamma^{(k)}} \cdots (F_{\tau(1)})_{\gamma^{(k)}}y\| < \frac{1}{8l}. \tag{3.11}$$

By Lemma 3.2, if

$$\begin{aligned}
\{H_j\}_{j=1}^{\infty} &\in B_{d_w}\left(\{(F_j)_{\gamma^{(k)}}\}_{j=1}^{\infty}, \frac{1}{16\,l\left(\max\left\{\sup_{j\in\mathbb{N}}\|(F_j)_{\gamma^{(k)}}\|_{\text{lip}},\, 1\right\}\right)^n}\right) \\
&\overset{(3.5)}{=} B_{d_w}\left(\{(F_j)_{\gamma^{(k)}}\}_{j=1}^{\infty}, \frac{1}{16\,l\left(\max\left\{(1-\gamma)^k\sup_{j\in\mathbb{N}}\|F_j\|_{\text{lip}},\, 1\right\}\right)^n}\right),
\end{aligned}$$

then

$$\sup_{x,y\in K} \|H_{\tau(n)} \cdots H_{\tau(1)}x - H_{\tau(n)} \cdots H_{\tau(1)}y\| < \frac{1}{l}.$$

Define $\mathcal{F} = \bigcap_{l=1}^{\infty} \mathcal{F}_l$, where

$$\mathcal{F}_l = \bigcup_{\{F_j\}_{j=1}^{\infty}\in\mathcal{B}_M} \bigcup_{\gamma\in(0,1)} B_{d_w}\left(\{(F_j)_{\gamma^{(k)}}\}_{j=1}^{\infty}, \frac{1}{16\,l\left(\max\left\{(1-\gamma)^k\sup_{j\in\mathbb{N}}\|F_j\|_{\text{lip}},\, 1\right\}\right)^{N(\{F_j\}_{j=1}^{\infty},\gamma,l)}}\right).$$

Then $\mathcal{F}$ is a countable intersection of open sets in the weak-distance topology. Also, for every $\{F_j\}_{j=1}^{\infty} \in \mathcal{B}_M$, by Lemma 3.2, we have

$$d_w(\{F_j\}_{j=1}^{\infty}, \{(F_j)_{\gamma^{(k)}}\}_{j=1}^{\infty}) \leq \left(1 - \frac{(1-\gamma)^2}{M}\right)\operatorname{diam}(K).$$

This completes the proof of (3.2). Let $\varepsilon \in (0,1)$, and choose $l = \left\lceil \frac{8}{\varepsilon} \right\rceil$, and $n \geq N(\{F_j\}_{j=1}^{\infty}, \gamma, l)$. Let

$$\begin{aligned}
\{H_j\}_{j=1}^{\infty} &\in B_{d_w}\left(\{(F_j)_{\gamma^{(k)}}\}_{j=1}^{\infty}, \frac{\varepsilon}{200\left(\max\left\{(1-\gamma)^k\sup_{j\in\mathbb{N}}\|F_j\|_{\text{lip}},\, 1\right\}\right)^n}\right) \\
&\subseteq B_{d_w}\left(\{(F_j)_{\gamma^{(k)}}\}_{j=1}^{\infty}, \frac{1}{16\,l\left(\max\left\{(1-\gamma)^k\sup_{j\in\mathbb{N}}\|F_j\|_{\text{lip}},\, 1\right\}\right)^n}\right).
\end{aligned}$$

Then by Lemma 3.2 and (3.11), we have that for every injective $\tau : \{1, \dots, n\} \to \mathbb{N}$,

$$\sup_{x,y\in K} \|H_{\tau(m)} \cdots H_{\tau(1)}x - H_{\tau(m)} \cdots H_{\tau(1)}y\| < \frac{8}{l} \leq \varepsilon,$$

which proves (3.3). In case $\{F_j\}_{j=1}^\infty$ is a constant sequence, we have by the choice of k,

$$(1-\gamma)^k \sup_{j\in\mathbb{N}} \|F\|_{\mathrm{lip}} = (1-\gamma)^k M \in \left[(1-\gamma)^2, 1-\gamma\right].$$

Hence, in this case we have $\max\left\{(1-\gamma)^k \sup_{j\in\mathbb{N}} \|F_j\|_{\mathrm{lip}},\, 1\right\} = 1$, and so we can choose

$$\{H_j\}_{j=1}^\infty \in B_{d_w}\left(\{(F_j)_{\gamma(k)}\}_{j=1}^\infty, \frac{\varepsilon}{200}\right).$$

Also, note that for (3.11) to hold true, by Lemma 3.1 we need to choose

$$N(\{F_j\}_{j=1}^\infty, \gamma, l) = \frac{-\log(8l) - \log\left(\mathrm{diam}(K)\right)}{\log\left(1-\frac{\gamma}{2}\right)} + 2.$$

Since $l \geq \frac{8}{\varepsilon}$, it is enough that we choose

$$n \geq \frac{\log\left(\frac{\varepsilon}{64\mathrm{diam}(K)}\right)}{\log\left(1-\frac{\gamma}{2}\right)} + 2.$$

This proves (3.4) and completes the proof of Theorem 3.1. $\square$

Remark 3.3. By (3.9) it follows that if $\{F_j\}_{j=1}^\infty$ is constant, then $\|(F_j)_{\gamma(k)}\|_{\mathrm{lip}} \leq 1-\frac{\gamma}{2}$ for all $j \in \mathbb{N}$. $\diamond$

4. Proof of Theorem 1.1

First, we begin with the following trivial observations.

Proposition 4.1. *Assume that $\alpha \in [0,1)$. Assume that $R \geq \frac{4}{1-\alpha}$ and $\rho = \sqrt{\frac{(1-\alpha)R}{2}}$. Then the following inequalities hold.*

$$R \geq \sqrt{\left(\frac{\rho^2+4}{2(1-\alpha)}\right)^2 + \rho^2 + 5}, \tag{4.1}$$

$$\sqrt{R^2-\rho^2} \geq \frac{R}{2}, \tag{4.2}$$

$$R - \sqrt{R^2-\rho^2} \geq \frac{1-\alpha}{4}. \tag{4.3}$$

Proof. Since $R \geq \frac{4}{1-\alpha}$, we have $1 \leq \frac{(1-\alpha)R}{4}$. Hence,

$$\begin{aligned}\left(\frac{\rho^2+4}{2(1-\alpha)}\right)^2 + \rho^2 + 5 &\leq \left(\frac{\frac{(1-\alpha)R}{2} + (1-\alpha)R}{2(1-\alpha)}\right)^2 + \frac{(1-\alpha)R}{2} + 5\frac{(1-\alpha)R}{4} \\ &= \frac{9R^2}{16} + \frac{7(1-\alpha)R}{4},\end{aligned}$$

and the last term is smaller than R^2 provided that $R \geq 4(1-\alpha)$, which is assumed to be the case. This proves (4.1). Next, in order to prove (4.2), note that it is equivalent to $R \geq \frac{2}{3}(1-\alpha)$, which is assumed to be the case. Finally, to prove (4.3), note that we have

$$R - \sqrt{R^2-\rho^2} \geq \frac{\rho^2}{2R} = \frac{1-\alpha}{4},$$

and the proof is complete. $\square$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. If we choose $R \geq \frac{4}{1-\alpha}$, and $\rho = \sqrt{\frac{(1-\alpha)R}{2}}$, then by (4.3) in Proposition 4.1, any $r > 0$ satisfying $r \leq \frac{1-\alpha}{4}$ also satisfies $r \leq R - \sqrt{R^2 - \rho^2}$. Also, again by Proposition 4.1, we have $R \geq \sqrt{\left(\frac{\rho^2+4}{2(1-\alpha)}\right)^2 + \rho^2 + 5}$. Hence, all the assumptions of Theorem 2.1 hold, and so there exists $F : \mathbb{R}^d \to \mathbb{R}^d$ that satisfies (2.2), (2.3), (2.4). Next, let $G = F_{\gamma^{(k)}}$, as defined in (3.1), and let

$$K = B\big[0, \sqrt{R^2 - \rho^2}\,\big].$$

Then $\operatorname{diam}(K) \leq 2R$. Also, by (4.2), we have $B[0, R/2] \subseteq B\big[0, \sqrt{R^2 - \rho^2}\big]$. Thus, by part (1) of Theorem 3.1 with $M = \frac{1}{1-\beta} + \frac{d\beta}{2}$, we have

$$\sup_{x \in B[0,R/2]} \|Fx - Gx\| \leq \sup_{x \in B\left[0, \sqrt{R^2-\rho^2}\right]} \|Fx - Gx\| \leq \left(1 - \frac{(1-\gamma)^2}{\frac{1}{1-\beta} + \frac{d\beta}{2}}\right) 2R.$$

Combining this with (2.3), we get

$$\sup_{x \in B[0,R/2] \setminus B(0,1-\beta)} \|T_{S,L_\alpha} x - Gx\| \leq \left(1 - \frac{(1-\gamma)^2}{\frac{1}{1-\beta} + \frac{d\beta}{2}}\right) 2R + \frac{r}{1-\beta} + \beta.$$

This proves (1.7). Next, assuming that $n \in \mathbb{N}$ satisfies (1.8), (1.9) follows from Theorem 3.1. To prove (1.10), note that by the choice of ρ, we have

$$K = B\big[0, \sqrt{R^2 - \rho^2}\big] = B\left[0, \sqrt{R^2 - \frac{1-\alpha}{2}R}\right].$$

Thus, by part (2) of Theorem 3.1 and (4.2) in Proposition 4.1, we have

$$\sup_{x,y \in B[0,R/2]} \|H^n x - H^n y\| \leq \sup_{x,y \in B\left[0, \sqrt{R^2 - \frac{1-\alpha}{2}R}\right]} \|H^n x - H^n y\| < \varepsilon,$$

and this completes the proof of Theorem 1.1. $\square$

Remark 4.1. By Remark 3.3 it follows that $\|G\|_{\mathrm{lip}} \leq 1 - \frac{\gamma}{2}$. $\diamond$

5. Other iteration schemes

5.1. **Families of iteration schemes.** The Douglas-Rachford operator, as defined in (1.1), is a part of a bigger family of operators. Given $A, B \subseteq \mathbb{R}^d$ and $s_1, s_2, s_3 \in [0,1]$, define

$$T_{A,B}^{s_1,s_2,s_3} = s_1 I + (1-s_1)\left(s_2 I + (1-s_2)R_B\right)\left(s_3 I + (1-s_3)R_A\right). \tag{5.1}$$

As before, I denotes the identity operator. See [BST15] for a more detailed discussion of this family of operators. The Douglas-Rachford operator defined in (1.1) corresponds to the case $s_1 = \frac{1}{2}$, $s_2 = s_3 = 0$. Focusing on the case $A = S$, $B = L_\alpha$ as defined in (1.3), we have the following analogue of Proposition (2.1).

Proposition 5.1. *Let $s_1, s_2, s_3 \in [0,1]$. If $\|x\| = \|y\|$ and $\|x\| \leq 1$, then*

$$\|T_{S,L_\alpha}^{s_1,s_2,s_3} x - T_{S,L_\alpha}^{s_1,s_2,s_3} y\| \leq \frac{\|x-y\|}{\|x\|}. \tag{5.2}$$

If $y = tx$, $t \in \mathbb{R} \setminus \{0\}$, then

$$\|T_{S,L_\alpha}^{s_1,s_2,s_3} x - T_{S,L_\alpha}^{s_1,s_2,s_3} y\| \leq \|x-y\|. \tag{5.3}$$

Proof. By (5.1) and the fact that $\|R_{L_\alpha}x - R_{L_\alpha}y\| = \|x-y\|$, we have

$$\begin{aligned}
&\|T^{s_1,s_2,s_3}_{S,L_\alpha}x - T^{s_1,s_2,s_3}_{S,L_\alpha}y\| \le s_1\|x-y\| + (1-s_1)(1-s_2)s_3\|R_{L_\alpha}x - R_{L_\alpha}y\| \\
&\quad + (1-s_1)s_2(1-s_3)\|R_Sx - R_sy\| + (1-s_1)(1-s_2)(1-s_3)\|R_{L_\alpha}R_Sx - R_{L_\alpha}R_Sy\| \\
&= \big(s_1 + (1-s_1)(1-s_2)s_3\big)\|x-y\| \\
&\quad + \big((1-s_1)s_2(1-s_3) + (1-s_1)(1-s_2)(1-s_3)\big)\|R_Sx - R_Sy\| \\
&= s\|x-y\| + (1-s)\|R_Sx - R_Sy\|,
\end{aligned}$$

where $s = s_1 + (1-s_1)(1-s_2)s_3$. If $\|x\| = \|y\|$, we have

$$\begin{aligned}
&\|T^{s_1,s_2,s_3}_{S,L_\alpha}x - T^{s_1,s_2,s_3}_{S,L_\alpha}y\| \le s\|x-y\| + (1-s)\|R_Sx - R_Sy\| \\
&= s\|x-y\| + (1-s)\left|\frac{2}{\|x\|} - 1\right|\|x-y\| = \left(s + (1-s)\left|\frac{2}{\|x\|}-1\right|\right)\|x-y\|.
\end{aligned}$$

Now, since $\|x\| \le 1$, we have

$$s + (1-s)\left|\frac{2}{\|x\|} - 1\right| = s + (1-s)\left(\frac{2}{\|x\|} - 1\right) = 2s - 1 + (1-s)\frac{2}{\|x\|} \le \frac{2s-1+2-2s}{\|x\|} = \frac{1}{\|x\|}.$$

This proves (5.2). The proof of (5.3) is exactly as in the proof of Proposition 2.1, and so the proof is complete. ☐

Now with Proposition 5.1 in hand, we can deduce the following proposition, the same way Proposition 2.2 was deduced from Proposition 2.1.

Proposition 5.2. *Assume that $\|x\| \ge 1-\beta$, $\|y\| \ge 1-\beta$, then*

$$\|T^{s_1,s_2,s_3}_{S,L_\alpha}x - T^{s_1,s_2,s_3}_{S,L_\alpha}y\| \le \frac{\|x-y\|}{1-\beta} + \beta.$$

While the family $T^{s_1,s_2,s_3}_{S,L_\alpha}$ has the same Lipschitz properties as the Douglas-Rachford operator, an analogue result to Corollary 2.1 regarding the invariance would be more difficult to obtain, and would not be discussed in this note.

5.2. **Von-Neumann iteration scheme.** Another interesting example arising from (5.1), is the Von-Neumann operator, corresponding to the case $s_1 = 0$, $s_2 = s_3 = \frac{1}{2}$. Since $R_A = 2P_A - I$, the Von-Neumann operator is given by

$$T^{0,\frac{1}{2},\frac{1}{2}}_{A,B}x = P_BP_Ax. \tag{5.4}$$

Focusing again on the case $A = S$, $B = L_\alpha$ as defined in (1.3), we obtain the following explicit formula,

$$P_{L_\alpha}P_Sx = \frac{x_1}{\|x\|}\mathbf{e}_1 + \alpha\mathbf{e}_2 + \frac{1}{\|x\|}\sum_{j=3}^{\infty} x_j\mathbf{e}_j. \tag{5.5}$$

Note that here we consider the infinite dimensional case (this is possible, as note in Remark 1.1). As will be shown below, the result still holds in this case. For the Von-Neumann operator, we have the following stronger result.

Proposition 5.3. *Let $\mathbb{H}$ be a Hilbert space. Then for every $x, y \in \mathbb{H}$,*

$$\|P_{L_\alpha}P_Sx - P_{L_\alpha}P_Sy\| \le \max\left\{\frac{1}{\|x\|}, \frac{1}{\|y\|}\right\}\|x-y\|.$$

In particular, if $\beta \in [0,1)$ and $\|x\| \geq 1-\beta$, $\|y\| \geq 1-\beta$, then

$$\|P_{L_\alpha} P_S x - P_{L_\alpha} P_S y\| \leq \frac{\|x-y\|}{1-\beta}.$$

Proof. By (5.5), we have

$$P_{L_\alpha} P_S x - P_{L_\alpha} P_S y = P_{\mathbf{e}_2^\perp}\left(\frac{x}{\|x\|} - \frac{y}{\|y\|}\right). \tag{5.6}$$

Next, assume without loss of generality that $\|x\| \leq \|y\|$. Then we have

$$\begin{aligned} &\frac{1}{\|x\|^2}\|x-y\|^2 - \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|^2 = \frac{\|y\|^2}{\|x\|^2} - 2\langle x, y\rangle \left(\frac{1}{\|x\|^2} - \frac{1}{\|x\|\|y\|}\right) - 1 \\ &\geq \frac{\|y\|^2}{\|x\|^2} - 2\|x\|\|y\| \left(\frac{1}{\|x\|^2} - \frac{1}{\|x\|\|y\|}\right) - 1 = \frac{\|y\|^2}{\|x\|^2} - 2\frac{\|y\|}{\|x\|} + 1 = \left(\frac{\|y\|}{\|x\|} - 1\right)^2 \geq 0. \end{aligned}$$

Therefore,

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \frac{1}{\|x\|}\|x-y\|. \tag{5.7}$$

Combining (5.6) and (5.7), we have

$$\|P_{L_\alpha} P_S x - P_{L_\alpha} P_S y\| = \left\|P_{\mathbf{e}_2^\perp}\left(\frac{x}{\|x\|} - \frac{y}{\|y\|}\right)\right\| \leq \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \frac{1}{\|x\|}\|x-y\|,$$

and the proof is complete. □

Proposition 5.3 implies the following.

Theorem 5.1. *Let $\mathbb{H}$ be a Hilbert space and let $\alpha \in [0,1]$. Then there exists $T : \mathbb{H} \to \mathbb{H}$ such that $\|T\|_{\mathrm{lip}} = \frac{1}{1-\beta}$, $Tx = P_{L_\alpha} P_S x$ for all $\|x\| \geq 1-\beta$, and we have*

$$T(\mathbb{H}) \subseteq B\big[0, 1+\sqrt{1+\alpha^2}\,\big]. \tag{5.8}$$

Proof. The existence and Lipschitz property of T follow from Proposition 5.3 and Kirszbraun's Theorem (Theorem 2.2, which holds in the infinite dimensional case as well). To prove (5.8), note that by (5.5), we have

$$\|P_{L_\alpha} P_S x\| \leq \sqrt{1+\alpha^2}. \tag{5.9}$$

Hence, by Proposition 2.5 (which still holds true since Kirszbraun's Theorem holds in the infinite dimensional case), it follows that

$$\sup_{x \in B[0,1-\beta]} \|Tx\| \leq \sup_{\|x\|=1-\beta} \|P_{L_\alpha} P_S x\| + \frac{1-\beta}{1-\beta} \leq 1 + \sqrt{1+\alpha^2}, \tag{5.10}$$

Since $T\big|_{\mathbb{H}\setminus B(0,1-\beta)} = P_{L_\alpha} P_S$, combining (5.9) and (5.10) completes the proof. □

Applying Theorem 3.1 to the map obtained in Theorem 5.1 implies the following.

Theorem 5.2. *Assume that $\mathbb{H}$ is a Hilbert space. Assume that $\alpha, \beta \in [0,1)$, $\gamma, \varepsilon \in (0,1)$, and $R \geq 1+\sqrt{1+\alpha^2}$. Then there exists a map $G : \mathbb{H} \to \mathbb{H}$ satisfying*

$$\sup_{x \in \mathbb{H}\setminus B(0,1-\beta)} \|P_{L_\alpha} P_S x - Gx\| \leq \big(1 - (1-\beta)(1-\gamma)^2\big)\,\big(2 + 2\sqrt{1+\alpha^2}\big). \tag{5.11}$$

and for every $n \in \mathbb{N}$ satisfying

$$n \geq \frac{\log\left(\frac{\varepsilon}{128R}\right)}{\log\left(1-\frac{\gamma}{2}\right)} + 2, \tag{5.12}$$

we have,

$$\sup_{x,y\in B[0,R]} \|G^n x - G^n y\| < \varepsilon.$$

Moreover, if $H : B[0,R] \to B[0,R]$ is such that

$$\sup_{x\in B[0,R]} \|Gx - Hx\| \leq \frac{\varepsilon}{200},$$

then for every $n \in \mathbb{N}$ satisfying (5.12), we have

$$\sup_{x,y\in B[0,R]} \|H^n x - H^n y\| < \varepsilon.$$

Proof. By Remark 3.2 and (5.8), (5.11) follows. The rest of the proof follows from applying Theorem 3.1 to the map obtained in Theorem 5.1. □

References

[AAB13] F. J. Aragón Artacho and J. M. Borwein, *Global convergence of a non-convex Douglas-Rachford iteration*, J. Global Optim. **57** (2013), no. 3, 753–769.

[Beg99] B. Begun, *A remark on almost extensions of Lipschitz functions*, Israel J. Math. **109** (1999), 151–155.

[Ben15] J. Benoist, *The Douglas–Rachford algorithm for the case of the sphere and the line*, J. Global Optim. **63** (2015), no. 2, 363–380.

[BL00] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[BS11] J. M. Borwein and B. Sims, *The Douglas-Rachford algorithm in the absence of convexity*, Fixed-point algorithms for inverse problems in science and engineering, Springer Optim. Appl., vol. 49, Springer, New York, 2011, pp. 93–109.

[BST15] J. M. Borwein, B. Sims, and M. K. Tam, *Norm convergence of realistic projection and reflection methods*, Optimization **64** (2015), no. 1, 161–178.

[Bou87] J. Bourgain, *Remarks on the extension of Lipschitz maps defined on discrete sets and uniform homeomorphisms*, Geometrical aspects of functional analysis (1985/86), Lecture Notes in Math., vol. 1267, Springer, Berlin, 1987, pp. 157–167.

[GNS12] O. Giladi, A. Naor, and G. Schechtman, *Bourgain's discretization theorem*, Ann. Fac. Sci. Toulouse Math. (6) **21** (2012), no. 4, 817–837.

[GK90] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.

[Opi67] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[RZ03] S. Reich and A. Zaslavski, *A weak ergodic theorem for infinite products of Lipschitzian mappings*, Abstr. Appl. Anal. **2** (2003), 67–74.

Computer Assisted Research Mathematics and its Applications (CARMA), The University of Newcastle, University Drive, Callaghan NSW 2308, Australia

E-mail address: jonathan.borwein@newcastle.edu.au, ohad.giladi@newcastle.edu.au